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# Geometric properties of normalized imaginary error function

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** The error function takes place in a wide range in the fields of mathematics, mathematical physics and natural sciences. The aim of the current paper is to investigate certain properties such as univalence and close-to-convexity of normalized imaginary error function, which its region is symmetric with respect to the real axis. Some other outcomes are also obtained.

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## 1. Introduction and preliminaries

The *error function* plays a significant role in different fields of science including statistics, probability, partial differential equations, and many engineering problems. Therefore, it is attracted much attentions in mathematics. Specially, several remarkable inequalities and related topics for the error function were reported, see for examples [10, 12, 14, 15]. The error function and its approximations are usually employed to forecast outcomes that hold with high or low probability.

The error function, which is denoted by the symbol erf and defined by [1, p. 297]

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2)\mathrm{d}x = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!} \tag{1.1}$$

for every complex number $z \in \mathbb{C}$ and is a subject of intensive studies and recently applications. The integral of relation (1.1) cannot be assessed in closed form in terms of elementary functions, but by wringing the integrand $e^{-z^2}$ as Maclaurin series and

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integrating term by term, the error function's Maclaurin series is obtained as it is shown above. Also, the *imaginary error function*, denoted by the symbol erfi, has a very similar Maclaurin series, which is defined by

$$\operatorname{erfi}(z)=\frac{2}{\sqrt{\pi}}\int_0^z \exp(x^2)\mathrm{d}x=\frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)k!}, \tag{1.2}$$

for every $z\in\mathbb{C}$. Some inequalities and properties of error function can be seen in [3, 11].

Before presenting our results, some basic definitions are first stated. Let $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and let $\mathcal{A}$ be the class of normalized analytic functions $f$ in $\mathbb{U}$ that has the following power series expansion

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j,\ z\in\mathbb{U}, \tag{1.3}$$

and denote by $\mathcal{S}$ the class of univalent functions that belong to $\mathcal{A}$.

In 2018, Ramachandran *et al.* [20] studied the normalized analytic error function, which is obtained from (1.1) and of the form

$$\operatorname{Erf}(z)=\frac{\sqrt{\pi z}}{2}\operatorname{erf}(\sqrt{z})=z+\sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{(2j-1)(j-1)!}z^j,$$

and defined a class of analytic functions having the following representation

$$\operatorname{Erf}*\mathcal{A}=\left\{\mathcal{L}:\mathcal{L}(z)=(\operatorname{Erf}*f)(z)=z+\sum_{j=2}^{\infty}\frac{(-1)^{n-1}a_j}{(2j-1)(j-1)!}z^j, f\in\mathcal{A}\right\},$$

where the symbol "$*$" represents the Hadamard (or convolution) product, while Erf denotes the class that consists of the single function Erf.

Let Erfi be the normalized form of the error function which is obtained from (1.2) and defined by

$$\operatorname{Erfi}(z)=\frac{\sqrt{\pi z}}{2}\operatorname{erfi}(\sqrt{z})=z+\sum_{j=2}^{\infty}\frac{z^j}{(2j-1)(j-1)!}. \tag{1.4}$$

We denote by $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ the classes of $\mathcal{A}$ consisting of functions which are *starlike of order* $\gamma$ and *convex of order* $\gamma$, that is,

$$\mathcal{S}^*(\gamma):=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf'(z)}{f(z)}>\gamma,\ z\in\mathbb{U}\right\}\ (0\le\gamma<1)$$

and

$$\mathcal{C}(\gamma):=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{(zf'(z))'}{f'(z)}>\gamma,\ z\in\mathbb{U}\right\}\ (0\le\gamma<1)$$

respectively. Specifically, $\mathcal{S}^*:=\mathcal{S}^*(0)$ and $\mathcal{C}:=\mathcal{C}(0)$ are the classes of *starlike functions* and *convex functions* in $\mathbb{U}$, respectively.

Furthermore, we represent by $\mathcal{K}(\gamma)$ the subclass of $\mathcal{A}$ consisting of functions which are *close-to-convex of order* $\gamma$, that is,

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > \gamma, \; z \in \mathbb{U}, \; (0 \leq \gamma < 1)$$

for some function $g \in \mathcal{C}$. In particular, $\mathcal{K} := \mathcal{K}(0)$ is the class of *close-to-convex functions* in $\mathbb{U}$.

In the recent years a serious attention was attracted on the geometric and other related properties as univalence, convexity and starlikeness of various special functions like the normalized forms of Bessel, Struve and Lommel functions of the first kind. In this area, some authors obtained many applications in the Geometric Functions Theory for these special functions, and see for examples the articles [2, 4, 5, 6, 7, 8, 9, 16, 18, 19]. The aim of the present paper is to investigate some properties such as univalence and close-to-convexity of normalized imaginary error function, which maps the open unit disk in a domain that is symmetric with respect to the real axis. Some other outcomes are also presented.

