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On the existence of positive solutions of a class of parabolic reaction diffusion systems

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Abstract. In this paper, we show the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions using techniques of functional analysis and potential analysis.

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1. Introduction

The modeling and the mathematical analysis of parabolic systems, in particular, reaction diffusion systems, has been the subject of in-depth studies of several mathematicians in recent years, as they appear in the modeling of a large variety of phenomena, not only in biology and chemistry, but also in engineering, economics and ecology, such as gas dynamics, fusion processes, cellular processes, disease propagation, industrial processes , catalytic transport of contaminants in the environment, population dynamics, flame spread and others.

For the mathematical analysis of this type of problem, various methods and elaborate techniques have been proposed, see for example Mesbahi et *al.* [1], [2], [16], [15], Gontara [9], Lions [10], Maâgli et *al.* [13], [12], Pierre [17] and Zhang [20], [19]. We refer the reader to Arakelian and Gauthier [3], Armitage and Gardiner [4] and Port [18] for more details on the potential arguments of the theory that interest us mainly in this work.

The subject of this paper is in this context, we will take care to study the existence of positive solutions of the following nonlinear parabolic reaction diffusion

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system

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \Delta u = \lambda p(x,t) f(v) \\ -\dfrac{\partial v}{\partial t} + \Delta v = \mu q(x,t) g(w) \\ -\dfrac{\partial w}{\partial t} + \Delta w = \eta r(x,t) h(z) \\ -\dfrac{\partial z}{\partial t} + \Delta z = \varrho e(x,t) k(u) \end{cases} \tag{1.1}$$

with $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and the initial conditions

$$\begin{cases} u(x,0) = \varphi(x) \quad , \quad v(x,0) = \psi(x) \\ w(x,0) = \gamma(x) \quad , \quad z(x,0) = \zeta(x) \end{cases} , \ \forall x \in \mathbb{R}^n \tag{1.2}$$

where $n \geq 3$, φ, ψ, γ and $\zeta : \mathbb{R}^n \to [0,\infty)$ are continuous, the constants λ, μ, η and ϱ are nonnegative, f, g, h and $k : (0,\infty) \to [0,\infty)$ are nondecreasing and continuous. p, q, r and $e : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ are measurable functions and satisfy an appropriate hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n)$ introduced in Zhang [19].

Before stating the main result of this work, it is worth mentioning that several mathematicians have dealt with this type of problem using various analytical and numerical techniques and methods, under different hypotheses as appropriate, see for example, Bachar et *al.* [5], Maâgli et *al.* [7], [6], [13]-[14], Ghergu and Radulescu [8], Gontara [9], Ma [11], Zhang [20], [19] and Zhao [21].

Concerning the problem $(1.1)-(1.2)$ in the case of a single equation of the form

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = q(x,t) u^{p+1} \quad , \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R}^n, \ n \geq 3 \end{cases} \tag{1.3}$$

Zhang in [20] discussed the existence and the asymptotic behavior of solutions to this problem, he proved the following result:

Theorem 1.1. *Suppose $p > 0$, $q \in P^\infty(\mathbb{R}^n)$. For any $M > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_0 \in C^2(\mathbb{R}^n)$ satisfying $\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq b_0$, there exists a positive and continuous solution u of* (1.3) *such that*

$$M^{-1} \int_{\mathbb{R}^n} G(x,t,y,0) u_0(y) dy \leq u(x,t) \leq M \int_{\mathbb{R}^n} G(x,t,y,0) u_0(y) dy$$

for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

G denotes the fundamental solution of the heat equation $\Delta u - \dfrac{\partial u}{\partial t} = 0$ in $\mathbb{R}^n \times (0,\infty)$ given for $t > s$ and $x, \ y \in \mathbb{R}^n$ by

$$G(x,t,y,s) = \frac{1}{[4\pi(t-s)]^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)$$

In [12], the authors considered the problem (1.3) with boundary condition u_0, not necessarily bounded function. The nonlinearity $u\varphi(.,u)$ is required to satisfy some conditions related to the parabolic Kato class $P^{\infty}(\mathbb{R}^n)$. They gave existence results and similar estimates on the solutions as in [20].

