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Sălăgean-type harmonic multivalent functions defined by q-difference operator

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Abstract. We introduce a new subclass of Sălăgean-type harmonic multivalent functions by using $q-$difference operator. We investigate sufficient coefficient estimates, distortion bounds, extreme points, convolution properties and neighborhood for the functions belonging to this function class.

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1. Introduction

The study of harmonic functions which are multivalent in the open unit disc

$$\mathbb{D} = \{z : |z| < 1\}$$

was initiated by Duren, Hengartner and Laugesen [4]. Let $\mathcal{H}(m), (m \geq 1)$ be the class of harmonic multivalent and sense-preserving functions $f = h+\overline{g}$, where h and g have the following power series

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1},\ g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1 \tag{1.1}$$

that are analytic and $m-$valent in $\mathbb{D}$. The class $\mathcal{H}(1)$ of harmonic univalent functions was studied by Clunie and Sheil-Small [3]. For more details of harmonic multivalent functions, one may refer to [2] and [6].

Jackson [7, 8] in 1909-1910 developed quantum calculus, popularly known as $q-$calculus. Since then it has found applications in physics, quantum mechanics, analytic number theory, Sobolev spaces, representation theory of groups, theta functions, gamma functions, operator theory, and more recently in geometric function theory. For definitions, properties and references of $q-$calculus one may refer to [1].

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In fact, $q-$calculus methodology is centered on the idea of deriving $q-$analogues results without the use of limits. Let us first recall certain notations and definitions of the $q-$calculus.

Definition 1.1. Let $q \in (0,1)$. The $q-$derivative (or $q-$difference operator) of a function f, defined on a subset Ω with $0 \in \Omega$ of $\mathbb{C}$, is given by

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0. \end{cases}$$

We note that $\lim_{q\to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z.

For the function $f(z) = z^n$, we observe that

$$D_q z^n = [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. Therefore, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$, then

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1}.$$

Clearly, for $q \to 1^-$, $[n]_q \to n$. For the definitions and properties of $q-$derivative and $q-$calculus, one may refer to [1, 5, 7, 8].

The $q-$Sălăgean differential operator of a $m-$valent function h given in (1.1) is formed by

$$\begin{aligned} L_q^0 h(z) &= h(z) \\ L_q^1 h(z) &= \frac{zD_q(h(z))}{[m]_q} \\ &\vdots \\ L_q^k h(z) &= L_q(L_q^{k-1}h(z)). \end{aligned}$$

Then

$$L_q^k h(z) = z^m + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k a_{n+m-1} z^{n+m-1}, \tag{1.2}$$

where $[n+m-1]_q^k = (\frac{1-q^{n+m-1}}{1-q})^k$, $q \in (0,1)$, $k = 0, 1, \ldots$. Clearly, when $q \to 1^-$ and $m = 1$, the equation (1.2) reduces to Sălăgean differential operator (see [12]).

Making use of (1.1) and (1.2), we define the $q-$Sălăgean differential operator for harmonic multivalent function $L_q^k f(z) : \mathcal{H}(m) \to \mathcal{H}(m)$ by

$$L_q^k f(z) = L_q^k h(z) + (-1)^k \overline{L_q^k g(z)}, \tag{1.3}$$

where $L_q^k h(z)$ is given by (1.2) and

$$L_q^k g(z) = \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k b_{n+m-1} z^{n+m-1}.$$

When $q \to 1^-$ the equation (1.3) reduces to Sălăgean differential operator for multivalent harmonic functions given in [11]. Motivated by definition of $q-$Sălăgean differential operator for harmonic multivalent functions, we create the following class.

Definition 1.2. For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $q \in (0,1)$ and $z \in \mathbb{D}$, a function $f \in \mathcal{H}(m)$, $(m \geq 1)$ is said to belong to the class $\mathcal{H}_q(m,k,\lambda,\alpha)$ if

$$\operatorname{Re}\left(\frac{L_q^{k+1}f(z)}{(1-\lambda)z^m + \lambda L_q^k f(z)}\right) \geq \alpha, \tag{1.4}$$

where $L_q^k f(z), (k = 0,1,...)$ is defined by (1.3). A function f in this class is called $q-$Sălăgean-type harmonic multivalent function of order α.

