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Existence of solutions for an impulsive boundary value problem with nonlinear derivative dependence on unbounded intervals via variational methods

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Abstract. In this paper, we employ the critical point theory and iterative methods to establish the existence of solutions for an impulsive boundary value problem with nonlinear derivative dependence on the half-line.

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1. Introduction

In this paper, we consider the solvability of an impulsive boundary value problem with nonlinear derivative dependence on the half-line. More precisely, we consider the problem

$$\begin{cases} -(p(t)u'(t))' &=& f(t,u(t),u'(t)), \quad \text{a.e. } t \geq 0,\ t \neq t_j, \\ u(0)=u(+\infty) &=& 0, \\ \triangle(p(t_j)u'(t_j)) &=& g(t_j)I_j(u(t_j)), \quad j \in \{1,2,\ldots\}, \end{cases} \tag{1.1}$$

where $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable in $t \in [0,+\infty)$ for each $(x,\xi) \in \mathbb{R}\times\mathbb{R}$, and continuous in $(x,\xi) \in \mathbb{R}\times\mathbb{R}$ for a.e. $t \in [0,+\infty)$. We assume that the impulsive functions $I_j : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous where $t_0 = 0 < t_1 < t_2 < \ldots < t_j < \ldots < t_m \to +\infty$, as $m \to \infty$, are the impulse points.
The coefficient $p : [0,+\infty) \longrightarrow (0,+\infty)$ satisfies $\frac{1}{p} \in L^1(0,+\infty)$, and

$$M = \int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

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We define the jump

$$\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-),$$

where $u'(t_j^+) = \lim_{t\to t_j^+} u'(t)$ and $u'(t_j^-) = \lim_{t\to t_j^-} u'(t)$ stand for the right and the left limits of u' at t_j, respectively. Finally $g : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function that satisfies

$$\sum_{j=1}^{+\infty} g(t_j) < +\infty.$$

Recently, in [2, 3], the authors obtained the existence of solutions for BVPs associated to impulsive equations on unbounded domains by using variational methods. In [4], de Figueiredo, Girardi and Matzeu proved the existence of solution for semilinear elliptic equations with dependence on the gradient through an iterative technique. However, there are few papers that have studied the existence of solutions for impulsive boundary value problems similar to the problem (1.1) by using variational methods coupled with the iterative methods.

In order to use variational methods, we consider a family of boundary value problems with no dependence on the derivative. Namely, for each $w \in H^1_{0,p}(0,+\infty)$, we consider the problem

$$\begin{cases} -(p(t)u'(t))' &= f(t,u(t),w'(t)), \quad \text{a.e. } t\geq 0,\ t\neq t_j,\\ u(0)=u(+\infty) &= 0,\\ \triangle(p(t_j)u'(t_j)) &= g(t_j)I_j(u(t_j)), \quad j\in\{1,2,\ldots\}. \end{cases} \tag{1.2}$$

The class of problems (1.2) is of variational type and we can resolve them by variational methods and the existence of a solution for the initial problem is obtained by iterative methods.

Now we need to define the following Banach space and this before giving the variational formulation of (1.2).

$$H^1_{0,p}(0,+\infty) = \{u\in AC[0,+\infty),\mathbb{R}) \mid u(0)=u(+\infty)=0,\ \sqrt{p}u'\in L^2(0,+\infty)\},$$

equipped with the norm

$$\|u\|_{0,p} = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt + \int_0^{+\infty} u^2(t)dt},$$

or the equivalent norm

$$\|u\|_p = \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2}.$$

Moreover the space $H^1_{0,p}(0,+\infty)$ is reflexive (see [2]).

Lemma 1.1. *On $H^1_{0,p}(0,+\infty)$, the quantity $\|u\| = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt}$ is a norm which is equivalent to the $H^1_{0,p}(0,+\infty)$-norm.*

Now let us recall the following essential embeddings (see [2]).

Lemma 1.2. *$(H^1_{0,p}(0,+\infty),\|\cdot\|)$ embeds in $(C_0[0,+\infty),\|u\|_\infty)$, where*

$$C_0[0,+\infty) = \{u\in C([0,+\infty),\mathbb{R}) \mid \lim_{t\to+\infty} u(t)=0\} \text{ and } \|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|.$$

Lemma 1.3. *$H^1_{0,p}(0,+\infty)$ embeds continuously in $C_0[0,+\infty)$ and in $L^2(0,+\infty)$.*

Lemma 1.4. *The embedding $H^1_{0,p}(0,+\infty) \hookrightarrow C_0[0,+\infty)$ is compact with*

$$\|u\|_\infty \leq M_1\|u\|,$$

where

$$M_1 = \sqrt{\|\frac{1}{p}\|_{L^1}}.$$

2. Preliminaries

First we recall some basic definitions and lemmas which are used in this paper.

Lemma 2.1. *(Minimization Principle[1]) Let X be a reflexive Banach space and J a functional defined on X such that*
(1) $\lim\limits_{\|u\|\to+\infty} J(u) = +\infty$ (coercivity condition),
(2) J is sequentially weakly lower semi-continuous.
Then J is lower bounded on X and achieves its lower bound at some point u_0.

Definition 2.2. Let X be a real Banach space, $J \in C^1(X,\mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \longrightarrow 0$ as $n \to +\infty$ in X' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (PS condition for brevity).

