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Well-posedness and exponential decay for a laminated beam with distributed delay term

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Abstract. In this paper, we study the well-posedness and the asymptotic behavior of a one-dimensional laminated beam system with a distributed delay term in the first equation, where the heat conduction is given by Fourier's law effective in the rotation angle displacements. We first give the well-posedness of the system by using the semigroup method. Then, we show that the system is exponentially stable under the assumption of equal wave speeds.

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1. Introduction

Recent advances in smart laminated composite structures in the past two decades resulted in the application of these new generation of structures in modern industries, including automotive, robot arms, aerospace and civil engineering. Such structures are mainly work in harsh dynamic conditions, particularly the design of their piezoelectric materials can be used as both actuators and sensors. Hansen and Spies in [8, 9] derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, the system is given by the following equations:

$$\begin{cases} \rho_1\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ \rho_2\left(3w - \psi\right)_{tt} - G\left(\psi - \varphi_x\right) - D\left(3w - \psi\right)_{xx} = 0, \\ \rho_2 w_{tt} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - D w_{xx} = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and $\varphi = \varphi(x,t)$ is the transversal displacement, $\psi = \psi(x,t)$ denotes the rotational displacement, and $w = w(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. The coefficients $\rho_1, G, \rho_2, D, \gamma, \beta > 0$ are the density of the beams, the shear stiffness, mass

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moment of inertia, flexural rigidity, adhesive stiffness of the beams and the adhesive damping parameter, respectively.

In recent years, an increasing interest has been developed to determine the asymptotic behavior of the solution of several laminated beam problems, we refer the reader to [3, 12, 13, 14, 15, 23, 24] and the references therein. In [23], Raposo considered system (1.1) with two frictional dampings of the form:

$$\begin{cases} \rho_1\varphi_{tt} + G\left(\psi - \varphi_x\right)_x + k_1\varphi_t = 0, \\ \rho_2\left(3w - \psi\right)_{tt} - G\left(\psi - \varphi_x\right) - D\left(3w - \psi\right)_{xx} + k_2\left(3w - \psi\right)_t = 0, \\ \rho_2 w_{tt} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and obtained the exponential decay result under appropriate initial and boundary conditions. In [24], Wang, Xu and Yung considered system (1.1) with the cantilever boundary conditions and two different wave speeds $(\sqrt{\frac{G}{\rho_1}}$ and $\sqrt{\frac{D}{\rho_2}})$. W. Liu and W. Zhao [14] considered a coupled system of a laminated beam with Fourier's type heat conduction, which has the form:

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3w - \psi\right)_{tt} - G\left(\psi - \varphi_x\right) - D\left(3w - \psi\right)_{xx} + \sigma\theta_x = 0, \\ I_\rho w_{tt} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma\left(3w - \psi\right)_{tx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, they used the energy method to prove an exponential decay result for the case of equal wave speeds. (See also [1, 5, 11, 16, 17]).

Time delays arise in many applications of most phenomena naturally modulate by partial differential equations problems, depending not only on the present state but also on some past occurrences. The presence of delay may be a source of instability. It may turn a well-behaved system into a wild one. For example, it was shown in [6, 7, 10, 20, 21, 25] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. In [21], Nicaise and Pignotti considered wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \triangle u + \mu_1 u_t + a\left(x\right)\int_{\tau_1}^{\tau_2} \mu_2\left(s\right) u_t\left(t-s\right) ds = 0, \text{ in } \Omega \times \left(0,\infty\right),$$

with initial and mixed Dirichlet-Neumann boundary conditions and a is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) ds < \mu_1.$$

Regarding the similar result concerning boundary distributed delay see [2, 18, 19]. Moreover, Nicaise, Pignotti and Valein [22] replaced the constant delay term in the boundary condition of [20] by a time-varying delay term and obtained an exponential decay result under an appropriate assumption on the weights of the damping and delay.

In this work, we consider the laminated beam system where the heat flux is given by Fourier's law with distributed delay term. The system is written as

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi - \varphi_x)_x + \mu_0\varphi_t + \int_{\tau_1}^{\tau_2} \mu(s)\,\varphi_t(x, t-s)\,ds = 0, \\ \rho_2(3w-\psi)_{tt} - G(\psi - \varphi_x) - D(3w-\psi)_{xx} + \sigma\theta_x = 0, \\ \rho_2 w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = 0, \end{cases} \tag{1.2}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and ρ_1, G, ρ_2, D, σ, γ, β, k, τ are positive constant coefficients, with the Dirichlet-Neumann boundary conditions:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty), \end{cases} \tag{1.3}$$

and the initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x), & x \in [0,1], \\ \varphi_t(x,-t) = f_0(x,t), & (x;t) \in (0,1) \times (0,\tau_2), \end{cases} \tag{1.4}$$

where τ_1 and τ_2 are two real numbers with $0 \le \tau_1 < \tau_2$, μ_0 is a positive constant, and $\mu : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is an L^∞ function, $\mu \ge 0$ almost everywhere, and the initial data $(\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, f_0)$ belong to a suitable Sobolev space.

Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption

$$\mu_0 \ge \int_{\tau_1}^{\tau_2} \mu(s)\,ds. \tag{1.5}$$

The rest of our paper is organized as follows. In Section 2, by using Hille-Yosida theorem, we state and prove the well posedness of problem (1.2)-(1.4). In Section 3, by using the perturbed energy method, we then establish the exponential result if and only if $\frac{\rho_1}{G} = \frac{\rho_2}{D}$.

