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Distortion theorems for homeomorphic Sobolev mappings of integrable p-dilatations

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Dedicated to the memory of Professor Gabriela Kohr – outstanding mathematician and person

Abstract. We study the distortion features of homeomorphisms of Sobolev class $W^{1,1}_{\rm loc}$ admitting integrability for p-outer dilatation. We show that such mappings belong to $W^{1,n-1}_{\rm loc}$, are differentiable almost everywhere and possess absolute continuity in measure. In addition, such mappings are both ring and lower Q-homeomorphisms with appropriate measurable functions Q. This allows us to derive various distortion results like Lipschitz, Hölder, logarithmic Hölder continuity, etc. We also establish a weak bounded variation property for such class of homeomorphisms.

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1. Introduction

Geometric Function Theory which lies at the core of two distinguished fields of Mathematics, namely, Geometry and Analysis, has various fundamental applications. One of appeals relates to the distortion theory of mappings.

The main claim of the present paper is developing the theory of mappings and solving some important problems in this field of geometric function theory of several real variables.

Various relations between absolute continuity, bounded variation, Sobolev spaces, etc. in higher dimensions attract an attention of many mathematicians during last decades. It is the well-known fact that homeomorphisms of Sobolev classes $W^{1,p}$ are differentiable almost everywhere (a.e.) under $p > n-1$. The border case $p = n-1$, in general case, fails to guarantee this crucial property, but assuming an appropriate

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additional restriction on mappings many far advanced regularity properties can be reached. We refer here to recent papers [2], [6], [7], [19], [24] and monograph [11].

The idea to study various properties of mappings involving only a geometric description is a key approach in Geometric Function Theory and goes back to the classical works of Grötzsch, Köbe and Ahlfors-Beurling. This method relies on the invariance/quasi-invariance of the conformal modulus under conformal/quasiconformal mappings. The classes of ring and lower Q-homeomorphisms provide a modern tool for studying various properties of mappings including regularity, removability, boundary correspondence and others; see, e.g. [1], [2], [4], [5], [9], [10], [13], [22], [23] and monograph [18].

We study the distortion features of homeomorphisms of Sobolev class $W^{1,1}_{\rm loc}$ in $\mathbb{R}^n$ admitting appropriate integrability for p-outer dilatation. It is shown that such mappings belong to $W^{1,n-1}_{\rm loc}$, are differentiable a.e., possess absolute continuities in measure with respect to the n-dimensional Lebesque measure m and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ in $\mathbb{R}^n$, and have a bounded variation. In addition, such mappings are both ring and lower Q-homeomorphisms with the corresponding measurable functions Q. This allows us to derive various distortion results like Lipschitz, Hölder, logarithmic Hölder continuity, etc. The range of real parameter p is the interval $[n, n+1/(n-2))$ for $n \geq 2$. It means that for the planer case we deal with $[2, \infty)$.

2. Sobolev classes and absolute continuity

2.1. Obviously the notion of absolute continuity is strongly connected with Sobolev classes in $\mathbb{R}^n$. We recall the definition of Sobolev spaces $W^{1,p}$, $p \geq 1$, following [11].

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{\rm loc}(\Omega)$. A function $v \in L^1_{\rm loc}(\Omega)$ is called a *weak derivative of* u if

$$\int\limits_{\Omega} \varphi(x) v(x)\, dm(x) \;=\; -\int\limits_{\Omega} u(x) \nabla\varphi(x)\, dm(x)$$

for every $\varphi \in \mathrm{C}^{\infty}_{C}(\Omega)$. The function v is referred to Du. For $1 \leq p \leq \infty$, the Sobolev space is defined by

$$W^{1,p}(\Omega) \;=\; \{u \in L^p(\Omega) : Du \in L^p(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} \;=\; \left(\int\limits_{\Omega} |u|^p + \int\limits_{\Omega} |Du|^p \right)^{1/p} .$$

Here C_C denotes the collection of continuous functions with compact support. A mapping $f : \Omega \to \mathbb{R}^n$ belongs to $W^{1,p}(\Omega)$ if its each component f_j, $j = 1, \ldots, n$ is a $W^{1,p}$-function.

A mapping $f \in L^1(\Omega)$ is of bounded variation, $f \in \mathrm{BV}(\Omega)$, if the coordinate functions of f belong to the space $\mathrm{BV}(\Omega)$. This means that the distributional derivatives of each coordinate function f_j are measures with finite total variations in Ω; see, e.g. [6].

2.2. It is well known that if $f \in W^{1,1}_{\mathrm{loc}}(D)$ of a domain $D \subset \mathbb{R}^n$, $n \geq 2$, then f has partial derivatives a.e. For $n = 2$, in addition, f is differentiable a.e. Thus, for any $f \in W^{1,1}_{\mathrm{loc}}(D)$ we denote by $f'(x)$ its Jacobi matrix. The quantities $\|f'(x)\| = \sup_{|h|=1} |f'(x)h|$ and $l(f'(x)) = \inf_{|h|=1} |f'(x)h|$ can be regarded as a maximal stretching and a minimal stretching of f at x, respectively. At a point of nondegenerate differentiability, i.e. $J_f(x) = \det f'(x) \neq 0$, the outer and inner dilatations are defined by

$$K_O(x, f) = \frac{\|f'(x)\|^n}{J_f(x)}\,, \qquad K_I(x, f) = \frac{J_f(x)}{l^n(f'(x))}\,,$$

respectively, extended to points where $J_f(x) = 0$ by $K_O(x, f) = K_I(x, f) = 1$.

Pick real p, $p \geq 1$, we consider p-counterparts of the above quantities determined as

$$K_{O,p}(x, f) = \frac{\|f'(x)\|^p}{J_f(x)}\,, \qquad K_{I,p}(x, f) = \frac{J_f(x)}{l^p(f'(x))}\,,$$

whereas $J_f(x) \neq 0$. Define $K_{O,p}(x, f) = K_{I,p}(x, f) = 1$, if $f'(x) = 0$, and $K_{O,p}(x, f) = K_{I,p}(x, f) = \infty$ otherwise. We call these quantities the *p-outer and p-inner dilatations* of f at x, respectively.

2.3. Let D be a domain in $\mathbb{R}^n$ for some $n \geq 2$. A mapping $f : D \to \mathbb{R}^n$ is called *quasiregular* (or a *mapping of bounded distortion* by Reshetnyak) if $f \in W^{1,n}(D)$ and there exists a constant $K \geq 1$ such that $K_O(x, f) \leq K$ a.e. in D.

