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The Fekete–Szegö problem for spirallike mappings and non-linear resolvents in Banach spaces

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. We study the Fekete–Szegö problem on the open unit ball of a complex Banach space. Namely, the Fekete–Szegö inequalities are proved for the class of spirallike mappings relative to an arbitrary strongly accretive operator, and some of its subclasses. Next, we consider families of non-linear resolvents for holomorphically accretive mappings vanishing at the origin. We solve the Fekete–Szegö problem over these families.

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1. Introduction

Let X be a complex Banach space equipped with the norm $\|\cdot\|$ and let X^* be the dual space of X. We denote by $\mathbb{B}$ the open unit ball in X. For each $x \in X \setminus \{0\}$, denote

$$T(x) = \{\ell_x \in X^* : \|\ell_x\| = 1 \text{ and } \ell_x(x) = \|x\|\} . \tag{1.1}$$

According to the Hahn–Banach theorem (see, for example, [25, Theorem 3.2]), $T(x)$ is nonempty and may consists of a singleton (for instance, in the case of Hilbert space), or, otherwise, of infinitely many elements. Its elements $\ell_x \in T(x)$ are called support functionals at the point x.

Let Y be a Banach space (possibly, different from X). The set of all holomorphic mappings from $\mathbb{B}$ into Y will be denoted by $\mathrm{Hol}(\mathbb{B}, Y)$. It is well known (see, for

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example, [20, 9, 15, 24]) that if $f \in \mathrm{Hol}(\mathbb{B}, Y)$, then for every $x_0 \in \mathbb{B}$ and all x in some neighborhood of $x_0 \in \mathbb{B}$, the mapping f admits the Taylor series representation:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x_0) \left[(x - x_0)^n\right], \tag{1.2}$$

where $D^n f(x_0) : \prod_{k=1}^{n} X \to Y$ is a bounded symmetric n-linear operator that is called the n-th Fréchet derivative of f at x_0. Also we write $D^n f(x_0) \left[(x - x_0)^n\right]$ for $D^n f(x_0)[x-x_0, \ldots, x-x_0]$. One says that f is normalized if $f(0) = 0$ and $Df(0) = \mathrm{Id}$, the identity operator on X.

Recall that a holomorphic mapping $f : \mathbb{B} \to X$ is called biholomorphic if the inverse f^{-1} exists and is holomorphic on the image $f(\mathbb{B})$. A mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ is said to be locally biholomorphic if for each $x \in \mathbb{B}$ there exists a bounded inverse for the Fréchet derivative $Df(x)$, see [9, 15].

In the one-dimensional case, where $X = \mathbb{C}$ and $\mathbb{B} = \mathbb{D}$ is the open unit disk in $\mathbb{C}$, one usually writes $a_n(x - x_0)^n$ instead of $\frac{1}{n!} D^n f(x) \left[(x - x_0)^n\right]$ in (1.2). The classical Fekete–Szegö problem [12] for a given subclass $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is to find

$$\sup_{f \in \mathcal{F}} \left| a_3 - \nu a_2^2 \right|, \quad \text{where } \ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots .$$

In multi-dimensional settings various analogs of the classical Fekete–Szegö problem for different classes of holomorphic mappings have been studied by many mathematicians. Nice survey of the current state of the art and references can be found in [19] and [22].

H. Hamada, G. Kohr and M. Kohr in [19] introduced a new quadratic functional that generalizes the Fekete–Szegö functional to infinite-dimensional settings. Moreover, they estimated this functional over several classes of holomorphic mappings, including starlike mappings and non-linear resolvents of normalized holomorphically accretive mappings.

The aim of this paper is to extend the method used in [19] and solve the Fekete–Szegö problem over the classes of spirallike mappings and resolvents of non-normalized holomorphically accretive mappings. Along the way we generalize some results in [19] and [6].

Spirallike mappings in Banach spaces were first introduced and studied in the mid 1970's by K. Gurganus and T. J. Suffridge. This study has evolved into a coherent theory thanks to the influential contributions of Gabriela Kohr and her co-authors (I. Graham, H. Hamada, M. Kohr and others) over the past decades (some details can be found below). As for non-linear resolvents, they seem to have been among the last issues that caught her attention. Progress on this topic is reflected in [13, 19].

2. Preliminaries

Recall that for a densely defined linear operator A with the domain $D_A \subset X$, the set $V(A) = \{\ell_x(Ax) : x \in D_A, \|x\| = 1, \ \ell_x \in T(x)\}$ is called the numerical range of A.