For special values of parameters q, m, k, λ and α, we obtain several new and known subclasses as special cases; for example:

(i) If $k = 0$, we get a new subclass $\mathcal{H}_q(m,\lambda,\alpha)$ as below

$$\operatorname{Re}\left(\frac{zD_q f(z)}{(1-\lambda)z^m + \lambda f(z)}\right) \geq \alpha.$$

(ii) If $k = 0$, $m = 1$, we get a new subclass $\mathcal{H}_q(\lambda,\alpha)$ as below

$$\operatorname{Re}\left(\frac{zD_q f(z)}{(1-\lambda)z + \lambda f(z)}\right) \geq \alpha.$$

(iii) If $k = 0$, $m = 1$, $q \to 1^-$, we get a known class $\mathcal{S}^*_{\mathcal{H}}(\lambda,\alpha)$ defined in [13] in the following

$$\operatorname{Re}\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)}\right) \geq \alpha.$$

(iv) If $k = 0$, $m = 1$, $\lambda = 1$, $q \to 1^-$, we get a known class $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ defined in [9] in the following

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha.$$

(v) If $m = 1$, $\lambda = 1$, $q \to 1^-$, we get a known class $\mathcal{S}_{\mathcal{H}}(k,\alpha)$ defined in [10] in the following

$$\operatorname{Re}\left(\frac{L^{k+1}f(z)}{L^k f(z)}\right) \geq \alpha.$$

We also introduce a new subclass of $q-$Sălăgean-type harmonic multivalent functions using negative coefficients. Let $\mathcal{TH}_q(m,k,\lambda,\alpha)$ denote a subclass of $\mathcal{H}(m)$ that consists of harmonic functions $f = h + \overline{g}$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}, \; g(z) = (-1)^k \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \; |b_m| < 1. \tag{1.5}$$

In Section 2, we first obtain coefficient characterization for our main class. Using this characterization, we obtain distortion and covering theorems. Finally, we obtain extreme points, convolution properties and neighborhood results for our class.

2. Main results

We first obtain two lemmas that we need for proving other results.

Lemma 2.1. *Let $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $q \in (0,1)$, $z \in \mathbb{D}$, and $f = h + \overline{g}$ with h and g of the form (1.1). If*

$$\sum_{n=2}^{\infty} \Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}| + \sum_{n=1}^{\infty} \Psi_q(m,k,\lambda,\alpha)|b_{n+m-1}| \leq 1-\alpha, \tag{2.1}$$

where

$$\Omega_q(m,k,\lambda,\alpha) = \left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} - \alpha\lambda\right] \tag{2.2}$$

and

$$\Psi_q(m,k,\lambda,\alpha) = \left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} + \alpha\lambda\right], \tag{2.3}$$

then $f \in \mathcal{H}_q(m,k,\lambda,\alpha)$.

Proof. In view of (1.4), and using the fact that $Re(w) \geq \alpha$ if and only if

$$|1-\alpha+w| > |1+\alpha-w|,$$

it suffices to show that

$$\left|1-\alpha+\frac{L_q^{k+1}f(z)}{(1-\lambda)z^m+\lambda L_q^k f(z)}\right| - \left|1+\alpha-\frac{L_q^{k+1}f(z)}{(1-\lambda)z^m+\lambda L_q^k f(z)}\right| \geq 0.$$

We observe that left side of this inequality

$$\begin{aligned}
&= \left|L_q^{k+1}f(z) + (1-\alpha)[(1-\lambda)z^m + \lambda L_q^k f(z)]\right| \\
&\quad - \left|L_q^{k+1}f(z) - (1+\alpha)[(1-\lambda)z^m + \lambda L_q^k f(z)]\right| \\
&= \Bigg|(2-\alpha)z^m + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} + (1-\alpha)\lambda\right] a_{n+m-1}z^{n+m-1} \\
&\quad -(-1)^k \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} - (1-\alpha)\lambda\right] \overline{b_{n+m-1}z^{n+m-1}}\Bigg| \\
&\quad - \Bigg| -\alpha z^m + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} - (1+\alpha)\lambda\right] a_{n+m-1}z^{n+m-1} \\
&\quad -(-1)^k \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} + (1+\alpha)\lambda\right] \overline{b_{n+m-1}z^{n+m-1}}\Bigg| \\
&\geq 2(1-\alpha)|z|^m - \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} + (1-\alpha)\lambda\right] |a_{n+m-1}||z|^{n+m-1} \\
&\quad - \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} - (1-\alpha)\lambda\right] |b_{n+m-1}||z|^{n+m-1} \\
&\quad -\alpha|z|^m - \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} - (1+\alpha)\lambda\right] |a_{n+m-1}||z|^{n+m-1}
\end{aligned}$$