A crucial extension of quasiregularity relates to the class of mappings of finite distortion where the uniform boundedness of $K_O(x, f)$ is relaxed by its finiteness. We say that a mapping $f : D \to \mathbb{R}^n$ has *finite distortion* if $f \in W^{1,1}(D)$, $J_f(x) \in L^1_{\mathrm{loc}}(D)$ and there is a function $K : D \to [1, \infty]$ with $K(x) < \infty$ a.e. such that $K_O(x, f) \leq K(x)$ a.e. in D.

Note that for homeomorphisms of finite distortion the condition $J_f(x) \in L^1_{\mathrm{loc}}(D)$ can be removed. Many analytic and topological properties for quasiregular mappings and mappings of finite distortion can be derived from their definitions. For the latter such properties obviously depend on appropriate restrictions on $K(x)$; see, e.g. [11], [14] and references therein.

Here we recall some analytic features for mappings of Sobolev spaces $W^{1,p}$. Each mapping f of $W^{1,p}$ has a representative g (i.e. $g = f$ a.e.) which is differentiable a.e. for the case when $p > n$ and $n \geq 2$. On the other hand, there are mappings of $W^{1,p}$, $p \leq n$, which are not continuous at any point and, therefore, are differentiable nowhere. Homeomorphisms of Sobolev classes $W^{1,p}$, $p > n-1$ for $n > 2$ or $p \geq 1$ for $n = 2$ provide differentiability a.e. So, the case when $p = n-1$ is crucial for higher dimensions.

The following recent results given in [24] are of special interest since they relate to the border case $p = n-1$.

Proposition 2.1. Suppose that D is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $f \in W^{1,n-1}_{\mathrm{loc}}(D)$ be a continuous, discrete and open mapping satisfying $K_O(\cdot, f) \in L^1_{\mathrm{loc}}(D)$. Then f is differentiable a.e. in D.

The second result in [24] ensuring differentiability a.e. relies on integrability of p-outer dilatation.

Proposition 2.2. Suppose that D is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $f \in W^{1,n-1}_{\mathrm{loc}}(D)$ be a continuous, discrete and open mapping satisfying $K_{O,q}(\cdot, f) \in L^1_{\mathrm{loc}}(D)$ for some $n-1 < q \leq n$. Then f is differentiable a.e. in D.

One more result on mappings of Sobolev class $W^{1,n-1}_{\mathrm{loc}}$ provided rich regularity properties for their inverses can be found in [12]; cf. [15].

Proposition 2.3. Let $f : D \to D'$ be a $W^{1,n-1}_{\mathrm{loc}}$-homeomorphism of finite inner distortion. If $K_I(\cdot, f) \in L^1_{\mathrm{loc}}(D)$ then $f^{-1} \in W^{1,n}_{\mathrm{loc}}(D')$.

2.4. Absolute continuity, more precisely, absolute continuity in measure is called the Lusin (N)-property, or preservation of sets of zero measure.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a mapping. We say that f possesses the *Lusin (N)-property* on a set $\Omega' \subset \Omega$ if the implication

$$mE = 0 \qquad \Longrightarrow \qquad mf(E) = 0$$

holds for each subset E of Ω'.

The above definition can be extended to the k-dimensional Hausdorff measure $\mathcal{H}^k$, $k = 1, \dots, n-1$, by replacing the n-dimensional Lebesgue measure m to $\mathcal{H}^k$. In this case, we deal with the Lusin (N)-property with respect to the k-dimensional Hausdorff measure.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $f : \Omega \to \mathbb{R}^n$ satisfies the *Lusin (N^{-1})-property* if for each $E \subset f(\Omega)$ such that $mE = 0$ we have $mf^{-1}(E) = 0$.

The Lusin (N)-property is satisfied for general Sobolev mappings for the case $p > n$. The limiting case $p = n$ for homeomorphisms also guarantees the Lusin (N)-property.

The following statement provides a sufficient condition for mappings of finite distortion to satisfy the Lusin (N^{-1})-property; see [11].

Proposition 2.4. Let a continuous mapping $f \in W^{1,1}(\Omega)$ be a mapping of finite distortion with $K_f^{1/(n-1)} \in L^1(\Omega)$. If the multiplicity of f is essentially bounded by a constant N and f is not constant, then $J_f(x) > 0$ a.e. in Ω, and hence f satisfies the Lusin (N^{-1})-property.

The exponent $1/(n-1)$ is crucial and cannot be reduced even for homeomorphisms.

Remark 2.5. Let $a < 1/(n-1)$. There exists a Lipschitz homeomorphism f of finite distortion $f \in W^{1,1}((-1,1)^n)$ and $K_f^a \in L^1((-1,1)^n)$, for which the Lusin (N^{-1})-property fails; see again [11].

For $K_{O,q}(x)$-distortion function, $q \leq n$, the Lusin (N^{-1})-property can be derived assuming $K_{O,q} \in L^{1/(q-1)}$; see [11, Thm 5.14].

Remark 2.6. Note that the Lusin (N^{-1})-property is equivalent that the Jacobian does not vanish a.e.; cf. [20].

The following important statements proved in [6] provide the Lusin (N)-property w.r.t. the $(n-1)$-dimensional Hausdorff measure for homeomorphisms of Sobolev classes with the border exponent.

Proposition 2.7. Let $f \in W^{1,n-1}_{\rm loc}((-1,1)^n)$ be a homeomorphism. Then for almost every $y \in (-1,1)$ the restriction of f on $(-1,1)^{n-1} \times \{y\}$ satisfies the $(n-1)$-dimensional Lusin (N)-property, i.e. for every $E \subset (-1,1)^{n-1} \times \{y\}$, $\mathcal{H}^{n-1}E = 0$ implies $\mathcal{H}^{n-1}f(E) = 0$.

Replacing the cube $(-1,1)^n$ to a ball $B(x_0,r)$ and the hyperplanes $(-1,1)^{n-1} \times \{y\}$ to spheres $S(x,r)$, one gets

Proposition 2.8. Let $f \in W^{1,n-1}(B(x_0,r))$ be a homeomorphism. Then for almost every $r \in (0,r_0)$ the mapping $f : S(x,r) \to \mathbb{R}^n$ satisfies the $(n-1)$-dimensional Lusin (N)-property, i.e. for every $E \subset S(x,r)$, $\mathcal{H}^{n-1}E = 0$ implies $\mathcal{H}^{n-1}f(E) = 0$.