Lemma 2.3. ([5, Theorem 2.2], [6, Theorem 3.1]) [Mountain Pass Theorem] *Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$ satisfying the (PS) condition. Suppose that $J(0) = 0$ and*
(1) there are constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$,
(2) there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$.
Then J possesses a critical value such that $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} J(u),$$

where

$$\Gamma = \big\{\gamma \in C([0,1],X) : \gamma(0) = 0,\ \gamma(1) = u_0\big\}.$$

3. Variational setting

Take $v \in H^1_{0,p}(0,+\infty)$, multiply the equation in problem (1.1) by v and integrate over $(0,+\infty)$, we obtain

$$-\int_0^{+\infty} (p(t)u'(t))'v(t)dt = \int_0^{+\infty} f(t,u(t),u'(t))v(t)dt.$$

The first term is

$$-\int_0^{+\infty}(p(t)u'(t))'v(t)dt = -\sum_{j=0}^{+\infty}\int_{t_j}^{t_{j+1}}(p(t)u'(t))'v(t)dt$$
$$= \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) + \int_0^{+\infty} p(t)u'(t)v'(t)dt.$$

Hence

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt = -\sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) + \int_0^{+\infty} f(t,u(t),u'(t))v(t)dt.$$

Definition 3.1. We say that a function $u \in H^1_{0,p}(0,+\infty)$ is a weak solution of Problem (1.1) if

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t),u'(t))v(t)dt = 0,$$

for every $v \in H^1_{0,p}(0,+\infty)$.

Proposition 3.2. *Suppose that the following conditions hold:*
(H_1) *There exists constant* $\sigma > 2$ *and two positive functions* φ, ψ *such that* $\varphi \in L^1(0,+\infty), \psi \in L^\infty(0,+\infty)$ *with*

$$|f(t,x,\xi)| \le \varphi(t)|x|^\sigma \psi(\xi), \quad \text{for a.e. } t \in [0,+\infty), x \in \mathbb{R}, \xi \in \mathbb{R}.$$

(I_0) *There exist positive constants* c_0 *and* ν *such that*

$$|I_j(x)| \le c_0|x|^\nu, \quad \forall x \in \mathbb{R},\ j \in \{1,2,\ldots\}.$$

Then, for each $w \in H^1_{0,p}(0,+\infty)$ *fixed, the functional* $J_w : H^1_{0,p}(0,+\infty) \longrightarrow \mathbb{R}$ *defined by*

$$J_w(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t),w'(t))dt,$$

where $F(t,u,\xi) = \int_0^u f(t,s,\xi)ds$, *is continuous, differentiable and*

$$(J'_w(u),v) = \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t),w'(t))v(t)dt, \tag{3.1}$$

for all $v \in H^1_{0,p}(0,+\infty)$.

Proof. Claim 1. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then J_w is Gâteaux-differentiable. Indeed, for all $v \in H^1_{0,p}(0,+\infty)$, we have

$$\begin{aligned}
J_w(u+hv) - J_w(u) &= \frac{1}{2}\int_0^{+\infty} p(t)(u'(t)+hv'(t))^2 dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)+hv(t_j)} I_j(\tau)d\tau \\
&- \int_0^{+\infty} F(t,u(t)+hv(t),w'(t))dt \\
&- \frac{1}{2}\int_0^{+\infty} p(t)u'^2(t)dt - \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau \\
&+ \int_0^{+\infty} F(t,u(t),w'(t))dt \\
&= h\int_0^{+\infty} p(t)u'(t)v'(t)dt + \frac{h^2}{2}\int_0^{+\infty} p(t)v'^2(t)dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)\left[\int_0^{u(t_j)+hv(t_j)} I_j(\tau)d\tau - \int_0^{u(t_j)} I_j(\tau)d\tau\right] \\
&- \int_0^{+\infty}\Big[F(t,u(t)+hv(t),w'(t)) - F(t,u(t),w'(t))\Big]dt
\end{aligned}$$

$$\begin{aligned}
J_w(u+hv) - J_w(u) &= h\int_0^{+\infty} p(t)u'(t)v'(t)dt + \frac{h^2}{2}\int_0^{+\infty} p(t)v'^2(t)dt \\
&+ h\sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j)+c_h v(t_j))v(t_j) \\
&- h\int_0^{+\infty} f(t,u(t)+\theta_h v(t),w'(t))v(t)dt,
\end{aligned}$$

where $0 < \theta_h < 1$ and $0 < c_h < 1$ from the Mean Value Theorem. Thus

$$\begin{aligned}
\frac{J_w(u+hv) - J_w(u)}{h} &= \int_0^{+\infty} p(t)u'(t)v'(t)dt + \frac{h}{2}\int_0^{+\infty} p(t)v'^2(t)dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j)+c_h v(t_j))v(t_j) \\
&- \int_0^{+\infty} f(t,u(t)+\theta_h v(t),w'(t))v(t)dt.
\end{aligned}$$

By (H_1), (I_0) and the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}\lim_{h\to 0}\frac{J_w(u+hv)-J_w(u)}{h} &= \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j)\\ &\quad - \int_0^{+\infty} f(t,u(t),w'(t))v(t)dt,\end{aligned}$$

so that, J_w is Gâteaux-differentiable and

$$\begin{aligned}(J_w'(u),v) &= \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j)\\ &\quad - \int_0^{+\infty} f(t,u(t),w'(t))v(t)dt,\end{aligned}$$

for all $v \in H^1_{0,p}(0,+\infty)$. Therefore a critical point of J_w is a weak solution of Problem (1.2).

Claim 2. J_w' is continuous.

Indeed, let (u_n) be a sequence in $H^1_{0,p}(0,+\infty)$ such that $u_n \longrightarrow u$ as $n \longrightarrow +\infty$. From Lemma 1.4, we have (u_n) converges uniformly to u on $[0,+\infty)$ as $n \longrightarrow +\infty$. Since f and I_j are continuous, then

$$f(t,u_n(t),w'(t)) \longrightarrow f(t,u(t),w'(t)), \quad I_j(u_n(t_j)) \longrightarrow I_j(u(t_j))$$

as $n \longrightarrow +\infty$ and it follows from (H_1) that

$$\begin{aligned}|f(t,u_n(t),w'(t))| &\leq \varphi(t)|u_n(t)|^\sigma|\psi(w'(t))|\\ &\leq \varphi(t)\|u_n\|_\infty^\sigma|\psi(w'(t))|\\ &\leq M_1^\sigma\varphi(t)\|u_n\|^\sigma|\psi(w'(t))|.\end{aligned}$$