In [13], a similar problem as (1.3) has been treated in the half space $\mathbb{R}^n_+$. The elliptical version of (1.3) was studied in [7]. Dans [9], the authors examined the problem (1.1) – (1.2) in the case of a system with two equations.

2. Statement of the main result

2.1. Assumptions

To study problem (1.1) – (1.2), we consider the following definition and hypotheses:

Definition 2.1. We say that a nonnegative superharmonic function ω satisfies condition (H_0) if ω is locally bounded in $\mathbb{R}^n$ $(n \geq 3)$ and the map $(x,t) \mapsto P\omega(x,t)$ is continuous in $\mathbb{R}^n \times (0,\infty)$, where P is defined below.

$(P_t)_{t>0}$ on $\mathbb{R}^n$ denotes the Gauss semigroup defined for each nonnegative measurable function Φ on $\mathbb{R}^n$ by

$$P_t\Phi(x) = P\Phi(x,t) = \int_{\mathbb{R}^n} G(x,t,y,0)\Phi(y)dy \quad , \; t>0, \; x \in \mathbb{R}^n$$

The family $(P_t)_{t>0}$ is a markovian semigroup. Moreover, a nonnegative superharmonic function ω on $\mathbb{R}^n$ satisfies for every $t>0$, $P_t\omega \leq \omega$, and consequently the mapping $t \mapsto P_t\omega$ is nonincreasing. We remark that for each nonnegative measurable function Φ on $\mathbb{R}^n$, the map $(x,t) \to P_t\Phi(x)$ is lower semicontinuous on $\mathbb{R}^n \times (0,\infty)$ and becomes continuous if Φ is further bounded.

Remark 2.2. We note that every bounded superharmonic function in $\mathbb{R}^n$ satisfies (H_0), see Gontara and Turki [9] and Mâagli et *al.* [12].

We fix four nonnegative superharmonic functions ω, θ, δ and ϕ satisfying condition (H_0). Let us introduce the required hypotheses on the initial values φ, ψ, γ and ζ the nonlinear terms:

(H_1) There exist four constants $c_i > 1$, $1 \leq i \leq 4$, such that

$$\frac{1}{c_1}\omega(x) \leq \psi(x) \leq c_1\omega(x) \quad , \quad \frac{1}{c_2}\theta(x) \leq \varphi(x) \leq c_2\theta(x)$$
$$\frac{1}{c_3}\delta(x) \leq \gamma(x) \leq c_3\delta(x) \quad , \quad \frac{1}{c_4}\phi(x) \leq \zeta(x) \leq c_4\phi(x)$$

and

$$\lim_{t\to 0} P_t\psi(x) = \psi(x) \quad , \quad \lim_{t\to 0} P_t\varphi(x) = \varphi(x)$$
$$\lim_{t\to 0} P_t\gamma(x) = \gamma(x) \quad , \quad \lim_{t\to 0} P_t\zeta(x) = \zeta(x)$$

(H_2) $f,\ g,\ h,\ k : (0,\infty) \to [0,\infty)$ are nondecreasing and continuous.

(H_3) The functions $p,\ q,\ r$ and e are measurable nonnegative and for each $c>0$, the functions

$$\tilde{p}_c = \frac{pf(cP\omega)}{P\theta} \quad , \quad \tilde{q}_c = \frac{qg(cP\delta)}{P\omega} \quad , \quad \tilde{r}_c = \frac{rh(cP\phi)}{P\delta} \quad , \quad \tilde{e}_c = \frac{ek(cP\theta)}{P\phi}$$

belong to the parabolic Kato class $P^{\infty}(\mathbb{R}^n)$.

To study (1.1) − (1.2), a basic assumptions on p, q, r and e requires to fix four superharmonic functions ω, θ, δ and ϕ on $\mathbb{R}^n$ satisfying condition (H_0).