3. Moduli of curve and surface families

3.1. For a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$, its *integral over a k-dimensional surface* $\mathcal{S}$ (a continuous mapping $\mathcal{S} : D_{\mathcal{S}} \to \mathbb{R}^n$, $D_{\mathcal{S}}$ is a domain in $\mathbb{R}^k$, $k = 1,\dots,n-1$) is determined by

$$\int\limits_{\mathcal{S}} \rho \, d\mathcal{A} := \int\limits_{\mathbb{R}^n} \rho(y)\, N(\mathcal{S},y)\, d\mathcal{H}^k y\,,$$

where $N(\mathcal{S},y)$ stands for the multiplicity function of $\mathcal{S}$, namely, the multiplicity of covering the point y by the surface $\mathcal{S}$, $N(\mathcal{S},y) = \operatorname{card} \mathcal{S}^{-1}(y)$, which is measurable with respect to the Hausdorff measure $\mathcal{H}^k$; see, e.g. [21, Theorem II (7.6)].

A Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family of k-dimensional surfaces Γ in $\mathbb{R}^n$, $k = 1,2,...,n-1$, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\mathcal{S}} \rho^k \, d\mathcal{A} \ge 1 \qquad \forall\, \mathcal{S} \in \Gamma\,. \tag{3.1}$$

By the *k-dimensional Hausdorff area* of a Borel set B in $\mathbb{R}^n$ (or simply *area* of B in the case $k = n-1$) associated with the surface $\mathcal{S} : \omega \to \mathbb{R}^n$, we mean

$$\mathcal{A}_{\mathcal{S}}(B) \;=\; \mathcal{A}^k_{\mathcal{S}}(B) \;:=\; \int\limits_B N(\mathcal{S},y)\, d\mathcal{H}^k y\,,$$

(cf. [8, Ch. 3.2.1]). The surface $\mathcal{S}$ is called *rectifiable (quadrable)*, if $\mathcal{A}_{\mathcal{S}}(\mathbb{R}^n) < \infty$ (see, e.g [18, Ch. 9.2]).

The *modulus of family* Γ (conformal modulus) is defined by

$$\mathcal{M}(\Gamma) \;:=\; \inf_{\rho \,\in\, \operatorname{adm}\Gamma} \int\limits_D \rho^n(x)\, dm(x)\,. \tag{3.2}$$

Replacing the exponent n in (3.2) by real p, $p \ge 1$, we arrive at the quantity which is called p-modulus $\mathcal{M}_p(\Gamma)$ of the family Γ.

We say that a property P holds for a.a. $\mathcal{S} \in \Gamma$, if the corresponding modulus of a subfamily Γ_* of $\mathcal{S} \in \Gamma$, for which P is not true, vanishes. Following [18], a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *extensively admissible* for the family Γ of k-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, abbr. $\rho \in \mathrm{ext}_p\mathrm{adm}\,\Gamma$, if the admissibility condition (3.1) is fulfilled only for a.a. $\mathcal{S} \in \Gamma$.

3.2. A significance of moduli of curve/surface families follows mainly from the fact that the conformal modulus remains invariant under conformal mappings, whereas p-modulus is invariant under isometries. Various inequalities for moduli form the basis for the geometric part of quasiconformality/quasiregularity. We recall that the quasiinvariance of the conformal modulus completely characterizes quasiconformality. The same property for the p-modulus provides quasiisometry.

The following notions successfully extend the above classes of mappings including quasiconformality/quasiisometry.

Denote by $\Delta(E, F; G)$ a family of all curves $\gamma : [0, 1] \to \mathbb{R}^n$, which join arbitrary sets E and F located in $G \subset \mathbb{R}^n$, i.e. $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in G$ for all $t \in (0, 1)$.

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0, \infty]$ be a measurable function. We say that a homeomorphism $f : D \to D'$ is a *ring Q-homeomorphism with respect to p-modulus at* $x_0 \in D$, $p > 1$, if the following inequality

$$\mathcal{M}_p\left(\Delta\left(f(S_1), f(S_2); f(D)\right)\right) \leq \int\limits_{\mathbb{A}} Q(x) \cdot \eta^p(|x - x_0|)\, dm(x) \tag{3.3}$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0 = \mathrm{dist}\,(x_0, \partial D)$ and any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1\,. \tag{3.4}$$

We also say that a homeomorphism $f : D \to \mathbb{R}^n$ is a *ring Q-homeomorphism with respect to p-modulus in* D, if inequality (3.3) is valid for any $x_0 \in D$.

Now instead of an upper bound for the modulus, we consider a lower integral estimate. Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $x_0 \in D$. Given a Lebesgue measurable function $Q : D \to [0, \infty]$, a homeomorphism $f : D \to \mathbb{R}^n$ is called a *lower Q-homeomorphism with respect to p-modulus at* x_0 if

$$\mathcal{M}_p(f(\Sigma_\varepsilon)) \geq \inf_{\rho \in \exp_p \mathrm{adm}\ \Sigma_\varepsilon} \int\limits_{D_\varepsilon(x_0)} \frac{\rho^p(x)}{Q(x)}\, dm(x)\,,$$

where

$$D_\varepsilon(x_0) = D \cap \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad 0 < \varepsilon < \varepsilon_0, \quad 0 < \varepsilon_0 < \sup_{x \in D} |x - x_0|,$$

and Σ_ε denotes the family of all pieces of spheres centered at x_0 of radii r, $\varepsilon < r < \varepsilon_0$, located in D.

Similarly to above, a homeomorphism $f : D \to \mathbb{R}^n$ is called a *lower Q-homeomorphism with respect to p-modulus in* D if it is a lower Q-homeomorphism at each point $x_0 \in D$.

The following relationship between the ring and lower Q-homeomorphisms with respect to p-modulus has been established in [9].

Proposition 3.1. Every lower Q-homeomorphism $f : D \to \mathbb{R}^n$ at $x_0 \in D$ with respect to p-modulus, $p > n-1$ and $Q \in L_{\mathrm{loc}}^{\frac{n-1}{p-n+1}}$, is a ring $\widetilde{Q}$-homeomorphism with respect to α-modulus at x_0 with $\widetilde{Q} = Q^{\frac{n-1}{p-n+1}}$ and $\alpha = p/(p-n+1)$.

4. Auxiliary results

4.1. As was mentioned above, homeomorphisms of Sobolev space $W^{1,n-1}$ are of special interest, since, in general case, they need not be differentiable a.e., although for any $W^{1,p}$, $p > n-1$, this crucial regularity property holds. We show that the integrability of p-outer dilatation with an appropriate degree for Sobolev homeomorphisms guaranties differentiability a.e.

Theorem 4.1. Let $f : D \to \mathbb{R}^n$ satisfying $f \in W_{\mathrm{loc}}^{1,1}(D)$ and $K_{O,p} \in L_{\mathrm{loc}}^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Then $f \in W_{\mathrm{loc}}^{1,n-1}(D)$ and f is differentiable in D a.e. Moreover, $f^{-1} \in W_{\mathrm{loc}}^{1,n}(f(D))$, and, therefore, f possesses the Lusin (N^{-1})-property with respect to the n-dimensional Lebesgue measure.