And by (I_0), we have

$$\begin{aligned}|I_j(u_n(t_j))| &\leq c_0|u_n(t_j)|^\nu\\ &\leq c_0\|u_n\|_\infty^\nu\\ &\leq M_1^\nu c_0\|u_n\|^\nu.\end{aligned}$$

Then from the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to+\infty}\int_0^{+\infty} f(t,u_n(t),w'(t))dt = \int_0^{+\infty} f(t,u(t),w'(t))dt,$$

and

$$\lim_{n\to+\infty}\sum_{j=1}^{+\infty} g(t_j)I_j(u_n(t_j)) = \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j)).$$

So

$$
\begin{aligned}
(J'_w(u_n) - J'_w(u), v) &= \int_0^{+\infty} p(t)(u'_n(t) - u'(t))v'(t)dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)\Big[I_j(u_n(t_j)) - I_j(u(t_j))\Big]v(t_j) \\
&- \int_0^{+\infty} \Big[f(t, u_n(t), w'(t)) - f(t, u(t), w'(t))\Big]v(t)dt.
\end{aligned}
$$

Passing to the limit in $(J'_w(u_n) - J'_w(u), v)$ when $n \longrightarrow +\infty$, using assumptions (H_1), (I_0) and the Lebesgue Dominated Convergence Theorem, we obtain that $J'_w(u_n) \longrightarrow J'_w(u)$, as $n \longrightarrow +\infty$.
Consequently, $J_w \in C^1(H^1_{0,p}(0, +\infty), \mathbb{R})$. □

4. Main results

4.1. Nontrivial weak solution

Theorem 4.1. *Assume that f satisfies (H_1), I_j satisfies (I_0) and the following hypotheses:*

(H_2) $\lim\limits_{x\to 0} \frac{f(t,x,\xi)}{x} = 0$, *uniformly in $t \in [0, +\infty)$ and $\xi \in \mathbb{R}$.*

(H_3) *There exist positive functions $c_1, c_2 \in L^1(0, +\infty)$, and $\mu > 2$ such that*

(a) $F(t, x, \xi) \geq c_1(t)|x|^\mu - c_2(t)$, for a.e. $t \geq 0$, and all $x \in \mathbb{R}$, $\xi \in \mathbb{R}$,

(b) $\mu F(t, x, \xi) \leq x f(t, x, \xi)$, for a.e. $t \geq 0$, and all $x \in \mathbb{R}$, $\xi \in \mathbb{R}$.

(I_1) *There exists $0 < \gamma \leq 2$ such that*

$$\gamma \int_0^x I_j(s)ds \geq x I_j(x) > 0, \ \forall\, x \in \mathbb{R} \setminus \{0\}, \ \forall\, j \in \{1, 2, \ldots\}.$$

Then there exist positive constants d_1, d_2 such that, for each $w \in H^1_{0,p}(0, +\infty)$, Problem (1.2) has at least one nontrivial weak solution u_w satisfying

$$d_1 \leq \|u_w\| \leq d_2.$$

Proof. Claim 1. Let $w \in H^1_{0,p}(0, +\infty)$ fixed. Then J_w satisfies the (PS) condition.
Indeed, let $(u_n) \subset H^1_{0,p}(0, +\infty)$ such that $(J_w(u_n))$ is bounded and $J'_w(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. Using $(H_3)(b)$ and (I_1), there exists some $d > 0$ such that

$$
\begin{aligned}
d &\geq \mu J_w(u_n) - (J'_w(u_n), u_n) \\
&\geq \left(\frac{\mu}{2} - 1\right) \|u_n\|^2 \\
&- \int_0^{+\infty} \Big(\mu F(t, u_n(t), w'(t)) - f(t, u_n(t), w'(t))u_n(t)\Big)dt \\
&+ \sum_{j=1}^{+\infty} g(t_j) \left(\mu \int_0^{u_n(t_j)} I_j(\tau)d\tau - I_j(u_n(t_j))u_n(t_j)\right) \\
&\geq \left(\frac{\mu}{2} - 1\right) \|u_n\|^2.
\end{aligned}
$$

Since $\mu > 2$, it follows that (u_n) is bounded in $H^1_{0,p}(0,+\infty)$.
Then there exists a subsequence of (u_n) still denoted (u_n) such that (u_n) converges weakly to some u in $H^1_{0,p}(0,+\infty)$ because (u_n) is bounded in the reflexive Banach space $H^1_{0,p}(0,+\infty)$. Lemma 1.4 implies that (u_n) converges uniformly to u on $[0,+\infty)$. Thus

$$\lim_{n\to+\infty}\sum_{j=1}^{+\infty} g(t_j)\Big(I_j(u_n(t_j)) - I_j(u(t_j))\Big)(u_n(t_j) - u(t_j)) = 0$$

and

$$\lim_{n\to+\infty}\int_0^{+\infty}\Big(f(t,u_n(t),w'(t)) - f(t,u(t),w'(t))\Big)(u_n(t) - u(t))\,dt = 0.$$

Since $\lim_{n\to+\infty} J'(u_n) = 0$ and (u_n) converges weakly to some u, we get

$$\lim_{n\to+\infty}(J'_w(u_n) - J'_w(u), u_n - u) = 0.$$

From (3.1), we have

$$\begin{aligned}&(J'_w(u_n) - J'_w(u), u_n - u) = \|u_n - u\|^2\\ &+\sum_{j=1}^{+\infty} g(t_j)\big(I_j(u_n(t_j)) - I_j(u(t_j))\big)(u_n(t_j) - u(t_j))\\ &-\int_0^{+\infty}\big(f(t,u_n(t),w'(t)) - f(t,u(t),w'(t))\big)(u_n(t) - u(t))\,dt.\end{aligned}$$