$$-\sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}+(1+\alpha)\lambda\right]|b_{n+m-1}||z|^{n+m-1}$$

$$\geq 2(1-\alpha)|z|^m-2\sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]|a_{n+m-1}||z|^{n+m-1}$$

$$-2\sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]|b_{n+m-1}||z|^{n+m-1}$$

$$\geq 2(1-\alpha)\left\{1-\sum_{n=2}^{\infty}\frac{\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]}{1-\alpha}|a_{n+m-1}|\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]}{1-\alpha}|b_{n+m-1}|\right\}\geq 0,$$

by (2.1). This completes the proof. □

The $q-$Sălăgean-type harmonic multivalent functions

$$f(z)=z^m+\sum_{n=2}^{\infty}\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}x_{n+m-1}z^{n+m-1}+\sum_{n=1}^{\infty}\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}\overline{y_{n+m-1}z^{n+m-1}},$$

where

$$\sum_{n=2}^{\infty}|x_{n+m-1}|+\sum_{n=1}^{\infty}|y_{n+m-1}|=1$$

shows that the coefficient bound given by (2.1) is sharp.

We now show that the condition (2.1) is also necessary for functions $f=h+\overline{g}$, where h and g are of the form (1.5)

Lemma 2.2. *Let $0\leq\lambda\leq 1$, $0\leq\alpha<1$, $q\in(0,1)$, $z\in\mathbb{D}$, and $f=h+\overline{g}$ with h and g of the form (1.5). Then $f\in\mathcal{TH}_q(m,k,\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}|+\sum_{n=1}^{\infty}\Psi_q(m,k,\lambda,\alpha)|b_{n+m-1}|\leq 1-\alpha, \tag{2.4}$$

where $\Omega_q(m,k,\lambda,\alpha)$ and $\Psi_q(m,k,\lambda,\alpha)$ are, respectively, given by (2.2) and (2.3).

Proof. Since $\mathcal{TH}_q(m,k,\lambda,\alpha)\subset\mathcal{H}_q(m,k,\lambda,\alpha)$, we only need to prove the "only if" part of this theorem. Let $f\in\mathcal{TH}_q(m,k,\lambda,\alpha)$, then it satisfies (1.4) or equivalently

$$\mathrm{Re}\left\{\frac{(1-\alpha)z^m-\sum_{n=2}^{\infty}\Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}|z^{n+m-1}}{\Theta_q(m,k,\lambda,\alpha)}\right.$$

$$\left.+\frac{(-1)^{2k-1}\sum_{n=2}^{\infty}\Psi_q(m,k,\lambda,\alpha)|b_{n+m-1}|\overline{z}^{n+m-1}}{\Theta_q(m,k,\lambda,\alpha)}\right\}\geq 0,$$

where

$$\Theta_q(z) = z^m - \lambda \sum_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |a_{n+m-1}| z^{n+m-1} + \lambda(-1)^{2k} \sum_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |b_{n+m-1}| \overline{z}^{n+m-1}.$$

The above required condition must hold for all values of $z \in \mathbb{D}$, $|z| = r < 1$. By choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$\frac{(1-\alpha) - \sum\limits_{n=2}^{\infty} \Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}|r^{n-1}}{1 - \lambda \sum\limits_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q}\right)^k |a_{n+m-1}|r^{n-1} + \lambda \sum\limits_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q}\right)^k |b_{n+m-1}|r^{n-1}}$$
$$- \frac{\sum\limits_{n=2}^{\infty} \psi_q(m,k,\lambda,\alpha)|b_{n+m-1}|r^{n-1}}{1 - \lambda \sum\limits_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q}\right)^k |a_{n+m-1}|r^{n-1} + \lambda \sum\limits_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q}\right)^k |b_{n+m-1}|r^{n-1}} \ge 0. \tag{2.5}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists a point $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f \in \mathcal{TH}_q(m,k,\lambda,\alpha)$ and so the proof is complete. □

We now obtain distortion bounds of the class $\mathcal{TH}_q(m,k,\lambda,\alpha)$.