Proof. Denote by E any compact set in D. Then the Hölder inequality with exponents $\alpha = \frac{p}{p-n+1}$ and $\beta = \frac{p}{n-1}$ provides

$$\int_E \|f'(x)\|^{n-1}\, dm(x) = \int_E K_{O,p}^{\frac{n-1}{p}}(x,f) \cdot J_f^{\frac{n-1}{p}}(x)\, dm(x)$$

$$\leq \left(\int_E K_{O,p}^{\frac{n-1}{p-n+1}}(x,f) dm(x) \right)^{\frac{p-n+1}{p}} \left(\int_E J_f(x)\, dm(x) \right)^{\frac{n-1}{p}} < \infty,$$

and, therefore, $f \in W_{\mathrm{loc}}^{1,n-1}(D)$.

Now pick $\alpha = p/(p-n+1)$ for arbitrary $p \in [n, n+1/(n-2))$. Then $n-1 < \alpha \leq n$ and at a point of nondegenerate differentiability, we have

$$K_{I,\alpha}(x,f) = \frac{J_f(x)}{l^\alpha(f'(x))} = \frac{J_f^{\frac{p}{p-n+1}}(x)}{J_f^{\frac{n-1}{p-n+1}}(x) l^{\frac{p}{p-n+1}}(f'(x))} \leq \frac{\|f'(x)\|^{\frac{n-1}{p-n+1}}}{J_f^{\frac{n-1}{p-n+1}}(x)} = K_{O,p}^{\frac{n-1}{p-n+1}}(x,f).$$

Thus, $K_{I,\alpha}(x,f)$ is locally integrable in D. Applying Proposition 2.2 (Tengvall's theorem from [24]), we reach that f is differentiable a.e. in D.

To reach the last assertions of Theorem 4.1, note first that for $p = n$ we obtain $K_O \in L_{\mathrm{loc}}^{n-1}$. Further by the well-known relations between K_I and K_O-distortion functions, namely $K_I \leq K_O^{n-1}$, one can apply Proposition 2.3. This yields $f^{-1} \in W_{\mathrm{loc}}^{1,n}$, and, hence f^{-1} possesses the Lusin (N)-property. □

4.2. Absolute continuity in measure for homeomorphisms of $W_{\mathrm{loc}}^{1,1}$ with integrable p-outer dilatation is established in the following statements.

Theorem 4.2. Let D be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n + 1/(n-2))$. Then f satisfies the Lusin (N)-property w.r.t. the $(n-1)$-dimensional Hausdorff measure on pieces $S_r \cap D$ of almost all spheres S_r centered at an arbitrary point $x_0 \in D$. In addition, on all such pieces $\mathcal{H}^{n-1} f(E) = 0$ holds whereas $f' = 0$ on a measurable set E.

Proof. By Theorem 4.2, the homeomorphism f is differentiable a.e. in D and belongs to $W^{1,n-1}_{\rm loc}(D)$. This allows us to apply Proposition 2.7 and obtain the Lusin (N)-property with respect to the $\mathcal{H}^{n-1}$-measure; cf. [2].

The last assertion of Theorem 4.2 follows from the equality

$$\mathcal{H}^{n-1} f(E) = \int\limits_E J_{n-1,f}(x)\, d\mathcal{A},$$

where $J_{n-1,f}(x)$ stands for the $(n-1)$-dimensional Jacobian of the mapping f on $S_r \cap D$, and $E \subset S_r \cap D$ is a measurable set. Then from the evident estimate we have

$$\mathcal{H}^{n-1} f(E) \le \int\limits_E \|f'(x)\|^{n-1}\, d\mathcal{A},$$

which completes the proof. □

The above theorem with Proposition 2.8 yields

Corollary 4.3. Let D be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n + 1/(n-2))$. Then f satisfies the Lusin (N)-property w.r.t. the $(n-1)$-dimensional Hausdorff measure on $\mathcal{P} \cap D$ of almost all hyperplanes $\mathcal{P}$ which are parallel to the coordinate hyperplanes. In addition, on all such intersections $\mathcal{H}^{n-1} f(E) = 0$ holds whereas $f' = 0$ on a measurable set E.

4.3. Here we obtain relationships between Sobolev homeomorphisms with integrable p-outer dilatation and classes of mappings admitting modular presentations (ring and lower Q-homeomorphisms).

Theorem 4.4. Let $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n+1/(n-2))$. Then f is a lower Q-homeomorphism with respect to p-modulus at arbitrary $x_0 \in D$ with $Q(x) = K_{O,p}(x, f)$ in D. Moreover, f is a ring $\widetilde{Q}$-homeomorphism with respect to α-modulus in D with $\widetilde{Q}(x) = K^{(n-1)/(p-n+1)}_{O,p}(x, f)$, where $\alpha = p/(p-n+1)$.

Proof. Due to Theorem 4.1, f is differentiable a.e. in D. Moreover, the Lusin (N^{-1})-property yields that $J_f(x)$ does not vanish a.e. in D. Denote by E a Borel set of all points x in D, where f has a total differential $f'(x)$ and Jacobian $J_f(x) \neq 0$, and by $\widetilde{E}$ a set of all points at which f has a total differential $f'(x)$ but $J_f(x) = 0$. Then both sets $E_0 := D \setminus (E \cup \widetilde{E})$ and $\widetilde{E}$ have zero n-dimensional Lebesgue measure.

Now applying a Kirszbraun type theorem one concludes that the set E can be presented as a countable union of piecewise distinct Borel sets E_l, $l = 1, 2, \ldots$, such that $f_l = f|_{E_l}$ are bi-Lipschitz homeomorphisms; see, e.g. Lemma 3.2.2 and Theorems 3.1.4 and 3.1.8 in [8]. Since the set E_0 has zero Lebesgue n-measure, applying [18, Theorem 9.1] yields that $\mathcal{H}^{n-1}(f(E_0)\cap S_r^{'}) = 0$ and $\mathcal{H}^{n-1}(f(\widetilde{E})\cap S_r^{'}) = 0$ for almost all images $S_r^{'} = f(S_r)$ of spheres S_r in the sense of p-modulus of surface families. Fix arbitrarily $x_0 \in D$ and note that

$$\mathcal{H}^{n-1}(f(E_0)\cap S_r^{'}) \;=\; 0 \quad \text{and} \quad \mathcal{H}^{n-1}(f(\widetilde{E})\cap S_r^{'}) = 0 \tag{4.1}$$

for almost all $r \in (\varepsilon, \varepsilon_0)$ by [13, Theorem 4.1].