Hence $\lim\limits_{n\to+\infty}\|u_n - u\| = 0$. Thus (u_n) converges strongly to u in $H^1_{0,p}(0,+\infty)$.
Consequently J_w satisfies the (PS) condition.
Claim 2. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then there exist $\rho > 0$ and $\alpha > 0$, independent of w, such that $J_w(u) \geq \alpha, \quad \forall u \in H^1_{0,p}(0,+\infty), \|u\| = \rho$.
Indeed, let $0 < \varepsilon < \frac{1}{M}$. By (H_2), there exists $\delta > 0$ such that

$$|x| \leq \delta \Longrightarrow |f(t,x,\xi)| \leq \varepsilon|x|, \quad \forall t \in [0,+\infty), \xi \in \mathbb{R}.$$

We have $\|u\|^2_{L^2} \leq M\|u\|^2$ (see [2]) , so we deduce that

$$\int_0^{+\infty}|F(t,u(t),w'(t))dt| \leq \frac{\varepsilon}{2}\|u\|^2_{L^2} \leq \frac{\varepsilon}{2}M\|u\|^2, \text{ for a.e. } t \geq 0,$$

whenever $\|u\|_\infty \leq \delta$.
By choosing $0 < \rho \leq \frac{\delta}{M_1}$ and $\alpha = \frac{1}{2}(1 - \varepsilon M)\rho^2$, hence for $\|u\| = \rho$ (note $\|u\|_\infty \leq \delta$), we get

$$\begin{aligned}J_w(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t),w'(t))dt\\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t),w'(t))dt\\ &\geq \frac{1}{2}(1 - \varepsilon M)\|u\|^2 = \alpha.\end{aligned}$$

So there are $\rho > 0$ and $\alpha > 0$ such that $J_w(u) \geq \alpha, \ \ \forall u \in H^1_{0,p}(0,+\infty)$ with $\|u\| = \rho$.
Claim 3. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then there exists $T_0 > 0$, independent of w, such that

$$J_w(\vartheta u^*) \leq 0, \ \forall \vartheta \geq T_0,$$

where $u^* \in H^1_{0,p}(0,+\infty)$ with $\|u^*\| = 1$.
Indeed, from (I_1), there exists $c_3 > 0$ such that

$$\int_0^x I_j(s)ds \leq c_3|x|^\gamma, \ \ \text{for every} \ \ x \in \mathbb{R}.$$

Take an arbitrary $u^* \in H^1_{0,p}(0,+\infty)$ with $\|u^*\| = 1$ and using Lemma 1.4, $(H_3)(a)$, we obtain

$$\begin{aligned}
J_w(\vartheta u^*) = {} & \frac{1}{2}\vartheta^2\|u^*\|^2 + \sum_{j=1}^{+\infty} g(t_j) \int_0^{\vartheta u^*(t_j)} I_j(\tau)d\tau \\
& - \int_0^{+\infty} F(t, \vartheta u^*(t), w'(t))dt \\
& \leq \frac{1}{2}\vartheta^2 + c_3|\vartheta|^\gamma \|u^*\|^\vartheta_\infty \sum_{j=1}^{+\infty} g(t_j) \\
& - |\vartheta|^\mu \int_0^{+\infty} c_1(t)|u^*(t)|^\mu dt + \int_0^{+\infty} c_2(t)dt \\
& \leq \frac{1}{2}\vartheta^2 + c_3|\vartheta|^\gamma M_1^\gamma \sum_{j=1}^{+\infty} g(t_j) \\
& - |\vartheta|^\mu \int_0^{+\infty} c_1(t)|u^*(t)|^\mu dt + \int_0^{+\infty} c_2(t)dt \leq 0,
\end{aligned}$$

when $\vartheta \geq T_0$ for some T_0 large, since $\mu > 2 \geq \gamma$.
By Proposition 3.2, the functional j_w is in $C^1(H^1_{0,p}(0,+\infty),\mathbb{R})$. Lemma 2.3 guarantees that J_w possesses a critical point which is a weak solution of Problem (1.2).
Claim 4. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then there is a constant $d_1 > 0$, independent of w, such that $\|u_w\| \geq d_1$, for all solution u_w obtained above.
Indeed, let u_w be a solution of Problem (1.2). Then

$$\|u_w\|^2 + \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))u_w(t_j) = \int_0^{+\infty} f(t, u_w(t), w'(t))u_w(t)dt.$$

It follows from (H_1) and (H_2) that,

$$|f(t,x,\xi)| \leq \varepsilon|x| + \varphi(t)|x|^\sigma\psi(\xi), \quad \text{for } t \in [0,+\infty), x \in \mathbb{R}, \xi \in \mathbb{R}.$$

Then

$$\begin{aligned}
\|u_w\|^2 &\leq \|u_w\|^2 + \sum_{j=1}^{+\infty} g(t_j) I_j(u_w(t_j)) u_w(t_j) \\
&= \int_0^{+\infty} f(t, u_w(t), w'(t)) u_w(t) dt \\
&\leq \varepsilon \int_0^{+\infty} |u_w(t)|^2 dt + \int_0^{+\infty} \varphi(t) |u_w(t)|^{\sigma+1} \psi(w'(t)) dt \\
&\leq \varepsilon M \|u_w\|^2 + \|\varphi\|_{L^1} \, \|\psi\|_{L^\infty} \|u_w\|_\infty^{\sigma+1} \\
&\leq \varepsilon M \|u_w\|^2 + M_1^{\sigma+1} \|\varphi\|_{L^1} \, \|\psi\|_{L^\infty} \|u_w\|^{\sigma+1},
\end{aligned}$$

which implies that

$$(1 - \varepsilon M)\|u_w\|^2 \leq M_1^{\sigma+1} \|\varphi\|_{L^1} \, \|\psi\|_{L^\infty} \|u_w\|^{\sigma+1}.$$