2. Well-posedness of the problem

In this section, we will prove that system (1.2)-(1.4) are well posed using semigroup theory by introducing the following new variable as in [21].

$$z(x,\rho,t,s) = \varphi_t(x, t-\rho s), \quad x \in (0,1), \ \rho \in (0,1), \ t > 0, \ s \in (\tau_1,\tau_2). \tag{2.1}$$

Then, we have

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x \in (0,1), \ \rho \in (0,1), \ t > 0, \ s \in (\tau_1,\tau_2). \tag{2.2}$$

Therefore, problem (1.2) takes the form:

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi - \varphi_x)_x + \mu_0\varphi_t + \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,t,s)ds = 0, \\ \rho_2(3w-\psi)_{tt} - G(\psi - \varphi_x) - D(3w-\psi)_{xx} + \sigma\theta_x = 0, \\ \rho_2 w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = 0, \end{cases} \tag{2.3}$$

with the Dirichlet-Neumann boundary conditions:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty), \end{cases} \tag{2.4}$$

and the initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ x \in [0,1], \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ x \in [0,1], \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ x \in [0,1], \\ \theta(x,0) = \theta_0(x),\ x \in [0,1], \\ \varphi_t(x,-t) = f_0(x,t),\ (x,t) \in (0,1) \times (0,\tau_2) \\ z(x,0,t,s) = \varphi_t(x,t) \text{ on } (0,1) \times (0,\infty) \times (\tau_1,\tau_2), \\ z(x,\rho,0,s) = f_0(x,\rho,s) \text{ on } (0,1) \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{2.5}$$

Introducing the vector function

$$U = (\varphi, \varphi_t, 3w-\psi, (3w-\psi)_t, w, w_t, \theta, z)^T,$$

problem (2.3)-(2.5) can be written as

$$\begin{cases} \partial_t U = AU, \\ U(x,0) = U^0(x) = (\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, f_0)^T. \end{cases} \tag{2.6}$$

Where the operator A is defined by

$$AU = \begin{pmatrix} \varphi_t \\ -\dfrac{G}{\rho_1}(\psi - \varphi_x)_x - \dfrac{\mu_0}{\rho_1}\varphi_t - \dfrac{1}{\rho_1}\int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\,ds \\ (3w-\psi)_t \\ \dfrac{G}{\rho_2}(\psi - \varphi_x) + \dfrac{D}{\rho_2}(3w-\psi)_{xx} - \dfrac{\sigma}{\rho_2}\theta_x \\ w_t \\ -\dfrac{G}{\rho_2}(\psi - \varphi_x) - \dfrac{4\gamma}{3\rho_2}w - \dfrac{4\beta}{3\rho_2}w_t + \dfrac{D}{\rho_2}w_{xx} \\ \dfrac{\tau}{\kappa}\theta_{xx} - \dfrac{\sigma}{\kappa}(3w-\psi)_{tx} \\ -s^{-1}z_\rho \end{pmatrix}$$

We consider the following spaces

$$\begin{aligned} H_*^1(0,1) &= \left\{\chi/\chi \in H^1(0,1) : \chi(0) = 0\right\}, \\ \widetilde{H}_*^1(0,1) &= \left\{\chi/\chi \in H^1(0,1) : \chi(1) = 0\right\}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{H} = {} & H_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \\ & \times L^2(0,1) \times L^2(0,1) \times L^2\left((0,1) \times (\tau_1,\tau_2), H^1(0,1)\right), \end{aligned}$$

be the Hilbert space equipped with the inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^1 [\rho_1\varphi_t\widetilde{\varphi}_t + \rho_2\left(3w-\psi\right)_t\left(3\widetilde{w}-\widetilde{\psi}\right)_t + 3\rho_2 w_t\widetilde{w}_t dx + k\theta\widetilde{\theta}\\ &\quad +4\gamma w\widetilde{w} + G\left(\psi-\varphi_x\right)\left(\widetilde{\psi}-\widetilde{\varphi}_x\right) + D\left(3w-\psi\right)_x\left(3\widetilde{w}-\widetilde{\psi}\right)_x\\ &\quad +3Dw_x\widetilde{w}_x]dx + \int_0^1\int_{\tau_1}^{\tau_2} s\mu\left(s\right)\int_0^1 z\left(x,\rho,s\right)\widetilde{z}\left(x,\rho,s\right)d\rho ds dx.\end{aligned}$$

The domain of A is

$$D\left(A\right)=\left\{\begin{array}{c} U\in\mathcal{H}\mid\varphi\in H^2\left(0,1\right)\cap H_*^1\left(0,1\right),\ \theta\in H_*^1\left(0,1\right),\\ 3w-\psi, w\in H^2\left(0,1\right)\cap\widetilde{H}_*^1\left(0,1\right),\\ \varphi_t\in H_*^1\left(0,1\right),\left(3w-\psi\right)_t, w_t\in\widetilde{H}_*^1\left(0,1\right),\\ \varphi_x\left(1,t\right)=\psi_x\left(0,t\right)=w_x\left(0,t\right)=0,\varphi_t\left(x\right)=z\left(x,0,s\right)in(0,1)\end{array}\right\}, \quad (2.7)$$

and it is dense in $\mathcal{H}$. The well-posedness of problem (2.6) is ensured by

Theorem 2.1. *Assume that $U^0\in\mathcal{H}$ and (1.5) holds, then problem (2.6) exists a unique weak solution $U\in C\left(\mathbb{R}^+;\mathcal{H}\right)$. Moreover, if $U^0\in D\left(A\right)$, then*

$$U\in C\left(\mathbb{R}^+;D\left(A\right)\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)\right). \quad (2.8)$$

Proof. To prove the well-posedness result, it suffices to show that $A: D\left(A\right)\to\mathcal{H}$ is a maximal monotone operator, which means A is dissipative and $Id-A$ is surjective. First, we prove that A is dissipative.
For any $U=\left(\varphi,\varphi_t,3w-\psi,\left(3w-\psi\right)_t,w,w_t,\theta,z\right)^T\in D\left(A\right)$, by using the inner product and integrating by parts, we have

$$\begin{aligned}\left\langle AU,U\right\rangle_{\mathcal{H}} &= -\mu_0\int_0^1\varphi_t^2\left(x\right)dx - \int_0^1\varphi_t\left(x\right)\left(\int_{\tau_1}^{\tau_2}\mu\left(s\right)z\left(x,1,s\right)ds\right)dx\\ &\quad -4\beta\int_0^1 w_t^2 dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu\left(s\right)z^2\left(x,1,s\right)dsdx\\ &\quad -\tau\int_0^1\theta_x^2 dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu\left(s\right)ds\int_0^1\varphi_t^2\left(x\right)dx.\end{aligned}$$