2.2. The main result

Now, we can state the main result of this work:

Theorem 2.3. *Assume* $(H_1)-(H_3)$. *Then there exist four constants* λ_0, μ_0, η_0 *and* ϱ_0 *such that for each* $\lambda \in [0,\lambda_0)$, $\mu \in [0,\mu_0)$, $\eta \in [0,\eta_0)$ *and* $\varrho \in [0,\varrho_0)$, *the problem* $(1.1)-(1.2)$ *has a positive continuous solution* (u,v,w,z) *in* $(\mathbb{R}^n \times (0,\infty))^4$ *satisfying for each* $t>0$ *and* $x \in \mathbb{R}^n$

$$\begin{cases} (1-\dfrac{\lambda}{\lambda_0})P\varphi(x,t) \leq u(x,t) \leq P\varphi(x,t) \\ (1-\dfrac{\mu}{\mu_0})P\psi(x,t) \leq v(x,t) \leq P\psi(x,t) \\ (1-\dfrac{\eta}{\eta_0})P\gamma(x,t) \leq w(x,t) \leq P\gamma(x,t) \\ (1-\dfrac{\varrho}{\varrho_0})P\zeta(x,t) \leq z(x,t) \leq P\zeta(x,t) \end{cases} \tag{2.1}$$

This document is organized as follows: In the next section, we give some technical results and to recall some theoretical tools that are essential to prove our main result. The last section is devoted to the proof of the main result, Theorem 2.3. The difficulties in this section are similar to those in [5]-[9], [13]-[14] and [20]-[21], and the techniques are of the same spirit, but specific new difficulties due to the nature of the system must be handled.

3. Preliminary results

We give here some essential results proved in [12], we can also see [19], [21], which were retained for the proof of our result. Now, we recall the definition of the Kato class $P^{\infty}(\mathbb{R}^n)$.

Definition 3.1. A Borel measurable function q in $\mathbb{R}^{n+1}$ belongs to the Kato class $P^{\infty}(\mathbb{R}^n)$ if for all $c>0$,

$$\lim_{\epsilon\to 0}\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}\int_{t-\epsilon}^{t+\epsilon}\int_{B(x,\sqrt{\epsilon})} G_c(x,|t-s|,y,0)\,|q(y,s)|\,dyds = 0$$

and

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} G_c(x,|t-s|,y,0)\,|q(y,s)|\,dyds < \infty$$

where

$$G_c(x,t,y,s) = \frac{1}{(t-s)^{\frac{n}{2}}}\exp(-c\frac{|x-y|^2}{t-s}) \quad \text{for } t>s \text{ and } x,\ y\in\mathbb{R}^n$$

In the following, we give a class of functions belonging to $P^{\infty}(\mathbb{R}^n)$.

Proposition 3.2. *(i)* $L^\infty(\mathbb{R}^n) \otimes L^1(\mathbb{R}) \subset P^\infty(\mathbb{R}^n)$.

(ii) Let $1 \le p < \infty$ *and* $q \ge 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $\sigma > \frac{np}{2}$ *and* $\tau < \frac{2}{p} - \frac{n}{\sigma} < \upsilon$, *we have*

$$\frac{L^\sigma(\mathbb{R}^n)}{|\cdot|^\tau (1+|\cdot|)^{\upsilon-\tau}} \otimes L^q(\mathbb{R}) \subset P^\infty(\mathbb{R}^n)$$

(iii) $P^\infty(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^{n+1})$.