Hence

$$\|u_w\| \geq d_1, \quad \text{for some } d_1 > 0.$$

Claim 5. Let $w \in H^1_{0,p}(0, +\infty)$ fixed. Then there is a constant $d_2 > 0$, independent of w, such that $\|u_w\| \leq d_2$, for all solution u_w obtained above.
Indeed, by the characterization of the critical point and (H_3), it follows that

$$|J_w(u_w)| \leq \max_{\vartheta \in [0, +\infty)} J_w(\vartheta u^*),$$

where u^* is given in Claim 3.
From $(H_3)(a)$, we get

$$\begin{aligned}
|J_w(u_w)| \leq & \max_{\vartheta \in [0,+\infty)} \left\{ \frac{1}{2}\vartheta^2 + c_3 |\vartheta|^\gamma M_1^\gamma \sum_{j=1}^{+\infty} g(t_j) - |\vartheta|^\mu \int_0^{+\infty} c_1(t) |u^*(t)|^\mu dt \right. \\
& \left. + \int_0^{+\infty} c_2(t) dt \right\}.
\end{aligned}$$

We define K on $[0, +\infty)$ such that

$$K(\vartheta) = \frac{1}{2}\vartheta^2 + c_3 |\vartheta|^\gamma M_1^\gamma \sum_{j=1}^{+\infty} g(t_j) - |\vartheta|^\mu \int_0^{+\infty} c_1(t) |u^*(t)|^\mu dt + \int_0^{+\infty} c_2(t) dt,$$

and since $\mu > 2$, $K(\vartheta)$ can achieve its maximum at some ϑ_0.
Hence

$$|J_w(u_w)| \leq K(\vartheta_0).$$

On the other hand, we have

$$\begin{aligned}\left(1-\frac{2}{\mu}\right)\|u_w\|^2 &= 2J_w(u_w)-\frac{2}{\mu}(J'_w(u_w),u_w)\\ &+ 2\int_0^{+\infty}\Big[F(t,u_w(t),w'(t))-\frac{u_w(t)}{\mu}f(t,u_w(t),w'(t))\Big]dt\\ &+ 2\sum_{j=1}^{+\infty}g(t_j)\Big[\frac{u_w(t_j)}{\mu}I_j(u_w(t_j))-\int_0^{u_w(t_j)}I_j(\tau)d\tau\Big].\end{aligned}$$

Using $(H_3)(b),(I_1)$ and $(J'_w(u_w),u_w)=0$, we obtain

$$\left(1-\frac{2}{\mu}\right)\|u_w\|^2\le K(\vartheta_0).$$

Hence

$$\begin{aligned}\|u_w\| &\le \left(\frac{K(\vartheta_0)}{1-\frac{2}{\mu}}\right)^{\frac{1}{2}}\\ &\le d_2,\end{aligned}\tag{4.1}$$

we can choose $d_2=\left(\frac{K(\vartheta_0)}{1-\frac{2}{\mu}}\right)^{\frac{1}{2}}$, which is independent of w. □

Theorem 4.2. *Assume hypotheses $(H_1)-(H_3),(I_0),(I_1)$ hold and*
(H_4) there exist positive constants L_1 and L_2 such that

$$\begin{aligned}|f(t,x,\xi)-f(t,y,\xi)|\le L_1|x-y|,\quad &\forall t\in[0,+\infty),\ x,y\in[0;M_1d_2],\ \xi\in\mathbb{R},\\ |f(t,x,\xi)-f(t,x,\xi')|\le L_2|\xi-\xi'|,\quad &\forall t\in[0,+\infty),\ x\in[0;M_1d_2],\ \xi,\xi'\in\mathbb{R},\end{aligned}$$

(I_2) there exist positive constants α_j such that

$$|I_j(x)-I_j(y)|\le\alpha_j|x-y|,\quad \forall x,y\in[0;M_1d_2],\ j\in\{1,2,\ldots\}.$$

Then Problem (1.1) has at least one nontrivial weak solution provided that

$$0<\frac{L_2M}{1-L_1M-M_1^2\sum_{j=1}^{+\infty}g(t_j)\alpha_j}<1.$$

Proof. We construct a sequence $(u_n)\subset H^1_{0,p}(0,+\infty)$ as solutions of the problem

$$(P_n)\begin{cases}-(p(t)u'_n(t))' = f(t,u_n(t),u'_{n-1}(t)), & \text{a.e. } t\ge 0,\ t\ne t_j,\\ u_n(0)=u_n(+\infty) = 0, & \\ \triangle(p(t_j)u'_n(t_j)) = g(t_j)I_j(u_n(t_j)), & j\in\{1,2,\ldots\},\end{cases}$$

given in Theorem 4.1, starting with an arbitrary $u_0\in H^1_{0,p}(0,+\infty)$.
It follows from (4.1) and Lemma 1.4 that

$$\|u_n\|_\infty\le M_1d_2.$$

Using (P_{n+1}) and (P_n), we obtain

$$\int_0^{+\infty} p(t)u'_{n+1}(t)(u'_{n+1}(t)-u'_n(t))dt = -\sum_{j=1}^{+\infty} g(t_j)I_j(u_{n+1}(t_j))(u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} f(t,u_{n+1}(t),u'_n(t))(u_{n+1}(t)-u_n(t))dt,$$

and

$$\int_0^{+\infty} p(t)u'_n(t)(u'_{n+1}(t)-u'_n(t))dt = -\sum_{j=1}^{+\infty} g(t_j)I_j(u_n(t_j))(u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} f(t,u_n(t),u'_{n-1}(t))(u_{n+1}(t)-u_n(t))dt.$$

By subtracting, we obtain

$$\|u_{n+1}-u_n\|^2 = -\sum_{j=1}^{+\infty} g(t_j)\Big[I_j(u_{n+1}(t_j))-I_j(u_n(t_j))\Big](u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} \Big[f(t,u_{n+1}(t),u'_n(t))-f(t,u_n(t),u'_{n-1}(t))\Big](u_{n+1}(t)-u_n(t))dt,$$

then

$$\begin{aligned}\|u_{n+1}-u_n\|^2 = {} & -\sum_{j=1}^{+\infty} g(t_j)\Big[I_j(u_{n+1}(t_j))-I_j(u_n(t_j))\Big](u_{n+1}(t_j)-u_n(t_j)) \\ & + \int_0^{+\infty} \Big[f(t,u_{n+1}(t),u'_n(t))-f(t,u_n(t),u'_n(t))\Big](u_{n+1}(t)-u_n(t))dt \\ & + \int_0^{+\infty} \Big[f(t,u_n(t),u'_n(t))-f(t,u_n(t),u'_{n-1}(t))\Big](u_{n+1}(t)-u_n(t))dt.\end{aligned}$$