Definition 2.1. Let $A \in L(X)$ be a bounded linear operator on X. Then A is called accretive if

$$\Re \ell_x(Ax) \geq 0$$

for all $x \in X \setminus \{0\}$, or, what is the same, if $m(A) \geq 0$, where

$$m(A) := \inf \{\Re \lambda : \lambda \in V(A)\}.$$

If for some $k > 0$,

$$\Re \ell_x(Ax) \geq k\|x\|$$

for all $x \in X \setminus \{0\}$, the operator A is called strongly accretive.

The notion of accretivity was extended by Harris [20] to involve holomorphic mappings (see also [24, 9]).

Definition 2.2. Let $h \in \mathrm{Hol}(\mathbb{B}, X)$. This mapping h is said to be holomorphically accretive if

$$m(h) := \liminf_{s \to 1^-} \big(\inf \{\Re \ell_x(h(sx)) : \ \|x\| = 1, \ \ell_x \in T(x)\}\big) \geq 0.$$

In the case where the last lower limit $m(h)$ is positive, h is called strongly holomorphically accretive.

Remark 2.3. According to [9, Proposition 2.3.2] if $h(0) = 0$ then $V(A) \subset \overline{\mathrm{conv}}\, V(h)$, where $A = Dh(0)$, in particular, $m(A) \geq m(h)$. Consequently, if h is holomorphically accretive, its linear part at zero A is accretive too. Furthermore, for such mappings Proposition 2.5.4 in [9] implies that h is holomorphically accretive if and only if $\Re \ell_x(h(x)) \geq 0$ for all $x \in \mathbb{B} \setminus \{0\}$.

The main feature of the class of holomorphically accretive mappings is that they generate semigroups of holomorphic self-mappings on $\mathbb{B}$, so they are of most importance in dynamical systems [24, 9]. A very fruitful characterization of holomorphically accretive mappings is:

Proposition 2.4 (Theorem 7.3 in [24]**, see also** [9]**).** *A mapping* $h \in \mathrm{Hol}(\mathbb{B}, X)$ *is holomorphically accretive if and only if it satisfies the so-called range condition (RC), that is,* $(\mathrm{Id} + rh)(\mathbb{B}) \supseteq \mathbb{B}$ *for each* $r > 0$*, and the inverse mapping* $J_r := (\mathrm{Id} + rh)^{-1}$ *is a well-defined holomorphic self-mapping of* $\mathbb{B}$.

The mapping J_r that occurs in this proposition is called the *non-linear resolvent of* h. In other words, the non-linear resolvent is the unique solution $w = J_r(x) \in \mathbb{B}$ of the functional equation

$$w + rh(w) = x \in \mathbb{B}, \quad r > 0.$$

Assuming $h(0) = 0$, one sees that $J_r(0) = 0$ for all $r > 0$.
If, in addition, $A = Dh(0)$, then $DJ_r(0) = (\mathrm{Id} + rA)^{-1}$. Furthermore, the accretivity of A mentioned in Remark 2.3, implies $DJ_r(0)$ is strongly contractive because $\|(\mathrm{Id} + rA)^{-1}\| < 1$.

We use the following classes (see [15] and references therein):

$$\begin{aligned}
\mathcal{N} &= \{h \in \mathrm{Hol}(\mathbb{B}, X) \colon h(0) = 0, \Re \ell_x(h(x)) > 0, \ x \in \mathbb{B} \setminus \{0\}, \ell_x \in T(x)\}, \\
\mathcal{M} &= \{h \in \mathcal{N}, Dh(0) = \mathrm{Id}\}
\end{aligned}$$

and (see [14])

$$\mathcal{N}_A \quad := \quad \{h \in \mathcal{N} : \ Dh(0) = A\}\,. \tag{2.1}$$

To proceed, we note that the inclusion $h \in \mathcal{N}$ can be expressed as $\ell_x(h(x)) \in g_0(\mathbb{D})$, $x \in \mathbb{B}\backslash\{0\}$, where $g_0(z) = \frac{1+z}{1-z}$. At the same time, $\overline{V(A)}$ is a compact subset of the open right half-plane, hence the inclusion $\ell_x(h(x)) \in g_0(\mathbb{D})$ is imprecise. It can be improved by using other functions $g \prec g_0$, bearing in mind that $g(\mathbb{D})$ should contain $V(A)$ by Remark 2.3.