We denote for any measurable function Φ on $\mathbb{R}^n \times (0,\infty)$, the potential

$$V\Phi(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x,t,y,s)\Phi(y,s)dyds = \int_0^t P_{t-s}(\Phi(.,s))(x)ds$$

Proposition 3.3. *Let* q *be a nonnegative function in* $P^\infty(\mathbb{R}^n)$, *then there exists a positive constant* α_q *such that for each superharmonic function* υ *in* $\mathbb{R}^n$, *we have*

$$V(qP\upsilon)(x,t) \le \alpha_q P\upsilon(x,t) \quad , \quad \textit{for} \ \ (x,t) \in \mathbb{R}^n \times (0,\infty)$$

Proposition 3.4. *Let* υ *be a superharmonic function in* $\mathbb{R}^n$ *satisfying* (H_0) *and* q *be a nonnegative function in* $P^\infty(\mathbb{R}^n)$. *Then the family of functions*

$$\left\{(x,t) \to Vf(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x,t,y,s)f(y,s)dyds, \ |f| \le qP\upsilon\right\}$$

is equicontinuous in $\mathbb{R}^n \times [0,\infty)$.

Moreover, for each $x \in \mathbb{R}^n$ *we have* $\lim_{t\to 0} Vf(x,t) = 0$, *uniformly on* f.

We therefore conclude the following result on the continuity needed to obtain the proof of Theorem 2.3.

Proposition 3.5. *Assuming the hypothesis* (H_1). *Then the functions* $P\varphi$, $P\psi$, $P\gamma$ *and* $P\zeta$ *are continuous in* $\mathbb{R}^n \times (0,\infty)$.

Proof. We prove that $P\varphi$ is continuous in $\mathbb{R}^n \times (0,\infty)$.
Let c_2 be the constant given in (H_1). We write for each $t > 0$ and $x \in \mathbb{R}^n$

$$c_2 P_t\theta(x) = P_t(c_2\theta - \varphi)(x) + P_t\varphi(x)$$

So, from (H_0) we have $(x,t) \mapsto P\theta(x,t)$ is continuous in $\mathbb{R}^n \times (0,\infty)$ and from the fact that $(x,t) \mapsto P_t(c_2\theta - \varphi)(x)$ and $(x,t) \mapsto P_t\varphi(x)$ are lower semicontinuous in $\mathbb{R}^n \times (0,\infty)$, we deduce that $(x,t) \mapsto P_t\varphi(x)$ is continuous in $\mathbb{R}^n \times (0,\infty)$.
Similarly, we can prove the continuity of $P\psi$, $P\gamma$ and $P\zeta$ in $\mathbb{R}^n \times (0,\infty)$. □

4. Proof of the main result

Let

$$\begin{aligned}
\lambda_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\varphi(x,t)}{V(pf(P\psi))(x,t)}\\
\mu_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\psi(x,t)}{V(qg(P\gamma))(x,t)}\\
\eta_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\gamma(x,t)}{V(rh(P\zeta))(x,t)}\\
\varrho_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\zeta(x,t)}{V(ek(P\varphi))(x,t)}
\end{aligned}$$

Proposition 4.1. *Suppose that the hypotheses $(H_1) - (H_3)$ are satisfied, then the constants λ_0, μ_0, η_0 and ϱ_0 are positive.*

Proof. The hypothesis (H_1) leads to $\psi \leq c_1\omega$.
From the fact that f is nondecreasing and p is nonnegative, we have

$$V(pf(P\psi)) \leq V(pf(c_1 P\omega))$$

Hence, by hypothesis (H_3) and Proposition 3.3, there exist $\tilde{p}_{c_1} \in P^\infty(\mathbb{R}^n)$ and a positive constant $\alpha_{\tilde{p}_{c_1}}$ such that for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$, we have

$$V(pf(P\psi))(x,t) \leq V(\tilde{p}_{c_1} P\theta)(x,t) \leq \alpha_{\tilde{p}_{c_1}} P\theta(x,t)$$

So, using again (H_1) we find for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$

$$\frac{P\varphi(x,t)}{V(pf(P\psi))(x,t)} \geq \frac{\frac{1}{c_1}P\theta(x,t)}{\alpha_{\tilde{p}_{c_1}} P\theta(x,t)} = \frac{1}{c_1\alpha_{\tilde{p}_{c_1}}} > 0$$