By (H_4) and (I_2), we get

$$\begin{aligned}\|u_{n+1}-u_n\|^2 \le {} & \sum_{j=1}^{+\infty} g(t_j)\alpha_j|u_{n+1}(t_j)-u_n(t_j)|^2 \\ & + L_1\int_0^{+\infty} |u_{n+1}(t)-u_n(t)|^2dt \\ & + L_2\int_0^{+\infty} |u'_n(t)-u'_{n-1}(t)|\,|u_{n+1}(t)-u_n(t)|dt.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\leq \|u_{n+1}-u_n\|_\infty^2 \sum_{j=1}^{+\infty} g(t_j)\alpha_j + L_1\|u_{n+1}-u_n\|_{L^2}^2 \\ &\quad + L_2\|u_n' - u_{n-1}'\|_{L^2}\,\|u_{n+1}-u_n\|_{L^2} \\ &\leq M_1^2\|u_{n+1}-u_n\|^2 \sum_{j=1}^{+\infty} g(t_j)\alpha_j + L_1 M\|u_{n+1}-u_n\|^2 \\ &\quad + L_2 M\|u_n-u_{n-1}\|\,\|u_{n+1}-u_n\|,\end{aligned}$$

which implies that

$$\|u_{n+1}-u_n\| \leq \frac{L_2 M}{1-L_1M-M_1^2\sum_{j=1}^{+\infty} g(t_j)\alpha_j}\|u_n-u_{n-1}\|.$$

Since

$$0<\frac{L_2 M}{1-L_1M-M_1^2\sum_{j=1}^{+\infty} g(t_j)\alpha_j}<1,$$

it follows that (u_n) is a Cauchy sequence in the reflexive Banach space $H^1_{0,p}(0,+\infty)$. Then the sequence (u_n) strongly converges in $H^1_{0,p}(0,+\infty)$ to some $u \in H^1_{0,p}(0,+\infty)$. Since $\|u_n\| \geq d_1,\ \forall n \in \mathbb{N}$, it follows that $u \neq 0$.
Consequently, we obtain a nontrivial solution for Problem (1.1). □

Now we prove the existence of a solution for the problem (1.1) by using the Minimization principle.

4.2. The sublinear case

Theorem 4.3. *Suppose that the following conditions hold:*
(H_5) *There exist a constant* $\alpha \in [0,1)$ *and positive functions* $a_1, b_1 \in L^1(0,+\infty)$ *such that*

$$|f(t,x,\xi)| \leq a_1(t)|x|^\alpha + b_1(t), \ \ \textit{for a.e. } t \in [0,+\infty) \textit{ and all } x \in \mathbb{R}, \xi \in \mathbb{R}.$$

(I_3) *There exist constants* $c_4>0$ *and* $\beta \in [0,1)$ *such that*

$$|I_j(s)| \leq c_4|s|^\beta,\ \forall\, s \in \mathbb{R},\ j \in \{1,2,\ldots\}.$$

Then there exists positive constant d_3 *such that, for each* $w \in H^1_{0,p}(0,+\infty)$*, Problem (*1.2*) has at least one weak solution* u_w *satisfying*

$$\|u_w\| \leq d_3.$$

Proof. Claim 1. Let $w \in H^1_{0,p}(0,+\infty)$ *fixed. The functional* J_w *is well defined.*
Indeed, take u in $H^1_{0,p}(0,+\infty)$. From (H_5), we deduce that

$$|F(t,u(t),w'(t))| \leq \frac{a_1(t)}{\alpha+1}|u(t)|^{\alpha+1} + b_1(t)|u(t)|.$$

Thus, by using Lemma 1.4

$$\begin{aligned}\left|\int_0^{+\infty} F(t,u(t),w'(t))dt\right| &\leq \|u\|_\infty^{\alpha+1}\int_0^{+\infty} a_1(t)dt + \|u\|_\infty \int_0^{+\infty} b_1(t)dt \\ &\leq \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\int_0^{+\infty} a_1(t)dt + M_1\|u\|\int_0^{+\infty} b_1(t)dt \\ &\leq \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\|a_1\|_{L^1} + M_1\|u\|\|b_1\|_{L^1}.\end{aligned}$$

It follows from (I_3) that

$$\begin{aligned}\left|\sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau\right| &\leq \frac{c_4}{\beta+1}\|u\|_\infty^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &\leq \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j).\end{aligned}$$

Hence

$$\begin{aligned}|J_w(u)| \leq \frac{1}{2}\|u\|^2 &+ \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &+ \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\|a_1\|_{L^1} + M_1\|u\|\|b_1\|_{L^1} \\ &< \infty.\end{aligned}$$

Claim 2. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. J_w is sequentially weakly lower semicontinuous. Indeed, let (u_n) be a sequence in $H^1_{0,p}(0,+\infty)$ such that $u_n \rightharpoonup u$ in $H^1_{0,p}(0,+\infty)$, as $n \to \infty$. Lemma 1.4 implies that (u_n) converges uniformly to u on $[0,+\infty)$ and by the fact that the norm is weakly lower semicontinuous, we have

$$\liminf_{n\to+\infty}\|u_n\| \geq \|u\|.$$

Using the Lebesgue Dominated Convergence Theorem and the continuity of the functions f and $I_j, j \in \{1,2,\dots\}$, we obtain

$$\begin{aligned}\liminf_{n\to+\infty} J_w(u_n) &= \liminf_{n\to+\infty}\left(\frac{1}{2}\|u_n\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u_n(t_j)} I_j(\tau)d\tau \right.\\ &\qquad \left. - \int_0^{+\infty} F(t,u_n(t),w'(t))dt\right) \\ &\geq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t),w'(t))dt \\ &= J(u).\end{aligned}$$

Consequently, J_w is sequentially weakly lower semicontinuous.