For an arbitrary admissible function $\varrho' \in \operatorname{adm} f(\Sigma_\varepsilon)$ extended by $\varrho' \equiv 0$ outside of $f(D)$, we define

$$\varrho(x) \;:=\; \varrho'(f(x))\|f'(x)\|$$

for $x \in E$ and set $\varrho \equiv 0$ otherwise.

Note that

$$f(\Sigma_\varepsilon) \;=\; f(D)\cap S_r^{'} \;=\; \bigcup_{l=0}^{\infty}(f(E_l)\cap S_r^{'}) \cup (f(\widetilde{E})\cap S_r^{'}).$$

Since $\varrho' \in \operatorname{adm} f(\Sigma_\varepsilon)$ and due to (4.1),

$$\begin{aligned} 1 \;\le\; \int\limits_{f(\Sigma_\varepsilon)} (\varrho'(y))^{n-1}\, d\mathcal{A}' \;&=\; \sum_{l=0}^{\infty}\; \int\limits_{f(E_l)\cap S_r'} (\varrho'(y))^{n-1} N(y,f,E_l\cap S_r)\, d\mathcal{H}^{n-1}y \\ &\quad+ \int\limits_{f(\widetilde{E})\cap S_r'} (\varrho'(y))^{n-1} N(y,f,\widetilde{E}\cap S_r)\, d\mathcal{H}^{n-1}y \\ &= \sum_{l=1}^{\infty}\; \int\limits_{f(E_l)\cap S_r'} (\varrho'(y))^{n-1} N(y,f,E_l\cap S_r)\, d\mathcal{H}^{n-1}y \end{aligned} \tag{4.2}$$

for almost all $r \in (\varepsilon, \varepsilon_0)$. Here $N(y,f,A)$ denotes the multiplicity function, i.e. $N(y,f,A) = \operatorname{card}\{x \in A | f(x) = y\}$. Recall that for homeomorphisms $N(y,f,A) = 1$.

Arguing piecewise on E_l, $l = 1,2,\ldots$, and using [8, Theorem 3.2.5] with the Lusin's (N)-property w.r.t. the $(n-1)$-dimensional Hausdorff measure (Theorem 4.2), we get

$$\begin{aligned} \int\limits_{E_l\cap S_r} \varrho^{n-1}(x)\, d\mathcal{A} \ge \int\limits_{E_l\cap S_r} (\varrho'(f(x)))^{n-1} J_{n-1,f}(x)\, d\mathcal{A} \\ = \int\limits_{f(E_l)\cap S_r'} (\varrho'(y))^{n-1} N(y,f,E_l\cap S_r)\, d\mathcal{H}^{n-1}y \end{aligned}$$

for a.a. $r \in (\varepsilon, \varepsilon_0)$, which together with (4.2) implies $\varrho \in \operatorname{ext}_p \operatorname{adm} \Sigma_\varepsilon$.

Now applying on each E_l the change of variables formula with the countable additivity of integrals, we have

$$\sum_l \int\limits_{E_l\cap S_r} \frac{\varrho^p(x)}{K_{O,p}(x,f)}\, dm(x) \;=\; \int\limits_{D\cap S_r} \frac{\varrho^p(x)}{K_{O,p}(x,f)}\, dm(x) \;\le\; \int\limits_{f(\Sigma_\varepsilon)} (\varrho^{'}(y))^p\, dm(y)\,.$$

Thus, f is a lower Q-homeomorphism with $Q(x) = K_{O,p}(x, f)$.

By Proposition 3.1, f is also a ring $\widetilde{Q}$-homeomorphism with respect to α-modulus in D with $\widetilde{Q}(x) = K_{O,p}^{(n-1)/(p-n+1)}(x, f)$, where $\alpha = p/(p-n+1)$, which completes the proof. □

4.4. Several properties related to absolute continuity and bounded variation can be derives as consequences from the above results.

Combining [6, Theorem 1.1] and Theorem 4.1, one derives

Corollary 4.5. Let $\Omega \in \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a homeomorphism of $W^{1,1}_{\rm loc}(\Omega)$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(\Omega)$. Then $f^{-1} \in {\rm BV}_{\rm loc}(f(\Omega))$.

Applying [6, Theorem 1.2] with Theorem 4.1 provides the following sufficient condition for a homeomorphism to be a bi-Sobolev mapping.

Corollary 4.6. Let $\Omega \in \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a homeomorphism of finite distortion with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(\Omega)$. Then $f^{-1} \in W^{1,1}_{\rm loc}(f(\Omega))$ and f^{-1} is of finite distortion.

The following corollaries can be derived from [19] and [7], respectively, replacing the condition $f \in W^{1,n-1}$ by $f \in W^{1,1}$ with the appropriate integrability of p-outer dilatation.

Corollary 4.7. *Let $f : D \to D'$ be a homeomorphism of $W^{1,1}_{\rm loc}(D)$ with finite inner distortion such that $K_{O,p} \in L^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Then $\|(f^{-1}(y))'\| \in L^n(D')$ and $\int_{D'} \|(f^{-1}(y))'\|^n\, dm(y) = \int_D K_I(x, f)\, dm(x)$.*

Corollary 4.8. Let $f : D \to D'$ be a homeomorphism of finite inner distortion and $f \in W^{1,1}(D)$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n+1/(n-2))$. Assume that $u \in W^{1,\infty}_{\rm loc}(D)$. Then $u \circ f^{-1} \in W^{1,1}_{\rm loc}(D')$.

5. Distortion theorems

In this section, we provide distortion type theorems whose proofs mainly rely on Theorem 4.4.

5.1. We start with Hölder's continuity. Theorem 4.4 yields that every homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ in $\mathbb{R}^n$, $n \geq 2$, with $K_{O,p}$ integrable in degree $(n-1)/(p-n+1)$ is a lower Q-homeomorphism with respect to p-modulus. Then by Theorem 4.2 in [22], one gets

Theorem 5.1. Let D and D' be two domains in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to D'$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in (n, n+1/(n-2))$. Assume that for some real $\lambda > 1$, $\sigma > 0$, and $C_{x_0} > 0$ the following condition holds

$$\varepsilon^\sigma \int\limits_{\varepsilon}^{\lambda\varepsilon} \frac{dr}{\|K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0, r)} \geq C_{x_0} \qquad \forall\, \varepsilon \in \left(0, \frac{{\rm dist}\,(x_0, \partial D)}{\lambda^2}\right),$$

where

$$\| K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0,r) \;=\; \left(\int\limits_{S(x_0,r)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\; d\mathcal{A}\right)^{\frac{p-n+1}{n-1}}. \qquad (5.1)$$

Then the estimate

$$|f(x)-f(x_0)| \;\le\; \nu_0\, C_{x_0}^{-\frac{1}{p-n}}\, |x-x_0|^{\frac{\sigma}{p-n}}$$

is valid for all $x \in B(x_0,\delta_0)$, where ν_0 is a positive constant depending only on n, p, λ and σ.