In the same way, we prove that

$$\frac{P\psi(x,t)}{V(qg(P\gamma))(x,t)} > 0 \quad , \quad \frac{P\gamma(x,t)}{V(rh(P\zeta))(x,t)} > 0 \quad , \quad \frac{P\zeta(x,t)}{V(ek(P\varphi))(x,t)} > 0$$

which implies that $\lambda_0 > 0$, $\mu_0 > 0$, $\eta_0 > 0$, $\varrho_0 > 0$. □

Proof. **(of Theorem 2.3).** Let $\lambda \in [0,\lambda_0)$, $\mu \in [0,\mu_0)$, $\eta \in [0,\eta_0)$ and $\varrho \in [0,\varrho_0)$. We define the sequences $(u_j)_{j\geq 0}$, $(v_j)_{j\geq 0}$, $(w_j)_{j\geq 0}$, and $(z_j)_{j\geq 0}$ by

$$\begin{cases}
v_0 = P\psi \quad , \quad z_0 = P\zeta\\
u_j = P\varphi - \lambda V(pf(v_j))\\
w_j = P\gamma - \eta V(rh(z_j))\\
z_{j+1} = P\zeta - \varrho V(ek(u_j))\\
v_{j+1} = P\psi - \mu V(qg(w_j))
\end{cases}$$

We are determined to prove for all $j \in \mathbb{N}$,

$$0 < (1 - \frac{\lambda}{\lambda_0})P\varphi \leq u_j \leq u_{j+1} \leq P\varphi \tag{4.1}$$

$$0 < (1 - \frac{\eta}{\eta_0})P\gamma \leq w_j \leq w_{j+1} \leq P\gamma \tag{4.2}$$

$$0 < (1 - \frac{\mu}{\mu_0})P\psi \leq v_{j+1} \leq v_j \leq P\psi \tag{4.3}$$

$$0 < (1 - \frac{\varrho}{\varrho_0})P\zeta \leq z_{j+1} \leq z_j \leq P\zeta \tag{4.4}$$

We note that according to the definition of λ_0, μ_0, η_0 and ϱ_0 that, for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$

$$\lambda_0 V(pf(P\psi))(x,t) \leq P\varphi(x,t) \tag{4.5}$$

$$\mu_0 V(qg(P\gamma))(x,t) \leq P\psi(x,t) \tag{4.6}$$

$$\eta_0 V(rh(P\zeta))(x,t) \leq P\gamma(x,t) \tag{4.7}$$

$$\varrho_0 V(ek(P\varphi))(x,t) \leq P\zeta(x,t) \tag{4.8}$$

From (4.5) and (4.7), we have

$$\begin{aligned} u_0 &= P\varphi - \lambda V(pf(P\psi)) \geq P\varphi - \frac{\lambda}{\lambda_0}P\varphi = (1 - \frac{\lambda}{\lambda_0})P\varphi > 0 \\ w_0 &= P\gamma - \eta V(rh(P\zeta)) \geq P\gamma - \frac{\eta}{\eta_0}P\gamma = (1 - \frac{\eta}{\eta_0})P\gamma > 0 \end{aligned}$$

Then

$$\begin{aligned} z_1 - z_0 &= -\varrho V(ek(u_0)) \leq 0 \\ v_1 - v_0 &= -\mu V(qg(w_0)) \leq 0 \end{aligned}$$

Since f and h are nondecreasing, we obtain

$$\begin{aligned} u_1 - u_0 &= \lambda V(p(f(v_0) - f(v_1))) \geq 0 \\ w_1 - w_0 &= \eta V(r(h(z_0) - h(z_1))) \geq 0 \end{aligned}$$