Theorem 2.3. *If a function f belongs to the class $\mathcal{TH}_q(m,k,\lambda,\alpha)$, then we have*

$$|f(z)| \le (1+|b_m|)r^m + \frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)} \left(1 - \frac{1+\alpha\lambda}{1-\alpha\lambda}|b_m|\right) r^{m+1} \tag{2.6}$$

and

$$|f(z)| \ge (1-|b_m|)r^m - \frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)} \left(1 - \frac{1+\alpha\lambda}{1-\alpha\lambda}|b_m|\right) r^{m+1}, \tag{2.7}$$

where

$$\theta_q(m,k,\lambda,\alpha) = \left(\frac{[m+1]_q}{[m]_q}\right)^k \left[\frac{[m+1]_q}{[m]_q} - \alpha\lambda\right],$$

and for all $z \in \mathbb{D}$.

Proof. Let $f \in \mathcal{TH}_q(m,k,\lambda,\alpha)$. Taking the absolute value of f and using Lemma 2.2, we obtain

$$\begin{aligned}
|f(z)| &\le (1+|b_m|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|)r^{n+m-1} \\
&\le (1+|b_m|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|)r^{m+1} \\
&\le (1+|b_m|)r^m + \frac{1-\alpha}{\theta_q(m,k,\lambda,\alpha)} \\
&\quad \times \sum_{n=2}^{\infty}\frac{\theta_q(m,k,\lambda,\alpha)}{1-\alpha}(|a_{n+m-1}|+|b_{n+m-1}|)r^{m+1} \\
&\le (1+|b_m|)r^m + \frac{1-\alpha}{\theta_q(m,k,\lambda,\alpha)} \\
&\quad \times \sum_{n=2}^{\infty}\left\{\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}| + \frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}|\right\}r^{m+1} \\
&\le (1+|b_m|)r^m + \left\{\frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)} - \frac{1+\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}|b_m|\right\}r^{m+1}.
\end{aligned}$$

The proof of the inequality (2.7) is similar to the proof of (2.6) and is omitted. □

The following covering result follows from the inequality (2.7) by letting r approaches to 1.

Corollary 2.4. *If $f \in \mathcal{TH}_q(m,k,\lambda,\alpha)$, then*

$$\left\{w : |w| < \left(1 - \frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}\right) - \left(1 - \frac{1+\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}\right)b_m\right\} \subset f(\mathbb{D}).$$

Next, we give the extreme points of this class.

Theorem 2.5. *Let $f = h + \overline{g}$ be given by (1.5). Then $f \in clco\mathcal{TH}_q(m,k,\lambda,\alpha)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty}(x_{n+m-1}h_{n+m-1}(z) + y_{n+m-1}g_{n+m-1}(z)),$$

where

$$h_m(z) = z^m, \quad h_{n+m-1}(z) = z^m - \frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}z^{n+m-1}, \ (n \ge 2)$$

$$g_{n+m-1}(z) = z^m + (-1)^k\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}\overline{z}^{n+m-1}, \ (n \ge 1)$$

and

$$\sum_{n=1}^{\infty}(x_{n+m-1}+y_{n+m-1}) = 1,$$

where $x_{n+m-1} \ge 0$ and $y_{n+m-1} \ge 0$. In particular, the extreme points of $\mathcal{TH}_q(m,k,\lambda,\alpha)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1}\}$.

Proof. For a function f of the form

$$f(z)=\sum_{n=1}^{\infty}(x_{n+m-1}h_{n+m-1}(z)+y_{n+m-1}g_{n+m-1}(z)),$$

where $\sum_{n=1}^{\infty}(x_{n+m-1}+y_{n+m-1})=1$, we have

$$f(z)=\sum_{n=1}^{\infty}(x_{n+m-1}+y_{n+m-1})z^m-\sum_{n=2}^{\infty}\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}x_{n+m-1}z^{n+m-1}$$
$$+(-1)^k\sum_{n=1}^{\infty}\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}y_{n+m-1}\overline{z}^{n+m-1}.$$

Then, $f\in clco\mathcal{TH}_q(m,k,\lambda,\alpha)$ because

$$\sum_{n=2}^{\infty}\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}|$$
$$=\sum_{n=2}^{\infty}x_{n+m-1}+\sum_{n=1}^{\infty}y_{n+m-1}=1-x_m\leq 1.$$