Throughout this paper we suppose that the following conditions hold

Assumption 1. A linear operator A is bounded and strongly accretive. A function $g = g_A \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfies $g \prec g_0$ and $\overline{V(A)} \subset g(\mathbb{D})$. Therefore $\Delta := g^{-1}(V(A))$ is compactly embedded in $\mathbb{D}$.

Definition 2.5 (cf. [2, 27]**).** Let A and g satisfy Assumption 1. Denote

$$\mathcal{N}_A(g) := \left\{ h \in \mathcal{N}_A : \frac{\ell_x(h(x))}{\|x\|} \in g(\mathbb{D}), x \in \mathbb{B}\backslash\{0\}, \ell_x \in T(x) \right\}. \tag{2.2}$$

We now consider specific choices of g providing some properties of semigroups generated by $h \in \mathcal{N}_A(g)$:

(a) $g_1^\alpha(z) := \left(\frac{1+z}{1-z}\right)^\alpha, \alpha \in (0,1)$: It can be shown that the semigroup generated by every $h \in \mathcal{N}_A(g_1^\alpha)$ can be analytically extended with respect to parameter t to the sector $|\arg t| < \frac{\pi(1-\alpha)}{2}$; for the one-dimensional case see [11];

(b) $g_2^\alpha(z) := \alpha + (1-\alpha)\frac{1+z}{1-z}, \alpha \in (0, m(A))$: it follows from Lemma 3.3.2 in [8] that the semigroup $\{u(t,x)\}_{t\geq 0}$ generated by any element of $\mathcal{N}_A(g_2^\alpha)$ satisfies the estimate $\|u(t,x)\| \leq e^{-t\alpha}\|x\|$ uniformly on the whole $\mathbb{B}$;

(c) $g_3^\alpha(z) := \frac{1-z}{1-(2\alpha-1)z}, \alpha \in (0,1)$, maps $\mathbb{D}$ onto a disk Δ tangent the imaginary axis. In a sense this choice is dual to the previous one (in the one-dimensional case such duality was investigated in [1]);

In what follows we will refer to these functions as $g_0, g_1^\alpha, g_2^\alpha, g_3^\alpha$.

Another area where holomorphically accretive mappings are widely used is geometric function theory. The study of spirallike mappings is a good example of this fruitful connection.

Definition 2.6 (see [26, 15, 8, 24]**).** Let A be a strongly accretive operator. A biholomorphic mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ is said to be spirallike relative to A if its image is invariant under the action of the semigroup $\{e^{-tA}\}_{t\geq 0}$, that is, $e^{-tA}f(x) \in f(\mathbb{B})$ for all $t \geq 0$ and $x \in \mathbb{B}$. The set of all spirallike mappings relative to A is denoted by $\widehat{S}_A(\mathbb{B})$.

If f is spirallike relative to $A = e^{-i\beta}\,\mathrm{Id}$ for some $|\beta| < \frac{\pi}{2}$, then f is said to be spirallike of type β. In the particular case where $\beta = 0$, spirallike mappings relative $A = \mathrm{Id}$ are called starlike.

The following result is well known (see, for example, Proposition 2.5.3 in [8] and references therein).

Proposition 2.7. *Let $A \in L(X)$ be strongly accretive, and let $f \in \operatorname{Hol}(\mathbb{B}, X)$ be a normalized and locally biholomorphic mapping. Then $f \in \widehat{S}_A(\mathbb{B})$ if and only if the mapping $h := (Df)^{-1}Af$ belongs to $\mathcal{N}_A$.*

This proposition *inter alia* implies that a spirallike mapping f relative to A linearizes the semigroup $u(t,x)$ generated by $h = (Df)^{-1}Af$ in the sense that $f \circ u(t, f^{-1}(x)) = e^{-tA}x$ on $f(\mathbb{B})$. In the one-dimensional case, any linear operator is scalar, hence can be chosen to be $A = e^{i\beta}\operatorname{Id}$. In this case the inclusion $h = (Df)^{-1}Af \in \mathcal{N}_A$ is equivalent to $\Re\left(e^{-i\beta}\frac{zf'(z)}{f(z)}\right) > 0$. This is the standard definition of spirallike functions of type β on $\mathbb{D}$ (see, for example, [5, 15]).