Now, using Young's and Cauchy–Schwarz' inequalities, we can estimate,

$$\begin{aligned}&-\int_0^1\varphi_t\left(x\right)\left(\int_{\tau_1}^{\tau_2}\mu\left(s\right)z\left(x,1,s\right)ds\right)dx\\ &\le \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu\left(s\right)ds\right)\int_0^1\varphi_t^2\left(x\right)dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu\left(s\right)z^2\left(x,1,s\right)dsdx.\end{aligned}$$

Therefore, from the assumption (1.5) we have

$$\begin{aligned}&\left\langle AU,U\right\rangle_{\mathcal{H}}\\ &\le -\tau\int_0^1\theta_x^2 dx - 4\beta\int_0^1 w_t^2 dx + \left(-\mu_0+\int_{\tau_1}^{\tau_2}\mu\left(s\right)ds\right)\int_0^1\varphi_t^2\left(x\right)dx \le 0.\end{aligned}$$

Consequently, A is a dissipative operator.
Next, we prove that the operator $Id-A$ is surjective.

Given $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)^T \in \mathcal{H}$, we prove that there exists a unique $U=(\varphi,\varphi_t,3w-\psi,(3w-\psi)_t,w,w_t,\theta,z)^T \in D(A)$ such that

$$(Id-A)\,U=F, \tag{2.9}$$

that is,

$$\left\{\begin{array}{c}
\varphi-\varphi_t=f_1,\\
(\rho_1+\mu_0)\,\varphi_t-G\varphi_{xx}-G\,(3w-\psi)_x+3Gw_x+\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,t,s)\,ds\\
=\rho_1 f_2,\\
(3w-\psi)-(3w-\psi)_t=f_3,\\
\rho_2\,(3w-\psi)_t+G\varphi_x+G\,(3w-\psi)-D\,(3w-\psi)_{xx}-3Gw+\sigma\theta_x\\
=\rho_2 f_4,\\
w-w_t=f_5,\\
\left(\rho_2+\frac{4\beta}{3}\right)w_t-G\varphi_x-G\,(3w-\psi)+\left(3G+\frac{4\gamma}{3}\right)w-Dw_{xx}=\rho_2 f_6,\\
k\theta-\tau\theta_{xx}+\sigma\,(3w-\psi)_{tx}=kf_7,\\
z+s^{-1}z_\rho=f_8.
\end{array}\right. \tag{2.10}$$

From $(2.10)_1$,$(2.10)_3$ and $(2.10)_5$ we have

$$\left\{\begin{array}{c}
\varphi_t=\varphi-f_1,\\
(3w-\psi)_t=(3w-\psi)-f_3,\\
w_t=w-f_5.
\end{array}\right. \tag{2.11}$$

Inserting (2.11) into $(2.10)_2$, $(2.10)_4$, $(2.10)_6$ and $(2.10)_7$, we get

$$\left\{\begin{array}{c}
(\mu_0+\rho_1)\,\varphi-G\varphi_{xx}-G\,(3w-\psi)_x+3Gw_x+\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,t,s)\,ds\\
=\rho_1\,(f_1+f_2)+\mu_0 f_1,\\[2ex]
\rho_2\,(3w-\psi)+G\varphi_x+G\,(3w-\psi)-D\,(3w-\psi)_{xx}-3Gw+\sigma\theta_x\\
=\rho_2\,(f_3+f_4),\\[2ex]
\left(\rho_2+\frac{4\beta}{3}\right)w-G\varphi_x-G\,(3w-\psi)+\left(3G+\frac{4\gamma}{3}\right)w-Dw_{xx}\\
=\rho_2\,(f_5+f_6)+\frac{4\beta}{3}f_5,\\[2ex]
k\theta-\tau\theta_{xx}+\sigma\,(3w-\psi)_x=\sigma\,(f_3)_x+kf_7,\\
z+s^{-1}z_\rho=f_8.
\end{array}\right. \tag{2.12}$$

Using (2.11) and the fact that $z(x,0,s)=\varphi_t(x)$, we get

$$z(x,\rho,s)=\varphi(x)e^{-\rho s}-f_1 e^{-\rho s}+se^{-\rho s}\int_0^\rho f_8(x,\delta,s)e^{\delta s}d\delta, \tag{2.13}$$

In order to solve (2.10), we consider the following variational formulation

$$B\left((\varphi,3w-\psi,w,\theta)^T,\left(\widetilde{\varphi},3\widetilde{w}-\widetilde{\psi},\widetilde{w},\widetilde{\theta}\right)^T\right)=L\left(\widetilde{\varphi},3\widetilde{w}-\widetilde{\psi},\widetilde{w},\widetilde{\theta}\right)^T, \tag{2.14}$$

where $B:\left[H_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times L^2(0,1)\right]^2\longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}
& B\left((\varphi,3w-\psi,w,\theta)^T,\left(\widetilde{\varphi},3\widetilde{w}-\widetilde{\psi},\widetilde{w},\widetilde{\theta}\right)^T\right)\\
= & \int_0^1 G(-\varphi_x+\psi)\left(-\widetilde{\varphi}_x+\widetilde{\psi}\right)dx+\int_0^1(\mu_0+\rho_1)\varphi\widetilde{\varphi}dx+\int_0^1 k\theta\widetilde{\theta}dx\\
& +\int_0^1\rho_2(3w-\psi)\left(3\widetilde{w}-\widetilde{\psi}\right)dx+\int_0^1(3\rho_2+4\beta+4\gamma)w\widetilde{w}dx\\
& +\int_0^1 D(3w-\psi)_x\left(3\widetilde{w}-\widetilde{\psi}\right)_x dx+\int_0^1 3Dw_x\widetilde{w}_x dx+\tau\int_0^1\theta_x\widetilde{\theta}_x dx\\
& +\sigma\int_0^1\theta_x\left(3\widetilde{w}-\widetilde{\psi}\right)dx+\sigma\int_0^1(3w-\psi)_x\widetilde{\theta}dx\\
& +\int_0^1\varphi\widetilde{\varphi}\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}dsdx,
\end{aligned}$$