Claim 3. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. J_w is coercive.
Indeed, From $(H_5), (I_3)$ and Lemma 1.4, we have

$$\begin{aligned} J_w(u) \geq \frac{1}{2}\|u\|^2 \; - \; \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ - \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\|a_1\|_{L^1} \; - \; M_1\|u\|\|b_1\|_{L^1}. \end{aligned} \tag{4.2}$$

Since $\alpha < 1$ and $\beta < 1$, then (4.2) implies that

$$\lim_{\|u\|\longrightarrow+\infty} J_w(u) = +\infty.$$

So, by Lemma 2.1, J_w has a minimum point u_w. Under hypothesis (H_5) and using the same ideas as in Proposition 3.2, we get, J_w is Gâteaux differentiable. Thus u_w is a critical point of J_w.
Claim 4. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then $\|u_w\| \leq d_3$, for some $d_3 > 0$, for all solutions u_w obtained above.
Indeed, let u_w be a solution of Problem (1.2). Then

$$\|u_w\|^2 = \int_0^{+\infty} f(t, u_w(t), w'(t))u_w(t)dt - \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))u_w(t_j).$$

By (H_5) and (I_3), we get

$$\begin{aligned} \|u_w\|^2 &\leq \int_0^{+\infty} a_1(t)|u_w(t)|^{\alpha+1}dt + \int_0^{+\infty} b_1(t)|u_w(t)|dt \\ &\quad +c_4\sum_{j=1}^{+\infty} g(t_j)|u_w(t_j)|^{\beta+1} \\ &\leq \|u_w\|_\infty^{\alpha+1}\int_0^{+\infty} a_1(t)dt + \|u_w\|_\infty\int_0^{+\infty} b_1(t)dt + c_4\|u_w\|_\infty^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &\leq M_1^{\alpha+1}\|u_w\|^{\alpha+1}\|a_1\|_{L^1} + M_1\|u_w\|\|b_1\|_{L^1} + c_4 M_1^{\beta+1}\|u_w\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j). \end{aligned}$$

Hence

$$\|u_w\| \leq d_3, \quad \text{for some } d_3 > 0.$$

Therefor u_w is a weak solution of Problem (1.2). □

Remark 4.4. In addition, if $u_w \in H^2_p(t_j, t_{j+1})$, for all $j \in \{1, 2, \ldots\}$, where

$$H^2_p(t_j, t_{j+1}) = \{u \in AC[0, +\infty), \mathbb{R}) : \; \sqrt{p}u' \in L^2(t_j, t_{j+1}),\ (pu')' \in L^2(t_j, t_{j+1})\},$$

then u_w will be called a strong solution of Problem (1.2).

Proposition 4.5. *In (H_5), assume that $a_1, b_1 \in L^2(0, +\infty)$. Then every weak solution is a strong solution of Problem (1.2).*

Proof. We know that $u_w \in H^1_{0,p}(0,+\infty)$ is a critical point of J_w. Then, for any $v \in H^1_{0,p}(0,+\infty)$, we have

$$\begin{aligned}\int_0^{+\infty} p(t)u_w'(t)v'(t)dt &+ \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))v(t_j) \\ &- \int_0^{+\infty} f(t,u_w(t),w'(t))v(t)dt = 0. \qquad (4.3)\end{aligned}$$

For $j \in \{1,2,\dots\}$, if $v \in H^1_{0,p}(t_j,t_{j+1})$ $(v = v_j)$, then

$$\int_{t_j}^{t_{j+1}} p(t)u_w'(t)v'(t)dt = \int_{t_j}^{t_{j+1}} f(t,u_w(t),w'(t))v(t)dt.$$

So $u_{w,j} \in H^1_{0,p}(t_j,t_{j+1})$ is a solution of the equation:

$$-(p(t)u_w')' = f(t,u_w(t),w'(t)),\ t \in (t_j,t_{j+1}), \qquad (4.4)$$

Since, $u_w \in C_0[0,+\infty)$, and by (H_5), we get

$$|f(t,u_w(t),w'(t))|^2 \le 2\left(a_1(t)^2\|u_w\|_\infty^{2\alpha} + b_1(t)^2\right),$$

thus $u_{w,j} \in H^2_p\,(t_j,t_{j+1})$. Then (4.4), implies that the limits
$u'(t_j^+), u'(t_j^-)$, $\quad j \in \{1,2,\dots\}$ exist.
Using the integration by parts in (4.3), we obtain

$$\begin{aligned} 0 &= -\sum_{j=0}^{j=+\infty}\int_{t_j}^{t_{j+1}} (p(t)u_w'(t))'v(t)dt - \sum_{j=1}^{+\infty} \triangle(p(t_j)u_w'(t_j))v(t_j) \\ &\quad + \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))v(t_j) - \int_0^{+\infty} f(t,u_w(t),w'(t))v(t)dt.\end{aligned}$$

Since u_w satisfies the equation in problem (1.2) a.e. on $[0,+\infty)$, we deduce that

$$\sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))v(t_j) = \sum_{j=1}^{+\infty} \triangle(p(t_j)u_w'(t_j))v(t_j), \quad \text{for all}\, v \in H^1_{0,p}(0,+\infty).$$

Thus

$$\triangle(p(t_j)u_w'(t_j)) = g(t_j)I_j(u_w(t_j)), \ \text{ for every } \ j \in \{1,2,\dots\}.$$

Actually, u_w is even a classical solution, i.e., $u \in C^2(t_j,t_{j+1})$, for all $j \in \{1,2,\dots\}$, when $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. □