Now, since v_0, z_0 are nonnegatives ($v_0 > 0 \Rightarrow u_0 \leq P\varphi,\ z_0 > 0 \Rightarrow w_0 \leq P\gamma$) and g, k are nondecreasing, we deduce from (4.6) and (4.8) that

$$\begin{aligned} z_1 &= P\zeta - \varrho V(ek(u_0)) \geq (1 - \frac{\varrho}{\varrho_0})P\zeta > 0 \\ v_1 &= P\psi - \mu V(qg(w_0)) \geq (1 - \frac{\mu}{\mu_0})P\psi > 0 \end{aligned}$$

which gives us

$$u_1 \leq P\varphi \quad \text{and} \quad w_1 \leq P\gamma$$

Finally, we find

$$\begin{cases} 0 < (1 - \frac{\lambda}{\lambda_0})P\varphi \leq u_0 \leq u_1 \leq P\varphi \\ 0 < (1 - \frac{\eta}{\eta_0})P\gamma \leq w_0 \leq w_1 \leq P\gamma \\ 0 < (1 - \frac{\mu}{\mu_0})P\psi \leq v_1 \leq v_0 \leq P\psi \\ 0 < (1 - \frac{\varrho}{\varrho_0})P\zeta \leq z_1 \leq z_0 \leq P\zeta \end{cases}$$

By induction, we suppose that (4.1), (4.2), (4.3)and (4, 4) hold for j. Since g, k are nondecreasing and $u_{j+1} \leq P\varphi,\ w_{j+1} \leq P\gamma$, we have

$$\begin{aligned} z_{j+2} - z_{j+1} &= \varrho V(e(k(u_j) - k(u_{j+1}))) \leq 0 \\ v_{j+2} - v_{j+1} &= \mu V(q(g(w_j) - g(w_{j+1}))) \leq 0 \end{aligned}$$

and

$$\begin{aligned} z_{k+2} &= P\zeta - \varrho V(ek(u_{k+1})) \geq P\zeta - \varrho V(ek(P\varphi)) \geq (1 - \frac{\varrho}{\varrho_0})P\zeta \\ v_{k+2} &= P\psi - \mu V(qg(w_{k+1})) \geq P\psi - \mu V(qg(P\gamma)) \geq (1 - \frac{\mu}{\mu_0})P\psi \end{aligned}$$

Using the two relations (4.6) and (4.8), we have

$$\begin{aligned} 0 &< (1 - \frac{\varrho}{\varrho_0})P\zeta \leq z_{j+2} \leq z_{j+1} \leq P\zeta \\ 0 &< (1 - \frac{\mu}{\mu_0})P\psi \leq v_{j+2} \leq v_{j+1} \leq P\psi \end{aligned}$$

Now, using that f, h are nondecreasing, we have

$$\begin{aligned} u_{j+2} - u_{j+1} &= \lambda V(p(f(v_{j+1}) - f(v_{j+2}))) \geq 0 \\ w_{j+2} - w_{j+1} &= \eta V(r(h(z_{j+1}) - h(z_{j+2}))) \geq 0 \end{aligned}$$

Since $z_{j+2} > 0$, $v_{j+2} > 0$, we obtain

$$\begin{aligned} 0 &< (1 - \frac{\lambda}{\lambda_0})P\varphi \leq u_{j+1} \leq u_{j+2} \leq P\varphi \\ 0 &< (1 - \frac{\eta}{\eta_0})P\gamma \leq w_{j+1} \leq w_{j+2} \leq P\gamma \end{aligned}$$

Therefore, the sequences $(u_j)_{j\geq 0}$, $(v_j)_{j\geq 0}$, $(w_j)_{j\geq 0}$ and $(z_j)_{j\geq 0}$ converge respectively to u, v, w and z satisfying (2.1). We claim that

$$u = P\varphi - \lambda V(pf(v)) \tag{4.9}$$

$$w = P\gamma - \eta V(rh(z)) \tag{4.10}$$

$$z = P\zeta - \varrho V(ek(u)) \tag{4.11}$$

$$v = P\psi - \mu V(qg(w)) \tag{4.12}$$

Since $v_j \leq P\psi$ and $z_j \leq P\zeta$ for all $j \in \mathbb{N}$, using hypotheses $(H_1), (H_3)$ and the fact that f, h are nondecreasing, there exist $\tilde{p}_{c_1}, \tilde{r}_{c_4} \in P^{\infty}(\mathbb{R}^n)$ such that