## 2. Main results

In this section we study some geometric and other related properties of normalized imaginary error function in the open unit disk. To prove our outcomes we require the following lemmas.

**Lemma 2.1.** [13], (see also [21, p. 59]) *Let* $h(z) = \sum_{k=1}^{\infty} b_k z^{k-1}$ *be analytic in* $\mathbb{U}$ *such that* $\{b_k\}_{k\geq 1}$ *is a sequence with* $b_1 = 1$ *and* $b_k \geq 0$ *for all* $k \geq 2$. *If* $\{b_k\}_{k\geq 2}$ *is a convex decreasing sequence, that is,*

$$0 \geq b_{k+2} - b_{k+1} \geq b_{k+1} - b_k \quad \textit{for all} \quad k \geq 2.$$

*Then*

$$\operatorname{Re}\left(\sum_{k=1}^{\infty} b_k z^{k-1}\right) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

**Lemma 2.2.** [17, Corollary 7] *Let* $h(z) = z + \sum_{k=2}^{\infty} b_k z^k$ *be analytic in* $\mathbb{U}$. *If*

$$1 \geq 2b_2 \geq \ldots \geq kb_k \geq (k+1)b_{k+1} \geq \ldots \geq 0,$$

*or*

$$1 \leq 2b_2 \geq \ldots \leq kb_k \leq (k+1)b_{k+1} \leq \ldots \leq 2,$$

*then the function* $h$ *is close-to-convex with respect to the convex function* $-\log(1-z)$ *in* $\mathbb{U}$.

Letting $q \to 1^-$ in Theorem 2.6 given in Sahoo and Sharma [22], we obtain the next lemma:

**Lemma 2.3.** *Let* $h(z) = z + \sum\limits_{k=1}^{\infty} b_{2k+1} z^{2k+1}$ *be an analytic and odd function* $\mathbb{U}$. *If*

$$1 \geq 3b_3 \geq \ldots \geq (2k+1)b_{2k+1} \geq \ldots \geq 0,$$

*or*

$$1 \leq 3b_3 \leq \ldots \leq (2k+1)b_{2k+1} \leq \ldots \leq 2,$$

*then the function* $h$ *is close-to-convex with respect to the function* $\dfrac{1}{2}\log\dfrac{1+z}{1-z}$ , *which is convex in* $\mathbb{U}$.

Now we obtain the following results applying the lemmas mentioned above.

**Theorem 2.4.** *The normalized error function* Erfi *is close-to-convex in* $\mathbb{U}$ *with respect to the convex function* $-\log(1-z)$.

*Proof.* From (1.4) it follows that

$$ja_j - (j+1)a_{j+1} = \frac{1}{j!}\left[\frac{j^2}{(2j-1)} - \frac{j+1}{(2j+1)}\right] = \frac{1}{(4k^2-1)j!}\nu(j),$$

where

$$\nu(j) = 2j^3 - j^2 - j + 1.$$

According to Lemma 2.2, it is enough to prove that $\nu(j) \geq 0$ for all $j \geq 1$. Setting

$$\varphi(x) := 2x^3 - x^2 - x + 1,\ x \geq 1,$$

the function $\varphi$ is strictly increasing on $[1, +\infty)$, hence $\min\{\varphi(x) : x \geq 1\} = \varphi(1) = 1$. Therefore, since

$$\nu(j) = 2j^3 - j^2 - j + 1 \geq 0,$$

for all $j \geq 1$, we get our result. $\square$

**Theorem 2.5.** *The normalized imaginary error odd function* $\operatorname{Eri}(z) = (\sqrt{\pi}/2)\operatorname{erfi}(z)$ *is close-to-convex in* $\mathbb{U}$ *with respect to the convex function* $(1/2)\log((1+z)/(1-z))$.

*Proof.* Since

$$\operatorname{Eri}(z) = \frac{\sqrt{\pi}}{2}\operatorname{erfi}(z) = z + \sum_{j=2}^{\infty} l_{2j-1} z^{2j-1} \quad (z \in \mathbb{U}),$$

where

$$l_{2j-1} = \frac{1}{(2j-1)(j-1)!},$$

then $l_{2k-1} > 0$ for all $j \geq 2$. To prove our result, from Lemma 2.3, it is sufficient to show that $\{(2j-1)l_{2j-1}\}_{j \geq 2}$ is a non-increasing sequence. If $j \geq 2$, a simple computation shows that

$$(2j-1)l_{2j-1} - (2j+1)l_{2j+1} = \frac{2j-1}{(2j-1)(j-1)!} - \frac{2j+1}{(2j+1)j!} = \frac{1}{j!}(j-1) \geq 0.$$

Therefore, the sequence $\{(2j-1)l_{2j-1}\}_{j \geq 2}$ is a non-increasing and form Lemma 2.3 our result follows. $\square$

**Remark 2.6.** According to Theorems 2.4 and 2.5, the both functions Erfi and $\mathrm{Eri}(z) = \frac{\sqrt{\pi}}{2}\,\mathrm{erfi}(z)$ are univalent in $\mathbb{U}$.