Theorem 4.6. *Assume that $(H_4),(H_5),(I_2)$ and (I_3) hold.*
Then Problem (1.1) has at least one classical solution provided that

$$0 < \frac{L_2 M}{1 - L_1 M - M_1^2 \sum_{j=1}^{+\infty} g(t_j)\alpha_j} < 1.$$

Proof. The proof is similar to the proof of Theorem 4.2. □

Example 4.7. Consider the impulsive boundary value problem

$$\begin{cases} -(e^t u'(t))' &= \frac{\sqrt{|u|}}{(1+t)^2}\cos u' + \frac{1}{(1+t)^3}, \quad \text{a.e. } t \geq 0,\ t \neq t_j, \\ u(0) = u(+\infty) &= 0, \\ \triangle(e^j u'(j)) &= \frac{\sqrt[3]{u(j)}}{1+j^2}, \qquad j \in \{1,2,\ldots\}. \end{cases} \tag{4.5}$$

We know that all hypotheses of Theorem 4.3 are satisfied with

$$f(t,x,\xi) = \frac{\sqrt{|x|}}{(1+t)^2}\cos\xi + \frac{1}{(1+t)^3},$$

$$\alpha = 1/2,\ a_1(t) = \frac{1}{(1+t)^2},\ b_1(t) = \frac{1}{(1+t)^3},$$

$$I_j(s) = s^{1/3},\ \beta = \frac{1}{3},\ c_4 = 1,$$

$$g(t) = \frac{1}{1+t^2} \text{ and } \sum_{j=1}^{\infty} g(j) = \frac{\pi}{4}.$$

Consequently, problem (4.5) has at least one solution.

4.3. The limit case $\alpha = 1$

Theorem 4.8. *Suppose that* (I_3) *holds and*
(H_6) *there exist positive functions* $a_2, b_2 \in L^1(0,+\infty)$ *with* $||a_2||_{L^1} < \frac{1}{M_1^2}$ *and*

$$|f(t,x,\xi)| \leq a_2(t)|x| + b_2(t), \quad \text{for a.e. } t \in [0,+\infty) \text{ and } \forall x \in \mathbb{R}, \xi \in \mathbb{R}.$$

Then there exists positive constant d_4 *such that, for each* $w \in H^1_{0,p}(0,+\infty)$, *Problem* (1.2) *has at least one weak solution* u_w *satisfying*

$$\|u_w\| \leq d_4.$$

Proof. Claim 1. Let $w \in H^1_{0,p}(0,+\infty)$ *fixed.* J_w *is sequentially weakly lower semicontinuous.*
Indeed, we use the same technique as in the proof of Theorem 4.3.
Claim 2. Let $w \in H^1_{0,p}(0,+\infty)$ *fixed.* J_w *is coercive.*
Indeed, by (H_6), we obtain

$$|F(t,u(t),w'(t))| \leq \frac{a_2(t)}{2}|u(t)|^2 + b_2(t)|u(t)|,$$

hence

$$\left|\int_0^{+\infty} F(t,u(t),w'(t))dt\right| \leq \int_0^{+\infty}\left(\frac{a_2(t)}{2}|u(t)|^2 + b_2(t)|u(t)|\right)dt$$
$$\leq \frac{M_1^2}{2}\|u\|^2\|a_2\|_{L^1} + M_1\|u\|\|b_2\|_{L^1}.$$

Thus

$$\begin{aligned} J_w(u) &\geq \frac{1}{2}\left(1 - M_1^{\,2}\|a_2\|_{L^1}\right)\|u\|^2 - M_1\|u\|\|b_2\|_{L^1} \\ &\quad - \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\,\beta+1}\sum_{j=1}^{+\infty} g(t_j). \end{aligned} \tag{4.6}$$

Since $\|a_2\|_{L^1} < \frac{1}{M_1^2}$ and $\beta < 1$, we pass to the limit in (4.6) when $n \to +\infty$, we get

$$\lim_{\|u\| \longrightarrow +\infty} J_w(u) = +\infty.$$

Therefore, J_w is coercive.
By applying Lemma 2.1, we find that J_w has a minimum point u_w. Under hypothesis (H_6) and using the same ideas as in Proposition 3.2, we get, J_w is Gâteaux differentiable. Then u_w is a critical point of J_w which is a weak solution of Problem (1.2).
Claim 3. Let $w \in H^1_{0,p}(0, +\infty)$ fixed. Then $\|u_w\| \leq d_4$, for some $d_4 > 0$, for all solutions u_w obtained above.
Indeed, let u_w be a solution of Problem (1.2). Then

$$\|u_w\|^2 = \int_0^{+\infty} f(t, u_w(t), w'(t))u_w(t)dt - \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))u_w(t_j).$$

It follows from (H_6) and (I_3) that

$$\begin{aligned} \|u_w\|^2 &\leq \int_0^{+\infty} a_2(t)|u_w(t)|^2 dt + \int_0^{+\infty} b_2(t)|u_w(t)|dt \\ &\quad + c_4\sum_{j=1}^{+\infty} g(t_j)|u_w(t_j)|^{\beta+1} \\ &\leq \|u_w\|_\infty^2 \int_0^{+\infty} a_2(t)dt + \|u_w\|_\infty \int_0^{+\infty} b_2(t)dt + c_4\|u_w\|_\infty^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &\leq M_1^2\|a_2\|_{L^1}\|u_w\|^2 + M_1\|u_w\|\|b_2\|_{L^1} + c_4 M_1^{\beta+1}\|u_w\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j). \end{aligned}$$

Thus

$$(1 - M_1^2\|a_2\|_{L^1})\|u_w\|^2 \leq M_1\|u_w\|\|b_2\|_{L^1} + c_4 M_1^{\beta+1}\|u_w\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j).$$

Hence

$$\|u_w\| \leq d_4, \quad \text{for some } d_4 > 0.$$

□

Theorem 4.9. *Assume that $(H_4), (H_6), (I_2)$ and (I_3) hold.*
Then Problem (1.1) has at least one weak solution provided that

$$0 < \frac{L_2 M}{1 - L_1 M - M_1^2\sum_{j=1}^{+\infty} g(t_j)\alpha_j} < 1.$$

Proof. Reasoning like in the proof of Theorem 4.2, we can prove that Problem (1.1) has at least one weak solution. □

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