Corollary 5.2. In particular, if for some $\lambda > 1$ and $C_{x_0} > 0$ the condition

$$\varepsilon^{p-n}\int\limits_{\varepsilon}^{\lambda\varepsilon} \frac{dr}{\| K_{O,p}(x,f)\|_{\frac{n-1}{p-n+1}}(x_0,r)} \;\ge\; C_{x_0}$$

holds for some $\varepsilon \in \left(0, \operatorname{dist}(x_0,\partial D)/\lambda^2\right)$, then f is Lipschitz continuous, i.e.

$$|f(x)-f(x_0)| \;\le\; \nu_0\, C_{x_0}^{-\frac{1}{p-n}}\, |x-x_0|$$

for all $x \in B(x_0,\delta_0)$; here $\nu_0 = \nu_0(n,p,\lambda) > 0$.

Now Theorem 4.4 together with Theorem 4.4 in [22] yields

Theorem 5.3. Let D and D' be two domains in $\mathbb{R}^n$, $n \ge 2$. Suppose that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in (n, n+1/(n-2))$. If $K_{O,p}(x,f) \in L^{\alpha}(B(x_0,\delta_0))$, $\delta_0 \le \operatorname{dist}(x_0,\partial D)/4$, $\alpha > n/(p-n)$, then

$$|f(x)-f(x_0)| \;\le\; \nu_0 \|K_{O,p}(f)\|_{\alpha}^{\frac{1}{p-n}}\, |x-x_0|^{1-\frac{n}{\alpha(p-n)}}$$

for all $x \in B(x_0,\delta_0)$, where $\|K_{O,p}(f)\|_{\alpha} = \left(\int_{B(x_0,\delta_0)} K^{\alpha}_{O,p}(x,f)\, dm(x)\right)^{1/\alpha}$ denotes an L^{α}-norm over $(B(x_0,\delta_0))$, and ν_0 stands for a positive constant depending only on n, p and α.

5.2. A logarithmic Hölder continuity is much weaker than the Hölder one; see, e.g. [10]. Here we first apply Theorem 4.4 and then Theorem 5.2 from [22], in order to reach a logarithmic type of distance distortions for homeomorphic Sobolev mappings of $W^{1,1}_{\rm loc}$ in $\mathbb{R}^n$, $n \ge 2$.

Theorem 5.4. Let D and D' be two domains in $\mathbb{R}^n$, $n \ge 2$, and $f : D \to D'$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$. If $p \in (n, n+1/(n-2))$, $\| K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0,r) \ne \infty$ for a.a. $r \in (0,d_0)$, $d_0 = \operatorname{dist}(x_0,\partial D)$, and for some real $\kappa \in [0, p/(p-n+1))$, $C_{x_0} > 0$, the upper bound

$$\int\limits_{\mathbb{A}(x_0,\varepsilon_1,\varepsilon_2)} \frac{[K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x)}{|x-x_0|^{\frac{p}{p-n+1}}} \;\le\; C_{x_0}\, \ln^{\kappa}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)$$

holds for any $0<\varepsilon_1<\varepsilon_2<d_0$, then

$$|f(x)-f(x_0)|\ \leq\ \nu_0\, C_{x_0}^{\gamma}\ \ln^{-\theta}\frac{1}{|x-x_0|}$$

for all $x\in B(x_0,\delta_0)$, where $\delta_0\leq\min\{1,\operatorname{dist}^4(x_0,\partial D)\}$,

$$\gamma=\frac{p-n+1}{(n-1)(p-n)},\quad \theta=\frac{p-\kappa(p-n+1)}{(n-1)(p-n)}$$

and ν_0 is a positive constant depending only on n, p and κ.

Theorem 5.4 with Corollary 5.1 and Theorem 5.3 in [22] imply two following statements.

Corollary 5.5. Let D and D' be two domains in $\mathbb{R}^n$, $n\geq 2$. Suppose that $f:D\to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p}\in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$. If $K_{O,p}(x,f)\in L^{n/(p-n)}(B(x_0,\delta_0))$, $\delta_0\ \leq\ \min\{1,\operatorname{dist}^4(x_0,\partial D)\}$ and $p\in(n,n+1/(n-2)))$, then

$$|f(x)-f(x_0)|\ \leq\ \nu_0\,\|K_{O,p}(f)\|^{\frac{1}{p-n}}_{\frac{n}{p-n}}\ \ln^{-\frac{p}{n(p-n)}}\frac{1}{|x-x_0|}$$

for all $x\in B(x_0,\delta_0)$, where

$$\|K_{O,p}(f)\|_{\frac{n}{p-n}}\ =\ \left(\ \int\limits_{B(x_0,\delta_0)} K_{O,p}^{\frac{n}{p-n}}(x,f)\,dm(x)\right)^{\frac{p-n}{n}}$$

stands for a norm in $L^{n/(p-n)}(B(x_0,\delta_0))$ and ν_0 is a positive constant depending only on n and p.

Corollary 5.6. Let D and D' be two domains in $\mathbb{R}^n$, $n\geq 2$. Suppose that $f:D\to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p}\in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$. If $p\in(n,n+1/(n-2))$ and for some $k_{x_0}>0$, the growth estimate

$$\|K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0,r)\ \leq\ k_{x_0}r$$

holds for a.a. $r\in(0,\delta_0)$, $\delta_0\leq\min\{1,\operatorname{dist}^4(x_0,\partial D)\}$, then

$$|f(x)-f(x_0)|\ \leq\ \nu_0\,\kappa_{x_0}^{\frac{1}{p-n}}\ \ln^{-\frac{1}{p-n}}\frac{1}{|x-x_0|}\,,$$

for all $x\in B(x_0,\delta_0)$, where $\nu_0>0$ depends only on n and p.

5.3. The finitely Lipschitz homeomorphisms have some very important and interesting properties. They can fail to belong to $W^{1,1}_{\rm loc}$, however, they possess the Lusin (N)-property with respect to the Hausdorff measure $\mathcal{H}^k$, $k=1,\dots,n$; see [18] (and [1] in more general settings).