**Theorem 2.7.** *For the function* Erfi*, the following inequality holds:*

$$\mathrm{Re}\,\frac{\mathrm{Erfi}(z)}{z} > \frac{1}{2} \quad (z \in \mathbb{U}).$$

*Proof.* To prove of the result, according to Lemma 2.1 it is sufficient to show that

$$\{a_j\}_{j=2}^{\infty} = \left\{\frac{1}{(2j-1)(j-1)!}\right\}_{j=2}^{\infty}$$

is a convex decreasing sequence, i.e., we need to prove that

$$a_j - a_{j+1} \geq 0 \tag{2.1}$$

and

$$a_{j+2} - a_{j+1} \geq a_{j+1} - a_j \tag{2.2}$$

for all $j \geq 2$.

First, we will prove (2.2) which is equivalent to

$$a_j - 2a_{j+1} + a_{j+2} \geq 0,\ j \geq 2.$$

A simple computation shows that

$$a_j - 2a_{j+1} = \frac{1}{(2j-1)(j-1)!} - \frac{2}{(2j+1)j!} = \frac{1}{(4j^2-1)j!}\mu(j),$$

where

$$\mu(j) := 2j^2 - 3j + 2.$$

Setting

$$\psi(x) := 2x^2 - 3x + 2,\ x \geq 2,$$

the function $f$ is strictly increasing on $[2, +\infty)$, hence $\min\{\psi(x) : x \geq 2\} = \psi(2) = 4$. Therefore, $a_j - 2a_{j+1} \geq 0$ for all $j \geq 2$. Using the fact that $a_{j+1} > 0$ for all $j \geq 2$, it follows $a_j - a_{j+1} \geq a_{j+1} > 0$ which is (2.1). From $a_{j+2} > 0$ for $j \geq 2$, the above inequality implies that $a_j - 2a_{j+1} + a_{j+2} \geq 0$ for all $j \geq 2$. Hence (2.2) holds. Therefore, from Lemma 2.1 we deduce that

$$\mathrm{Re}\left(\sum_{j=1}^{\infty} a_j z^{j-1}\right) > \frac{1}{2} \quad (z \in \mathbb{U}),$$

which gives

$$\mathrm{Re}\,\frac{\mathrm{Erfi}(z)}{z} > \frac{1}{2} \quad (z \in \mathbb{U})$$

and hence the proof is completed. □

**Theorem 2.8.** *For the function* Erfi*, the following inequality holds:*

$$\mathrm{Re}\left(\mathrm{Erfi}(z)\right)' > \frac{1}{2},\ z \in \mathbb{U}.$$

*Proof.* From (1.4) we have

$$\left(\operatorname{Erfi}(z)\right)' = 1 + \sum_{j=2}^{\infty} \frac{j}{(2j-1)(j-1)!} z^{j-1} \quad (z \in \mathbb{U}).$$

Denoting

$$a_j = \frac{j}{(2j-1)(j-1)!}, \tag{2.3}$$

and according to Lemma 2.1 it is sufficient to show that the inequalities (2.1) and (2.2) hold for $a_j$ given by (2.3). Using the same method like in the proof of Theorem 2.5, it is enough to prove that the difference of two term, that is,

$$a_j - 2a_{j+1} = \frac{j}{(2j-1)(j-1)!} - \frac{2(j+1)}{(2j+1)j!} = \frac{1}{(4j^2-1)j!}\lambda(j),$$

where

$$\lambda(j) := 2j^3 - 3j^2 - 2j + 2,$$

is positive. Setting

$$\chi(x) := 2x^3 - 3x^2 - 2x + 2, \ x \geq 2,$$

we see that the function $\chi$ is strictly increasing on $[2, +\infty)$, and hence $\min\{\chi(x) : x \geq 2\} = \chi(2) = 2 > 0$. Since $a_j > 0$ for all $j \geq 2$, using the same methods like in the last part of the proof of Theorem 2.7, we obtain the desired result. □

## 3. Conclusion

In this paper we have considered the normalized imaginary error function in the open unit disk and we obtained some geometric properties including close-to-convexity for this function. Moreover, it was proved that the normalized imaginary error function and the normalized error function are univalent (close-to-convex) in the open unit disk.

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