and $L:\left[H_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times L^2(0,1)\right]\longrightarrow \mathbb{R}$ is the linear form defined by

$$\begin{aligned}
& L\left(\widetilde{\varphi},3\widetilde{w}-\widetilde{\psi},\widetilde{w},\widetilde{\theta}\right)^T\\
= & \int_0^1\rho_1(f_1+f_2)\widetilde{\varphi}dx+\int_0^1\mu_0 f_1\widetilde{\varphi}dx+\int_0^1\rho_2(f_3+f_4)\left(3\widetilde{w}-\widetilde{\psi}\right)dx\\
& +\int_0^1 3\rho_2(f_5+f_6)\widetilde{w}dx+\int_0^1 4\beta f_5\widetilde{w}dx+\int_0^1\sigma(f_3)_x\widetilde{\theta}dx+\int_0^1 kf_7\widetilde{\theta}dx\\
& -\int_0^1\widetilde{\varphi}\int_{\tau_1}^{\tau_2}\mu(s)z_0(x,s)dsdx.
\end{aligned}$$

Now, for $V=H_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times L^2(0,1)$ equipped with the norm

$$\|(\varphi,3w-\psi,w,\theta)\|_V^2=\|-\varphi_x+\psi\|_2^2+\|\varphi\|_2^2+\|3w_x-\psi_x\|_2^2+\|w_x\|_2^2+\|\theta_x\|_2^2.$$

It is easy to verify that $B(.,.)$ is continuous and coercive, and $L(.)$ is continuous. So applying the Lax-Milgram theorem, problem (2.14) admits a unique solution

$$\varphi\in H_*^1(0,1),\ \ (3w-\psi)\in \widetilde{H}_*^1(0,1),\ \ w\in \widetilde{H}_*^1(0,1),\ \ \theta\in L^2(0,1).$$

The substitution of $\varphi, 3w-\psi$ and w into (2.11), we obtain

$$\varphi_t\in H_*^1(0,1),\ \ (3w-\psi)_t\in \widetilde{H}_*^1(0,1),\ \ w_t\in \widetilde{H}_*^1(0,1).$$

Applying the classical elliptic regularity, it follows from (2.12) that

$$\begin{aligned}
\varphi &\in H^2(0,1)\cap H_*^1(0,1),(3w-\psi)\in H^2(0,1)\cap \widetilde{H}_*^1(0,1),\theta\in H_*^1(0,1),\\
w &\in H^2(0,1)\cap \widetilde{H}_*^1(0,1),\varphi_x(1)=(3w-\psi)_x(0)=w_x(0)=0.
\end{aligned}$$

Therefore, the operator $Id-A$ is surjective. Consequently, the well-posedness result stated in Theorem 2.1 follows from the Hille–Yosida theorem (see [4]). $\square$

3. Exponential stability of solution

In this section, we show that, under the assumption $\mu_0 \geq \int_{\tau_1}^{\tau_2} \mu(s)\,ds$ and for $\frac{\rho_1}{G} = \frac{\rho_2}{D}$, the solution of problem (2.3)-(2.5) decays exponentially to the study state. To achieve our goal we use the energy method to produce a suitable Lyapunov functional. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := \frac{1}{2}\int_0^1 \Big[& \rho_1\varphi_t^2 + \rho_2(3w_t-\psi_t)^2 + 3\rho_2 w_t^2 + G(\psi-\varphi_x)^2 + 4\gamma w^2 + k\theta^2 \\ & + D(3w_x-\psi_x)^2 + 3Dw_x^2\Big]dx + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\, z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that $\frac{\rho_1}{G} = \frac{\rho_2}{D}$ and (1.5) holds. Let $U^0 \in \mathcal{H}$, then there exists positive constants c_0, c_1 such that the energy $E(t)$ associated with problem (2.3)-(2.5) satisfies,*

$$E(t) \leq c_0 e^{-c_1 t}, t \geq 0. \tag{3.2}$$

In order to prove this result, we need the following lemmas.

Lemma 3.2. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5) and assume (1.5) holds. Then the energy functional, defined by (3.1) satisfies*

$$\frac{d}{dt}E(t) \leq -4\beta\int_0^1 w_t^2 dx - \tau\int_0^1 \theta_x^2 dx - \left(\mu_0 - \int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^1 \varphi_t^2 dx \leq 0. \tag{3.3}$$

Proof. Multiplying $(2.3)_1$, $(2.3)_2$, $(2.3)_3$ and $(2.3)_4$ by $\varphi_t, 3(w-\psi)_t$, $3w_t$ and θ, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions in (2.4), we get

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\left(\rho_1\int_0^1 \varphi_t^2 dx + G\int_0^1(\psi-\varphi_x)^2 dx\right)\right] \\ & = G\int_0^1(\psi-\varphi_x)\psi_t dx - \mu_0\int_0^1\varphi_t^2 dx - \int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu(s)\varphi_t(x,t-s)\,ds\,dx, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\left(\rho_2\int_0^1(3w_t-\psi_t)^2 dx + D\int_0^1(3w_x-\psi_x)^2 dx\right)\right] \\ = \; & G\int_0^1(\psi-\varphi_x)(3w-\psi)_t dx - \sigma\int_0^1\theta_x(3w-\psi)_t dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\left(3\rho_2\int_0^1 w_t^2 dx + 4\gamma\int_0^1 w^2 dx + 3D\int_0^1 w_x^2 dx\right)\right] \\ = \; & -3G\int_0^1(\psi-\varphi_x)w_t dx - 4\beta\int_0^1 w_t^2 dx, \end{aligned} \tag{3.6}$$

and

$$\frac{d}{dt}\left[\frac{1}{2}k\int_0^1\theta^2 dx\right] = \sigma\int_0^1(3w-\psi)_t\theta_x dx - \tau\int_0^1\theta_x^2 dx. \tag{3.7}$$