Recall that a mapping is called Lipschitz in a domain $D \subset \mathbb{R}^n$, if there exists a constant L such that $|f(x) - f(y)| \le L|x - y|$ for ant $x, y \in D$. Consider a quantity

$$L(x_0, f) \;=\; \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}\,,$$

and say that a mapping f is finitely Lipschitz in D if $L(x_0, f) < \infty$ at any $x_0 \in D$; see, e.g. [18]. The quantity $L(x_0, f)$ can be treated as a maximal stretching of f at x_0, and the condition $L(x_0, f) < \infty$ at any $x_0 \in D$ provides (by the well-known Stepanoff's theorem) differentiability a.e.

Theorem 5.7. Let D and D' be two domains in $\mathbb{R}^n$, $n \ge 2$. Assume that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$. If $p \in (n, n + 1/(n-2))$ and

$$k_p(x_0) \;=\; \limsup_{\varepsilon \to 0} \left(\fint_{B(x_0,\varepsilon)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x) \right)^{\frac{p-n+1}{n-1}} \;<\; \infty\,,$$

then

$$L(x_0, f) \;\le\; \nu_0\, k_p^{\frac{1}{p-n}}(x_0) < \infty\,, \tag{5.2}$$

where ν_0 is a positive constant depending on n and p.

The proof of this theorem follows from Theorem 4.4 and Theorem 6.1 with Lemma 5.3 in [23]. For the reader convenience, we provide here the main ideas of proof.

Sketch of the proof. By Theorem 4.4, f is a lower Q-homeomorphism with respect to p-modulus with $Q(x) = K_{O,p}(x, f)$, and f is a ring $\widetilde{Q}$-homeomorphism with respect to α-modulus in D with $\widetilde{Q}(x) = K^{(n-1)/(p-n+1)}_{O,p}(x, f)$, where $\alpha = p/(p - n + 1)$.

Pick arbitrary $x_0 \in D$. Then for arbitrary spherical ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0 = \mathrm{dist}\,(x_0\,, \partial D)$ one obtains

$$\mathcal{M}_\alpha\left(\Delta\left(f(S_1), f(S_2); f(D)\right)\right) \;\le\; \left(\int\limits_{r_1}^{r_2} \frac{dr}{\|K_{O,p}(x,f)\|_{(n-1)/(p-n+1)}(r)} \right)^{\frac{1-n}{p-n+1}}\,, \tag{5.3}$$

applying the relation between p-modulus of the family of $(n-1)$-dimensional separating surfaces and α-modulus of the family of joining curves in $f(\mathbb{A})$ together with [9, Thm 6.1]. Here $\|K_{O,p}(x, f)\|_{(n-1)/(p-n+1)}(r)$ is defined by (5.1).

By Hölder's inequality, the right-hand side in (5.3) can be estimated from above by $(r_2 - r_1)^{p/(n-1-p)} \int_{\mathbb{A}} K^{(n-1)/(p-n+1)}_{O,p} dm(x)$. Choosing first $r_1 = 2\varepsilon$ and $r_2 = 4\varepsilon$ and applying the well-known connection between α-capacity of condenser and α-modulus together with the lower bound in terms of the n-dimensional Lebesgue measure, one gets

$$\frac{mf(B(x_0, 2\varepsilon))}{mB(x_0, 2\varepsilon)} \;\le\; c_1 \left(\fint_{B(x_0,4\varepsilon)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x) \right)^{\frac{n(p-n+1)}{n(p-n+1)-p}}\,.$$

Now we pick $r_1 = \varepsilon$ and $r_2 = 2\varepsilon$ and apply again the connection between α-capacity of condenser and α-modulus together with the lower bound in terms of the diameter of $f(B(x_0, r))$, then

$$\frac{\operatorname{diam} f(B(x_0,\varepsilon)}{\varepsilon} \le c_2 \left(\frac{mf(B(x_0,2\varepsilon))}{mB(x_0,2\varepsilon)}\right)^{j_1} \left(⨍_{B(x_0,4\varepsilon)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x)\right)^{j_2},$$

where $j_1 = ((1-n)(p-n+1)+p)/p$, $j_2 = (n-1)(p-n+1)/p$, and c_1 and c_2 are constants.

Finally, combining two last estimates and passing to the limsup as $\varepsilon \to 0$, we obtain the desirable conclusion (5.2). □

Corollary 5.8. *Let D and D' be two domains in $\mathbb{R}^n$, $n \ge 2$. Assume that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$. If $p \in (n, n+1/(n-2))$ and*

$$\limsup_{\varepsilon\to 0} ⨍_{B(x_0,\varepsilon)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x) < \infty$$

for all $x_0 \in D$, then f is finitely Lipschitz in D.

The finiteness of $k_p(x_0)$ is a necessary condition (in somewhat sense). One can illustrate it by the following example.

Example. Assume that $n \ge 3$ and $p \in (n, n+1/(n-2))$, and consider an automorphism $f : \mathbb{B}^n \to \mathbb{B}^n$ of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ of such a form

$$f(x) = \frac{x}{|x|}\left(\int_{|x|}^{1} \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/t)}\right)^{-\frac{1}{p-n}}, \quad x \ne 0, \tag{5.4}$$

extended by $f(0) = 0$.

Passing to the spherical coordinates in the image (ρ, ψ_i) and in the inverse image (r, φ_i), $i = 1, \dots, n-1$, one can rewrite (5.4) by

$$f(x) = \left\{\rho = \left(\int_{r}^{1} \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/t)}\right)^{-\frac{1}{p-n}}, 0 < r < 1,\ \rho(0) = 0\right\}.$$

Since ρ depends only on r, $\psi_i = \varphi_i$, whereas $0 \le \varphi_i \le \pi, i = 1, \dots, n-2$, and $0 \le \varphi_{n-1} \le 2\pi$. In this case, the stretchings are equal

$$\frac{d\rho}{dr}, \frac{\rho}{r}\frac{d\psi_1}{d\varphi_1}, \dots, \frac{\rho}{r}\frac{\sin\psi_i}{\sin\varphi_i}\frac{d\psi_1}{d\varphi_1}, \quad i = 2, \dots, n-1;$$

see, e.g. [16]. By a direct calculation,

$$\frac{\rho}{r} = \frac{1}{r}\left(\int_{r}^{1} \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/t)}\right)^{-\frac{1}{p-n}},$$

$$\frac{d\rho}{dr} = \frac{1}{(p-n)r^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/r)} \left(\int_r^1 \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/t)} \right)^{-\frac{p-n+1}{p-n}} .$$

Since $\rho/r \geq d\rho/dr$ and

$$\left(\frac{\rho}{r}\right)^{p-n+1} = \frac{d\rho}{dr}(p-n)\ln^{\frac{p-n+1}{n-1}}(e/r) ,$$

one gets,

$$K_{O,p}(x,f) = \frac{(\rho/r)^p}{(\rho/r)^{n-1}d\rho/dr} = (p-n)\ln^{\frac{p-n+1}{n-1}}(e/|x|) .$$

Clearly,

$$\limsup_{\varepsilon\to 0} \fint_{B_\varepsilon} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x) = \infty ,$$

where $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$.