Moreover, according to Proposition 2.7, it is relevant to consider biholomorphic functions $g \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ satisfying Assumption 1 and to distinguish subclasses of $\widehat{S}_A(\mathbb{B})$ letting

$$\widehat{S}_g(\mathbb{B}) := \left\{ f \in \widehat{S}_A(\mathbb{B}) : (Df)^{-1}Af \in \mathcal{N}_A(g) \right\}. \tag{2.3}$$

In particular, $\widehat{S}_{g_0}(\mathbb{B}) = \widehat{S}_A(\mathbb{B})$. Further, $\widehat{S}_{g_1^\alpha}(\mathbb{B})$ consists of mappings that are spirallike relative to operator $e^{i\beta}A$ with any $|\beta| < 1-\alpha$. The classes $\widehat{S}_{g_2^\alpha}(\mathbb{B})$ and $\widehat{S}_{g_3^\alpha}(\mathbb{B})$ are also of specific interest. For instance, if $A = e^{i\beta}\operatorname{Id}$ and $\alpha = \lambda\cos\beta$, the class $\widehat{S}_{g_3^\alpha}(\mathbb{B})$ of spirallike mappings of type β of order λ is a widely studied object. The intersection $\widehat{S}_{g_2^\alpha}(\mathbb{B}) \bigcap \widehat{S}_{g_3^\alpha}(\mathbb{B})$ consists of strongly spirallike mappings (for an equivalent definition and properties of these mappings see [17, 18, 3]).

3. Auxiliary lemmata

Our first auxiliary result essentially coincides with Theorem 2.12 in [19]. We present it in a somewhat more general form.

Lemma 3.1. *Let $p(z) = a + p_1z + p_2z^2 + o(z^2)$ and $\phi(z) = a + b_1z + b_2z^2 + o(z^2)$ be holomorphic functions on $\mathbb{D}$ such that $\phi \prec p$. Then for every $\mu \in \mathbb{C}$ the following sharp inequality holds:*

$$|b_2 - \mu b_1^2| \le \max\left(|p_1|, |p_2 - \mu p_1^2|\right).$$

Proof. Since $\phi \prec p$, there is a function $\omega \in \Omega$ such that $\phi = p \circ \omega$.
Let $\omega(z) = c_1z + c_2z^2 + o(z^2)$. Then

$$b_1 = p_1c_1 \quad \text{and} \quad b_2 = p_2c_1^2 + p_1c_2.$$

Therefore

$$b_2 - \mu b_1^2 = (p_2 - \mu p_1^2)c_1^2 + p_1c_2.$$

Because the inequality $|c_2| \le 1 - |c_1|^2$ holds and is sharp (see, for example, [5]), one concludes that $|b_2 - \mu b_1^2|$ is bounded by a convex hull of $|p_1|$ and $|p_2 - \mu p_1^2|$. The result follows. $\square$

Lemma 3.2. *Let $h \in \mathrm{Hol}(\mathbb{B}, X)$ with $h(0) = 0$ and $B \in L(X)$ with $\rho := \|B\| \le 1$. For any $x \in \partial\mathbb{B}$ and $\ell \in X^*$ denote*

$$\varphi(t) := \frac{\ell\left(h(tBx)\right)}{t}, \qquad t \in \mathbb{D} \setminus \{0\}.$$

(i) *The function φ can be analytically extended to the disk $\frac{1}{\rho}\mathbb{D}$ with the Taylor expansion $\varphi(t) = b_0 + b_1 t + b_2 t^2 + o(t^2)$, where $b_0 = \ell(Dh(0)Bx)$,*

$$b_1 = \frac{1}{2!}\,\ell\left(D^2h(0)[(Bx)^2]\right) \quad \text{and} \quad b_2 = \frac{1}{3!}\,\ell\left(D^3h(0)[(Bx)^3]\right). \tag{3.1}$$

(ii) *If, in addition, $\ell \in T(Bx)$ and $h \in \mathcal{N}_A(g)$, then $\varphi(\mathbb{D}) \subset \rho\widehat{g}(\rho\mathbb{D})$, where*

$$\widehat{g}(t) = g\left(\frac{\tau - t}{1 - t\overline{\tau}}\right) \text{ and } \tau = g^{-1}\left(\frac{\ell(Dh(0)Bx)}{\|Bx\|}\right).$$