On the other hand, multiplying (2.2) by $\mu(s)\, z(x,\rho,s,t)$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\, z(x,\rho,s,t)\, z_t(x,\rho,s,t)\, dsd\rho dx$$

$$+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \mu(s)\, z(x,\rho,s,t)\, z_\rho(x,\rho,s,t)\, dsd\rho dx = 0,$$

thus, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\, z^2(x,\rho,s,t)\, dsd\rho dx\\
= \;& -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,1,s,t)\, dsdx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s)\, ds\int_0^1 \varphi_t^2 dx. \qquad (3.8)
\end{aligned}$$

Summing up (3.4)-(3.8), we arrive at

$$\begin{aligned}
\frac{d}{dt}E(t) \;=\;& -4\beta\int_0^1 w_t^2 dx - \left(\mu_0 - \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)\, ds\right)\right)\int_0^1 \varphi_t^2 dx\\
& -\tau\int_0^1 \theta_x^2 dx - \int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu(s)\, z(x,1,s,t)\, dsdx\\
& -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,1,s,t)\, dsdx.
\end{aligned} \qquad (3.9)$$

Young's and Cauchy–Schwarz' inequalities applied to the fourth term on the right-hand side yield

$$\begin{aligned}
-\int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu(s)\, z(x,1,s,t)\, dsdx \;\leq\;& \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)\, ds\right)\int_0^1 \varphi_t^2 dx\\
& +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,1,s,t)\, dsdx.
\end{aligned} \qquad (3.10)$$

Simple substitution of (3.10) into (3.9) and using (1.5) give (3.3), which concludes the proof. □

Now, we are going to construct a Lyapunov functional equivalent to the energy. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.3. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_1(t) := -\rho_1\int_0^1 \varphi\varphi_t dx \qquad (3.11)$$

satisfies the estimate

$$\begin{aligned} F_1'(t) &\leq -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + C_1\int_0^1 (\psi-\varphi_x)^2 dx + C_2\int_0^1 (3w_x-\psi_x)^2 dx \\ &+C_3\int_0^1 w_x^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s) z^2(x,1,s,t) dsdx. \qquad (3.12) \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{3G}{2} + \frac{\mu_0^2}{\rho_1} + \int_{\tau_1}^{\tau_2}\mu(s) ds, \quad C_2 = G + \frac{2\mu_0^2}{\rho_1} + 2\int_{\tau_1}^{\tau_2}\mu(s) ds, \\ C_3 &= 9G + \frac{18\mu_0^2}{\rho_1} + 18\int_{\tau_1}^{\tau_2}\mu(s) ds. \end{aligned}$$

Proof. Taking the derivative of $F_1(t)$ with respect to t, using the first equation in (2.3), and integrating by parts, gives

$$\begin{aligned} F_1'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx - G\int_0^1 (\psi-\varphi_x)\varphi_x dx + \mu_0\int_0^1 \varphi_t\varphi dx \\ &+ \int_0^1 \varphi\int_{\tau_1}^{\tau_2}\mu(s) z(x,1,t,s) dsdx. \end{aligned}$$

Note that

$$-G\int_0^1 (\psi-\varphi_x)\varphi_x dx = G\int_0^1 (\psi-\varphi_x)^2 dx - G\int_0^1 \psi(\psi-\varphi_x) dx.$$

Then, we deduce that

$$\begin{aligned} F_1'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx + G\int_0^1 (\psi-\varphi_x)^2 dx - G\int_0^1 \psi(\psi-\varphi_x) dx \\ &+\mu_0\int_0^1 \varphi_t\varphi dx + \int_0^1 \varphi\int_{\tau_1}^{\tau_2}\mu(s) z(x,1,t,s) dsdx. \end{aligned}$$

We then use Young's inequality, we obtain

$$\begin{aligned} F_1'(t) &\leq -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + \frac{3G}{2}\int_0^1 (\psi-\varphi_x)^2 dx + \frac{G}{2}\int_0^1 \psi_x^2 dx \\ &+\left(\frac{\mu_0^2}{2\rho_1} + \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s) ds\right)\int_0^1 \varphi^2 dx \\ &+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s) z^2(x,1,s,t) dsdx. \end{aligned}$$

By using (1.5) and the trivial relation

$$\int_0^1 \varphi^2 dx \leq 2\int_0^1 (\psi-\varphi_x)^2 dx + 2\int_0^1 \psi_x^2 dx,$$

we obtain

$$
\begin{aligned}
F_1'(t) \leq & -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + \left(\frac{3G}{2} + \frac{\mu_0^2}{\rho_1} + \int_{\tau_1}^{\tau_2} \mu(s)\, ds\right) \int_0^1 (\psi - \varphi_x)^2\, dx \\
& + \left(\frac{G}{2} + \frac{\mu_0^2}{\rho_1} + \int_{\tau_1}^{\tau_2} \mu(s)\, ds\right) \int_0^1 \psi_x^2 dx \\
& + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z^2(x,1,s,t)\, ds dx.
\end{aligned}
$$

Note that

$$
\int_0^1 \psi_x^2 dx = \int_0^1 (\psi_x - 3w_x + 3w_x)^2\, dx \leq 2\int_0^1 (3w_x - \psi_x)^2\, dx + 18\int_0^1 w_x^2 dx.
$$