Conversely, suppose $f\in clco\mathcal{TH}_q(m,k,\lambda,\alpha)$. Then, by Lemma 2.2

$$|a_{n+m-1}|\leq\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}$$

and

$$|b_{n+m-1}|\leq\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}.$$

Putting

$$x_{n+m-1}=\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}|,$$
$$y_{n+m-1}=\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}|,$$

and $x_m=1-\sum_{n=2}^{\infty}x_{n+m-1}-\sum_{n=1}^{\infty}y_{n+m-1}\geq 0$, we obtain

$$f(z)=z^m-\sum_{n=2}^{\infty}a_{n+m-1}z^{n+m-1}+(-1)^k\sum_{n=1}^{\infty}\overline{b_{n+m-1}z^{n+m-1}}$$
$$=z^m-\sum_{n=2}^{\infty}\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}x_{n+m-1}z^{n+m-1}$$
$$+(-1)^k\sum_{n=1}^{\infty}\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}\overline{y_{n+m-1}z^{n+m-1}}$$
$$=z^m+\sum_{n=1}^{\infty}(h_{n+m-1}(z)-z^m)x_{n+m-1}+\sum_{n=1}^{\infty}(g_{n+m-1}(z)-z^m)y_{n+m-1}.$$

Consequently, we obtain $f(z) = \sum_{n=1}^{\infty} (x_{n+m-1}h_{n+m-1}(z) + y_{n+m-1}g_{n+m-1}(z))$ as required. □

Using definition of convolution and Lemma 2.2, we show that the class $\mathcal{TH}_q(m,k,\lambda,\alpha)$ is closed under convolution. Recall that the convolution of two complex-valued harmonic multivalent functions

$$f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}|z^{n+m-1} + (-1)^k \sum_{n=1}^{\infty} |b_{n+m-1}|\overline{z}^{n+m-1}$$

and

$$F(z) = z^m - \sum_{n=2}^{\infty} |A_{n+m-1}|z^{n+m-1} + (-1)^k \sum_{n=1}^{\infty} |B_{n+m-1}|\overline{z}^{n+m-1}$$

is defined by

$$(f * F)(z) = z^m + \sum_{n=2}^{\infty} |a_{n+m-1}||A_{n+m-1}|z^{n+m-1} + (-1)^k \sum_{n=1}^{\infty} |b_{n+m-1}||B_{n+m-1}|\overline{z}^{n+m-1}.$$

Theorem 2.6. *For $0 \le \beta \le \alpha < 1$, suppose*

$$f \in \mathcal{TH}_q(m,k,\lambda,\alpha) \text{ and } F \in \mathcal{TH}_q(m,k,\lambda,\beta).$$

Then

$$f * F \in \mathcal{TH}_q(m,k,\lambda,\alpha) \subset \mathcal{TH}_q(m,k,\lambda,\beta).$$

Proof. Let

$$f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}|z^{n+m-1} + (-1)^k \sum_{n=1}^{\infty} |b_{n+m-1}|\overline{z}^{n+m-1}$$

be in $\mathcal{TH}_q(m,k,\lambda,\alpha)$ and

$$F(z) = z^m - \sum_{n=2}^{\infty} |A_{n+m-1}|z^{n+m-1} + (-1)^k \sum_{n=1}^{\infty} |B_{n+m-1}|\overline{z}^{n+m-1},$$

be in $\mathcal{TH}_q(m,k,\lambda,\beta)$. Since $F \in \mathcal{TH}_q(m,k,\lambda,\beta)$, we note that $|A_{n+m-1}| \le 1$ and $|B_{n+m-1}| \le 1$. We want to show that if $f * F$ satisfy the condition given in Lemma 2.2, then

$$\sum_{n=2}^{\infty} \frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}||A_{n+m-1}| + \sum_{n=1}^{\infty} \frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}||B_{n+m-1}|$$
$$\le \sum_{n=2}^{\infty} \frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}| \le 1.$$

In view of Lemma 2.2, it follows that $f * F \in \mathcal{TH}_q(m,k,\lambda,\alpha) \subset \mathcal{TH}_q(m,k,\lambda,\beta)$. □

Finally, we define $q-\delta-$neighborhood and then investigate a containment property. The $q-\delta-$neighborhood of a function $f=h+\overline{g}$ in $\mathcal{H}_q(m,k,\lambda,\alpha)$ is defined as the set:

$$N_{q,\delta}(f)=\Bigg\{F(z)=z^m+B_m\overline{z}^m+\sum_{n=2}^{\infty}(A_{n+m-1}z^{n+m-1}+B_{n+m-1}\overline{z}^{n+m-1}):$$
$$\sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\Bigg\{\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]|a_{n+m-1}-A_{n+m-1}|+$$
$$\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]|b_{n+m-1}-B_{n+m-1}|\Bigg\}+(1+\alpha\lambda)|b_m-B_m|\le(1-\alpha)\delta,\ \delta>0\Bigg\}.$$