On the other hand, by L'Hôpital's rule,

$$\lim_{x\to 0} \frac{|f(x)|}{|x|} = \infty ,$$

thus, f fails to be finitely Lipschitz at the origin.

6. Bounded variation and differentiability a.e.

In this section, we show the connection between Sobolev topological mappings with integrable p-outer dilations and homeomorphisms of a weaker bounded variation.

6.1. In 1999, Jan Malý [17] introduced a multidimensional bounded variation by the following way. Given a mapping $f : \Omega \to \mathbb{R}^m$ and an open set $G \subset \Omega$, the *n-variation* of f on G is defined by

$$V^n(f,G) = \sup\Big\{ \sum_j \big(\mathrm{osc}_{B(x_j,r_j)} f\big)^n : \{B(x_j,r_j)\} \text{ is a disjoint family of balls in } G \Big\},$$

where $\mathrm{osc}_B f = \sup\{|f(x)-f(y)| : x,y \in B\}$ for a ball B. We say that f has a bounded *n-variation* in Ω if $V^n(f,\Omega) < \infty$. By $\mathrm{BV}^n(\Omega)$ we denote the class of all mappings with bounded n-variation with the seminorm $||f||_{\mathrm{BV}^n(\Omega)} = (V_n(f,\Omega))^{1/n}$.

This class provides a proper subset of Sobolev class $W^{1,n}$, and, moreover, has rich regularity properties as differentiability a.e., the Lusin (N)-condition, etc.

Later the Malý's definition has been extended to a p-counterpart of n-variation in [3], $1 \leq p < n$.

We say that f has bounded *p-variation* (abbr., $f \in \mathrm{BV}^p(\Omega)$) if there exist $M > 0$ and η such that

$$\sum_j (\mathrm{osc}_{B(x_j,r_j)} f)^p r_j^{n-p} < M$$

for each disjoint system of balls $\{B(x_j,r_j)\}$ in Ω such that $r_j < \eta$.

Several important properties and relationships between BV^p and Sobolev classes are also established in [3]. One of them is differentiability a.e. Note that although $p \geq 1$ has been assumed, in fact, it is enough to require for p to be only positive.

6.2. Here we show that homeomorphisms of $W^{1,1}_{\rm loc}$ with $K_{O,p}\in L^{\frac{n-1}{p-n+1}}_{\rm loc}$ belong to $\mathrm{BV}^{n-\alpha}$, $\alpha=p/(p-n+1)$.

Theorem 6.1. Let $f:D\to\mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p}\in L^{\frac{n-1}{p-n+1}}(D)$, $p\in[n,n+1/(n-2))$. Then $f\in\mathrm{BV}^{n-\alpha}(D)$, $\alpha=p/(p-n+1)$.

Proof. Due to Theorem 4.4, f is a ring $\widetilde{Q}$-homeomorphism in D with $\widetilde{Q}(x)=K^{\alpha-1}_{O,p}(x,f)$, where $\alpha=p/(p-n+1)$. Since for $\eta(r)=1/(r_2-r_1)$, condition (3.4) holds, we have for $r_1=2r$ and $r_2=4r$,

$$\mathcal{M}_\alpha\left(\Delta\left(f(S_1),f(S_2);f(D)\right)\right)\leq(2r)^{-\alpha}\int\limits_{\mathbb{A}}K^{\alpha-1}_{O,p}(x,f)\,dm(x)\,. \tag{6.1}$$

On the other hand, by estimate (15) in [10], one gets the following lower bound for the above modulus (or, equivalently, for the α-capacity of condenser)

$$\mathcal{M}_\alpha\left(\Delta\left(f(S_1),f(S_2);f(D)\right)\right)\geq C_1\left[mf(B_{2r})\right]^{\frac{n-\alpha}{n}}\,, \tag{6.2}$$

where C_1 is a positive constant depending only on n and α. Here and throughout the proof, we denote (by simplicity) by B_ε a ball in $\mathbb{R}^n$ of radius ε. Combining both (6.1)–(6.2), we reach the estimate for the image of B_{2r},

$$mf(B_{2r})\leq C_2r^{\frac{\alpha n}{\alpha-n}}\left(\int\limits_{B_{4r}}K^{\alpha-1}_{O,p}(x,f)\,dm(x)\right)^{\frac{n}{n-\alpha}}\,; \tag{6.3}$$

here $C_2=C_2(n,\alpha)>0$.

Now we apply the following double inequality letting $r_1=r$ and $r_2=2r$,

$$\begin{aligned}C_3\left[\frac{(\operatorname{diam}f(B_r))^\alpha}{(m\,f(B_{2r}))^{1-n+\alpha}}\right]^{1/(n-1)}&\leq\mathcal{M}_\alpha\left(\Delta\left(f(S_1),f(S_2);f(D)\right)\right)\\&\leq r^{-\alpha}\int\limits_{\mathbb{A}}K^{\alpha-1}_{O,p}(x,f)\,dm(x)\end{aligned}$$

with a positive constant C_3 depending only on n and α; cf. (18) in [10]. This derives the following upper bound

$$\operatorname{diam}f(B_r)\leq C_4\left[m\,f(B_{2r})\right]^{\frac{\alpha-n+1}{\alpha}}r^{1-n}\left(\int\limits_{B_{2r}}K^{\alpha-1}_{O,p}(x,f)\,dm(x)\right)^{\frac{n-1}{\alpha}}\,,$$

which together with (6.3) gives

$$\operatorname{diam}f(B_r)\leq C_5r^{\frac{\alpha}{\alpha-n}}\left(\int\limits_{B_{4r}}K^{\alpha-1}_{O,p}(x,f)\,dm(x)\right)^{\frac{1}{n-\alpha}}\,;$$

$C_5 = C_5(n, \alpha) > 0$. Rewriting the last inequality as

$$[\operatorname{diam} f(B_r)]^{n-\alpha}\, r^{\alpha} \;\le\; C_6 \int\limits_{B_{4r}} K_{O,p}^{\alpha-1}(x, f)\, dm(x)\,,$$

and summarizing over each disjoint system of balls $\{B(x_j, 4r_j)\}$, we complete the proof. □

Taking into account our remark on differentiability a.e. of homeomorphisms of BV^p, $0 < p \le n$, we obtain an alternative proof of the corresponding part of Theorem 4.1.

Corollary 6.2. Let $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n + 1/(n-2))$. Then f is differentiable a.e. in D.

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