Then the estimate (3.12) is established. □

Lemma 3.4. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$
F_2(t) := \rho_2 \int_0^1 (3w - \psi)(3w - \psi)_t\, dx \tag{3.13}
$$

satisfies the estimate

$$
\begin{aligned}
F_2'(t) \leq & -\frac{D}{2}\int_0^1 (3w_x - \psi_x)^2\, dx + \rho_2 \int_0^1 (3w_t - \psi_t)^2\, dx \\
& + \frac{G^2}{D}\int_0^1 (\psi - \varphi_x)^2\, dx + \frac{\sigma^2}{D}\int_0^1 \theta^2 dx.
\end{aligned} \tag{3.14}
$$

Proof. By differentiating $F_2(t)$ with respect to t, then exploiting the second equation in (2.3), and integrating by parts, we obtain

$$
\begin{aligned}
F_2'(t) = & -D\int_0^1 (3w_x - \psi_x)^2\, dx + \rho_2 \int_0^1 (3w_t - \psi_t)^2\, dx \\
& + G\int_0^1 (\psi - \varphi_x)(3w - \psi)\, dx + \sigma \int_0^1 (3w - \psi)_x\, \theta dx.
\end{aligned} \tag{3.15}
$$

Using Young's inequality, we obtain estimate (3.14). □

Lemma 3.5. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$
F_3(t) := \rho_2 \int_0^1 w w_t dx \tag{3.16}
$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$
\begin{aligned}
F_3'(t) \leq & -\left(\frac{4\gamma}{3} - \varepsilon_1\right)\int_0^1 w^2 dx - D\int_0^1 w_x^2 dx + C_4(\varepsilon_1)\int_0^1 w_t^2 dx \\
& + \frac{G^2}{2\varepsilon_1}\int_0^1 (\psi - \varphi_x)^2\, dx.
\end{aligned} \tag{3.17}
$$

where

$$
C_4(\varepsilon_1) = \rho_2 + \frac{8\beta^2}{9\varepsilon_1}.
$$

Proof. By differentiating $F_3(t)$ with respect to t, then exploiting the third equation in (2.3), and integrating by parts, we obtain

$$\begin{aligned} F_3'(t) &= \rho_2 \int_0^1 w_t^2 dx - G \int_0^1 w(\psi - \varphi_x) dx - \frac{4}{3}\gamma \int_0^1 w^2 dx \\ &\quad - \frac{4}{3}\beta \int_0^1 w w_t dx - D \int_0^1 w_x^2 dx. \end{aligned}$$

Using Young's inequality with $\varepsilon_1 > 0$, we obtain estimate (3.17). □

Lemma 3.6. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_4(t) := \frac{k\rho_2}{\sigma} \int_0^1 (3w - \psi)_t \int_0^x \theta dy dx \tag{3.18}$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned} F_4'(t) &\leq -\frac{\rho_2}{2} \int_0^1 (3w_t - \psi_t)^2 dx + C_5(\varepsilon_2) \int_0^1 \theta^2 dx + \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2 dx \\ &\quad + \varepsilon_2 \int_0^1 (3w_x - \psi_x)^2 dx + \frac{\tau\rho_2}{2\sigma^2} \int_0^1 \theta_x^2 dx, \end{aligned} \tag{3.19}$$

where

$$C_5(\varepsilon_2) = k + \frac{k^2 D^2}{4\varepsilon_2 \sigma^2} + \frac{k^2 G^2}{4\varepsilon_2 \sigma^2}.$$

Proof. By differentiating $F_4(t)$ with respect to t, using the second and the fourth equations in (2.3), and integrating by parts, we obtain

$$\begin{aligned} F_4'(t) &= -\rho_2 \int_0^1 (3w_t - \psi_t)^2 dx + \frac{kG}{\sigma} \int_0^1 (\psi - \varphi_x) \int_0^x \theta dy dx \\ &\quad - \frac{kD}{\sigma} \int_0^1 (3w - \psi)_x \theta dx + k \int_0^1 \theta^2 dx + \frac{\tau\rho_2}{\sigma} \int_0^1 (3w - \psi)_t \theta_x dx. \end{aligned} \tag{3.20}$$

Then, using Young's and Poincaré inequalities with $\varepsilon_2 > 0$, we arrive at (3.19). □

Lemma 3.7. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_5(t) := \rho_2 \int_0^1 w_t(\psi - \varphi_x) dx + \rho_2 \int_0^1 w_t \varphi_x dx - \frac{D\rho_1}{G} \int_0^1 (w_x \varphi_t - w_{xt}\varphi) dx \tag{3.21}$$

satisfies, for any $\varepsilon_3 > 0$, the estimate

$$\begin{aligned} F_5'(t) &\leq -\frac{G}{2} \int_0^1 (\psi - \varphi_x)^2 dx + \varepsilon_3 \int_0^1 (3w_t - \psi_t)^2 dx + \frac{16\gamma^2}{9G} \int_0^1 w^2 dx \\ &\quad + C_6 \int_0^1 w_x^2 dx + C_7(\varepsilon_3) \int_0^1 w_t^2 dx + \frac{D\mu_0}{2G} \int_0^1 \varphi_t^2 dx \\ &\quad + \frac{D}{2G} \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x, 1, s, t) ds dx, \end{aligned} \tag{3.22}$$

where $C_6 = \frac{D\mu_0}{2G} + \frac{D}{2G}\int_{\tau_1}^{\tau_2} \mu(s) ds$, $C_7(\varepsilon_3) = \frac{16\beta^2}{9G} + \frac{\rho_2^2}{2\varepsilon_3} + 9\varepsilon_3$.