Theorem 2.7. *If f given by (1.1) satisfies the condition (2.1) and*

$$\delta\le\left[1-\frac{1}{\frac{[m+1]_q}{[m]_q}-\alpha\lambda}\right]\left(1-\frac{1+\alpha\lambda}{1-\alpha}|b_m|\right),\tag{2.8}$$

then $N_{q,\delta}(f)\subset\mathcal{H}_q(m,k,\lambda,\alpha)$.

Proof. For any $f\in\mathcal{H}_q(m,k,\lambda,\alpha)$, suppose

$$F(z)=z^m+B_m\overline{z}^m+\sum_{n=2}^{\infty}(A_{n+m-1}z^{n+m-1}+B_{n+m-1}\overline{z}^{n+m-1})$$

belongs to $N_{q,\delta}(f)$. Then we have

$$\begin{aligned}
&(1+\alpha\lambda)|B_m|+\sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\Bigg\{\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]|A_{n+m-1}|+\\
&\qquad\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]|B_{n+m-1}|\Bigg\}\\
&\le(1+\alpha\lambda)|B_m-b_m|+(1+\alpha\lambda)|b_m|+\sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\\
&\qquad\Bigg\{\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]|A_{n+m-1}-a_{n+m-1}|\\
&\qquad+\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]|B_{n+m-1}-b_{n+m-1}|\Bigg\}\\
&\qquad+\sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\Bigg\{\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]|a_{n+m-1}|\\
&\qquad+\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]|b_{n+m-1}|\Bigg\}\\
&\le(1-\alpha)\delta+(1+\alpha\lambda)|b_m|+\frac{1}{\frac{[m+1]_q}{[m]_q}-\alpha\lambda}[(1-\alpha)-(1+\alpha\lambda)b_m]\le1-\alpha,
\end{aligned}\tag{2.9}$$

by given condition (2.8). Therefore, it follows that $F\in\mathcal{H}_q(m,k,\lambda,\alpha)$. This completes the proof this theorem. □

References

[1] Ahuja, O.P., Çetinkaya, A., *Use of quantum calculus approach in mathematical sciences and its role in geometric function theory,* National Conference on Mathematical Analysis and Computing, 2018, AIP Conference Proceedings 2095, 020001 (2019).

[2] Ahuja, O.P., Jahangiri, J.M., *Multivalent harmonic starlike functions*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **55**(2001), no. 1, 1-13.

[3] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **9**(1984), 3-25.

[4] Duren, P.L., Hengartner, W., Laugesen, R.S., *The argument principle for harmonic functions*, Amer. Math. Monthly, **103**(1996), no. 5, 411-415.

[5] Gasper, G., Rahman, M., *Basic Hypergeometric Series*, Cambridge University Press, 2004.

[6] Güney, H.Ö., Ahuja, O.P., *Inequalities involving multipliers for multivalent harmonic functions*, J. Ineq. Pure Appl. Math., **7**(2006), no. 5, 1-9.

[7] Jackson, F.H., *On $q-$functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, **46**(1909), 253-281.

[8] Jackson, F.H., *$q-$difference equations*, Amer. J. Math., **32**(1910), 305-314.

[9] Jahangiri, J.M., *Harmonic functions starlike in the unit disc*, J. Math. Anal. Appl., **235**(1999), 470-477.

[10] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Sălăgean-type harmonic univalent functions*, Southwest J. Pure Appl. Math., **2**(2002), 77-82.

[11] Jahangiri, J.M., Şeker, B., Sümer Eker, S., *Sălăgean-type harmonic multivalent functions,* Acta Univ. Apulensis Math. Inform., **18**(2009), 233-244.

[12] Sălăgean, G.S., *Subclasses of univalent functions,* In: Complex Analysis, Fifth Romanian Finnish Seminar, Part I (Bucharest, 1981), Lecture Notes in Math., 1013, Springer, Berlin, (1983), 362-372.

[13] Vijaya, K., *Studies on certain subclasses of harmonic functions*, Ph.D. Thesis, VIT University, Vellore, 2014.

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