$$pf(v) \leq pf(c_1 P\omega) \leq \tilde{p}_{c_1} P\theta \tag{4.13}$$

$$rh(z) \leq rh(c_4 P\phi) \leq \tilde{r}_{c_4} P\delta \tag{4.14}$$

and so

$$\begin{aligned} p\left|f(v_j) - f(v)\right| &\leq 2\tilde{p}_{c_1} P\theta \text{ , for all } j \in \mathbb{N} \\ r\left|h(z_j) - h(z)\right| &\leq 2\tilde{r}_{c_4} P\delta \text{ , for all } j \in \mathbb{N} \end{aligned}$$

Now, from Proposition 3.4 and by Lebesgue's theorem, we can deduce

$$\begin{aligned} \lim_{k\to\infty} V(pf(v_k)) &= V(pf(v)) \\ \lim_{k\to\infty} V(rh(z_k)) &= V(rh(z)) \end{aligned}$$

So, letting $j \to \infty$ in equations

$$u_j = P\varphi - \lambda V(pf(v_j)), \qquad w_j = P\gamma - \eta V\left(rh\left(z_j\right)\right)$$

we have (4.9) and (4.10). Similarly, we obtain (4.11) and (4.12).
Next, we affirm that (u, v, w, z) satisfies

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = \lambda pf(v) \\ \Delta v - \dfrac{\partial v}{\partial t} = \mu qg(w) \\ \Delta w - \dfrac{\partial w}{\partial t} = \eta rh(z) \\ \Delta z - \dfrac{\partial z}{\partial t} = \varrho ek(u) \end{cases} \tag{4.15}$$

Since θ, δ satisfies (H_0) and $\tilde{p}_{c_1}, \tilde{r}_{c_4} \in P^\infty(\mathbb{R}^n)$, using Proposition 3.2, we have

$$\tilde{p}_{c_1} P\theta,\ \tilde{r}_{c_4} P\delta \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

Moreover (4.13), (4.14) and Proposition 3.4 imply that

$$pf(v),\ rh(z) \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

and

$$V(pf(v)),\ V(rh(z)) \in C(\mathbb{R}^n \times (0, \infty)) \subset L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

Similarly

$$qg(w),\ V(qg(w)),\ ek(u),\ V(ek(u)) \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

Now, applying the heat operator $\Delta - \dfrac{\partial}{\partial t}$ in (4.9), (4.10), (4.11) and (4.12), (u, v, w, z) is clearly a positive solution (in the sense of distributions) of (4.15).
Furthermore since $V(pf(v))$, $V(qg(w))$, $V(rh(z))$ and $V(ek(u))$ are continuous in $\mathbb{R}^n \times (0, \infty)$ and using Proposition 3.5, we deduce from (4.9), (4.10), (4.11) and (4.12) that

$$(u, v, w, z) \in \left(C(\mathbb{R}^n \times (0, \infty))\right)^4$$

which implies according to hypothesis (H_1) and proposition 3.4 that

$$\begin{aligned} \lim_{t\to 0} u(x,t) &= \lim_{t\to 0} P\varphi(x,t) = \varphi(x)\ , \quad x \in \mathbb{R}^n \\ \lim_{t\to 0} v(x,t) &= \lim_{t\to 0} P\psi(x,t) = \psi(x)\ , \quad x \in \mathbb{R}^n \\ \lim_{t\to 0} w(x,t) &= \lim_{t\to 0} P\gamma(x,t) = \gamma(x)\ , \quad x \in \mathbb{R}^n \\ \lim_{t\to 0} z(x,t) &= \lim_{t\to 0} P\zeta(x,t) = \zeta(x)\ , \quad x \in \mathbb{R}^n \end{aligned}$$

This completes the proof of our theorem. □

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