Proof. Using the first and the third equations in (2.3), and integrating by parts, we obtain

$$\begin{aligned}
& \frac{d}{dt}\left\{\rho_2\int_0^1 w_t\left(\psi-\varphi_x\right)dx\right\} \\
= \; & \frac{D\rho_1}{G}\left\{\frac{d}{dt}\int_0^1\left(w_x\varphi_t-w_{xt}\varphi\right)dx-\int_0^1 w_{tt}\varphi_x dx\right\}+\frac{\mu_0 D}{G}\int_0^1 w_x\varphi_t dx \\
& +\frac{D}{G}\int_0^1 w_x\int_{\tau_1}^{\tau_2}\mu\left(s\right)\varphi_t\left(x,t-s\right)dsdx-G\int_0^1\left(\psi-\varphi_x\right)^2 dx \\
& -\frac{4\gamma}{3}\int_0^1 w\left(\psi-\varphi_x\right)dx-\frac{4\beta}{3}\int_0^1 w_t\left(\psi-\varphi_x\right)dx+\rho_2\int_0^1 w_t\psi_t dx \\
& -\frac{d}{dt}\left\{\rho_2\int_0^1 w_t\varphi_x dx\right\}+\rho_2\int_0^1 w_{tt}\varphi_x dx
\end{aligned}$$

We conclude for

$$\begin{aligned}
F_5'\left(t\right) \; = \; & D\left(\frac{\rho_2}{D}-\frac{\rho_1}{G}\right)\int_0^1 w_{tt}\varphi_x dx+\frac{\mu_0 D}{G}\int_0^1 w_x\varphi_t dx \\
& +\frac{D}{G}\int_0^1 w_x\int_{\tau_1}^{\tau_2}\mu\left(s\right)z\left(x,1,s,t\right)dsdx-G\int_0^1\left(\psi-\varphi_x\right)^2 dx \\
& -\frac{4\gamma}{3}\int_0^1 w\left(\psi-\varphi_x\right)dx-\frac{4\beta}{3}\int_0^1 w_t\left(\psi-\varphi_x\right)dx+\rho_2\int_0^1 w_t\psi_t dx.
\end{aligned}$$

Using Young's inequality and $\frac{\rho_2}{D}=\frac{\rho_1}{G}$, we obtain (3.22). □

Lemma 3.8. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.3)-(2.5) and (2.2). Then the functional*

$$F_6(t):=\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\mu\left(s\right)z^2\left(x,\rho,s,t\right)dsd\rho dx \tag{3.23}$$

satisfies, for some positive constant n, the following estimate

$$\begin{aligned}
F_6'(t) \; \leq \; & -n\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu\left(s\right)z^2\left(x,\rho,s,t\right)dsd\rho dx \\
& -n\int_0^1\int_{\tau_1}^{\tau_2}\mu\left(s\right)z^2\left(x,1,s,t\right)dsdx+\mu_0\int_0^1\varphi_t^2 dx.
\end{aligned} \tag{3.24}$$

Proof. By differentiating $F_6\left(t\right)$ with respect to t, and using the equation (2.2), we obtain

$$\begin{aligned}
F_6'(t) \; = \; & -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\mu\left(s\right)z\left(x,\rho,s,t\right)z_\rho\left(x,\rho,s,t\right)dsd\rho dx \\
= \; & -\int_0^1\int_{\tau_1}^{\tau_2}\mu\left(s\right)\left[e^{-s}z^2\left(x,1,s,t\right)-z^2\left(x,0,s,t\right)\right]dsdx \\
& -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\mu\left(s\right)z^2\left(x,\rho,s,t\right)dsd\rho dx.
\end{aligned} \tag{3.25}$$

Using the fact that $z(x,0,s,t)=\varphi_t$ and $e^{-s}\leq e^{-s\rho}\leq 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} F_6^{'}(t) \quad &\leq \quad -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\mu(s)\, z^2(x,1,s,t)\, dsdx+\int_{\tau_1}^{\tau_2}\mu(s)\, ds\int_0^1\varphi_t^2dx \\ &\quad -n_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,\rho,s,t)\, dsd\rho dx. \end{aligned} \tag{3.26}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\leq -e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $n=e^{-\tau_2}$ and recalling (1.5), we obtain (3.24). □

Next, we define a Lyapunov functional $L(t)$ and show that it is equivalent to the energy functional $E(t)$.

Lemma 3.9. *Let $N,N_2,N_3,N_4,N_5,N_6>0$ and $\frac{\rho_1}{G}=\frac{\rho_2}{D}$, we define*

$$L(t):=NE(t)+F_1(t)+\sum_{i=2}^{i=6}N_iF_i(t) \tag{3.27}$$

For two positive constants β_1 and β_2, we have

$$\beta_1E(t)\leq L(t)\leq\beta_2E(t),\forall t\geq 0. \tag{3.28}$$

Proof. Now, let

$$\mathcal{L}(t)=F_1(t)+\sum_{i=2}^{i=6}N_iF_i(t)$$

$$\begin{aligned} |\mathcal{L}(t)| \quad &\leq \quad \rho_1\int_0^1|\varphi\varphi_t|\, dx+N_2\rho_2\int_0^1|(3w-\psi)(3w-\psi)_t|\, dx \\ &\quad +N_3\rho_2\int_0^1|ww_t|\, dx+N_4\frac{k\rho_2}{\sigma}\int_0^1\left|(3w-\psi)_t\int_0^x\theta dy\right|dx \\ &\quad +N_5\rho_2\int_0^1|w_t(\psi-\varphi_x)|\, dx+N_5\frac{D\rho_1}{G}\int_0^1|(w_x\varphi_t-w_{xt}\varphi)|\, dx \\ &\quad +N_5\rho_2\int_0^1|w_t\varphi_x|\, dx \\ &\quad +N_6\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\left|se^{-s\rho}\mu(s)\, z^2(x,\rho,s,t)\right|dsd\rho dx. \end{aligned}$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, (3.1), and the fact that $e^{-s\rho}\leq 1$ for all $\rho\in[0,1]$, we obtain

$$\begin{aligned} |\mathcal{L}(t)|\leq c\textstyle\int_0^1\Big[&\varphi_t^2+(3w_t-\psi_t)^2+w_t^2+(\psi-\varphi_x)^2+(3w_x-\psi_x)^2+w_x^2+w^2 \\ &+\theta^2\Big]\, dx+c\textstyle\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\, z^2(x,\rho,z,t)\, dsd\rho dx\leq cE(t). \end{aligned}$$

Consequently, $|L(t)-NE(t)|\leq cE(t)$, which yields

$$(N-c)\, E(t)\leq L(t)\leq (N+c)\, E(t).$$

Choosing such that $(N-c)>0$, we obtain estimate (3.28). □

Now, we are ready to state and prove the main result of this section.

Proof. (of Theorem 3.1). By differentiating (3.27) and recalling (3.12), (3.14), (3.17), (3.19), (3.22) and (3.24), we obtain

$$\begin{aligned}
L'(t) \leq & -\left[\left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)\,ds\right) N + \frac{\rho_1}{2} - \frac{D\mu_0}{2G} N_5 - \mu_0 N_6\right] \int_0^1 \varphi_t^2 dx \\
& -\left[\frac{4\gamma}{3} N_3 - \varepsilon_1 N_3 - \frac{16\gamma^2}{9G} N_5\right] \int_0^1 w^2 dx \\
& -\left[\tau N - \frac{\tau\rho_2}{2\sigma^2} N_4\right] \int_0^1 \theta_x^2 dx \\
& -\left[D N_3 - C_3 - C_6 N_5\right] \int_0^1 w_x^2 dx + \left[\frac{\sigma^2}{D} N_2 + C_5(\varepsilon_2) N_4\right] \int_0^1 \theta^2 dx \\
& -\left[\frac{G}{2} N_5 - C_1 - \frac{G^2}{D} N_2 - \frac{G^2}{2\varepsilon_1} N_3 - \varepsilon_2 N_4\right] \int_0^1 (\psi - \varphi_x)^2 dx \\
& -\left[\frac{\rho_2}{2} N_4 - \rho_2 N_2 - \varepsilon_3 N_5\right] \int_0^1 (3w_t - \psi_t)^2 dx \\
& -\left[4\beta N - C_4(\varepsilon_1) N_3 - C_7(\varepsilon_3) N_5\right] \int_0^1 w_t^2 dx \\
& -\left[\frac{D}{2} N_2 - C_2 - \varepsilon_2 N_4\right] \int_0^1 (3w_x - \psi_x)^2 dx \\
& -\left[n N_6\right] \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)\, z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\
& -\left[n N_6 - \frac{1}{2} - \frac{D}{2G} N_5\right] \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z^2(x,1,s,t)\,ds\,dx. \qquad (3.29)
\end{aligned}$$

At this point, we need to choose our constants very carefully. First, we take N_2 large enough, such that

$$\frac{D}{2} N_2 - C_2 \geq 0.$$

Then, we choose N_4 and N_5 large enough, so that

$$\frac{\rho_2}{2} N_4 - \rho_2 N_2 \geq 0, \quad \frac{G}{2} N_5 - C_1 - \frac{G^2}{D} N_2 \geq 0.$$

Next, we pick ε_1 small and choose N_3 large enough, such that

$$D N_3 - C_3 - C_6 N_5 \geq 0, \ \frac{4\gamma}{3} N_3 - \varepsilon_1 N_3 - \frac{16\gamma^2}{9G} N_5 \geq 0.$$

Then, we select N_3 even smaller (if needed) and ε_2, ε_3 small enough, so that

$$\frac{D}{2} N_2 - C_2 - \varepsilon_2 N_4 \geq 0, \ \frac{\rho_2}{2} N_4 - \rho_2 N_2 - \varepsilon_3 N_5 \geq 0,$$

$$\frac{G}{2} N_5 - C_1 - \frac{G^2}{D} N_2 - \frac{G^2}{2\varepsilon_1} N_3 - \varepsilon_2 N_4 \ \geq 0.$$

Furthermore, we choose N_6 large enough, so that

$$nN_6 - \frac{D}{2G}N_5 - \frac{1}{2} \geq 0.$$

Finally, we choose N so large such that

$$\begin{aligned} \left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)\, ds\right) N + \frac{\rho_1}{2} - \frac{D\mu_0}{2G}N_5 - \mu_0 N_6 &\geq 0, \\ 4\beta N - C_4(\varepsilon_1) N_3 - C_7(\varepsilon_3) N_5 &\geq 0. \end{aligned}$$

Thus, we deduce that there exist positive constants α_1 and α_2 such that (3.29) becomes

$$\begin{aligned} L'(t) &\leq -\alpha_1 E(t) - \left[\tau N - \frac{\tau\rho_2}{2\sigma^2}N_4\right] \int_0^1 \theta_x^2 dx + \alpha_2 \int_0^1 \theta^2 dx \\ &\quad - \left[nN_6 - \frac{1}{2} - \frac{D}{2G}N_5\right] \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z^2(x,1,s,t) ds dx \\ &\leq -\alpha_1 E(t) + \alpha_2 \int_0^1 \theta_x^2 dx. \end{aligned}$$

By (3.3), we obtain

$$L'(t) \leq -\alpha_1 E(t) - \alpha_3 E'(t), \tag{3.30}$$

for some $\alpha_3 > 0$. It is obvious that

$$\mathfrak{L}(t) = L(t) + \alpha_3 E(t) \sim E(t).$$

Next, exploiting (3.30), we get

$$\mathfrak{L}'(t) = L'(t) + \alpha_3 E'(t) \leq -\alpha_1 E(t) \leq -c_1 \mathfrak{L}(t), \tag{3.31}$$

for some $c_1 > 0$. Integration (3.31) over $(0,t)$, leads to

$$\mathfrak{L}(t) \leq \mathfrak{L}(0)\, e^{-c_1 t}, \ \forall\, t \geq 0. \tag{3.32}$$

It gives the desired result theorem 3.1 when combined with the equivalence of $L(t)$ and $E(t)$. □

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