Proof. The function φ is holomorphic whenever $\|tBx\| < 1$, that is, for $|t| < \frac{1}{\rho} \le \frac{1}{\|Bx\|}$. Represent h by the Taylor series (1.2). A straightforward calculation proves (i). Recall that $h \in \mathcal{N}_A(g)$, hence Definition 2.5 implies $\frac{\varphi(t)}{\|Bx\|} \in g(\mathbb{D}) = \widehat{g}(\mathbb{D})$ as $|t| < \frac{1}{\rho}$. Therefore the function $\widehat{g}^{-1}(\frac{\varphi(\cdot)}{\|Bx\|})$ maps the disk of radius $\frac{1}{\rho}$ into $\mathbb{D}$ and preserves zero. By the Schwarz Lemma $\widehat{g}^{-1}(\frac{\varphi(t)}{\|Bx\|}) \le \rho|t|$. Thus $\varphi \prec \|Bx\|\widehat{g}(\rho\,\cdot)$. The proof is complete. □

A mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ is said to be of one-dimensional type if it takes the form $f(x) = s(x)x$ for some $s \in \mathrm{Hol}(\mathbb{B}, \mathbb{C})$. Such mappings were studied by many authors (see, for example, [23, 10, 4] and references therein).

Lemma 3.3. *Let $f \in \mathrm{Hol}(\mathbb{B}, X)$ be a mapping of one-dimensional type. Then for every $n \in \mathbb{N}$ the entire mapping $x \mapsto D^n f(0)[x^n]$ is also of one-dimensional type. Therefore for any $x \in \partial\mathbb{B}$, $\ell_x \in T(x)$ and constants $\mu_j \in \mathbb{C}$, $j = 1, 2, \ldots$, we have*

$$\left|\ell_x\left(\sum_{j=1}^{n} \mu_j D^j f(0)[x^j]\right)\right| = \left\|\sum_{j=1}^{n} \mu_j D^n f(0)[x^j]\right\|.$$

Proof. The first assertion is evident (for detailed calculation see [7]). To prove the second one we note that there is a function $F \in \mathrm{Hol}(X, \mathbb{C})$ such that

$$\sum_{j=1}^{n} \mu_j D^j f0)[x^j] = F(x)x.$$

Thus for any $x \in \partial\mathbb{B}$ we have

$$\left\|\sum_{j=1}^{n} \mu_j D^j f(0)[x^j]\right\| = |F(x)|\|x\| \qquad \text{and}$$

$$\ell_x\left(\sum_{j=1}^{n} \mu_j D^j f(0)[x^j]\right) = F(x)\ell_x(x) = F(x),$$

which completes the proof. □

4. Fekete–Szegö inequalities for spirallike mappings

In what follows A and g satisfy Assumption 1, and the class $\widehat{S}_g(\mathbb{B})$ is defined by formula (2.3).

Theorem 4.1. *Let $x \in \partial\mathbb{B}$, $\ell_x \in T(x)$ and $\tau = g^{-1}(\ell_x(Ax))$. Assume that*

$$g\left(\frac{\tau - z}{1 - z\overline{\tau}}\right) = q_0 + q_1 z + q_2 z^2 + o(z^2).$$

Given $f \in \mathrm{Hol}(\mathbb{B}, X)$ denote

$$\begin{aligned}
\widetilde{a}_2^2 &= \frac{1}{2}\ell_x\left(D^2 f(0)\left[x, D^2 f(0)[x, Ax]\right] - \frac{1}{2}D^2 f(0)\left[x, AD^2 f(0)[x^2]\right]\right), \\
a_2 &= \frac{1}{2!}\ell_x\left(2D^2 f(0)[x, Ax] - AD^2 f(0)[x^2]\right), \qquad (4.1)\\
a_3 &= \frac{1}{2\cdot 3!}\ell_x\left(3D^3 f(0)[x^2, Ax] - AD^3 f(0)[x^3]\right).
\end{aligned}$$

If $f \in \widehat{S}_g(\mathbb{B})$, then for any $\nu \in \mathbb{C}$ we have

$$\left|a_3 - (\nu - 1)a_2^2 - \widetilde{a}_2^2\right| \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\nu - 1)q_1\right|\right\}. \qquad (4.2)$$

Remark 4.2. It can be directly calculated that $q_1 = -g'(\tau)(1 - |\tau|^2)$ and

$$\frac{q_2}{q_1} = \overline{\tau} - \frac{g''(\tau)}{2g'(\tau)}(1 - |\tau|^2).$$

Thus the right-hand side in (4.2) can be expressed by the hyperbolic and pre-Schwarzian derivatives of g.

Proof. Let $h(x) = [Df(x)]^{-1} Af(x)$. Recall that f is a normalized biholomorphic mapping. Let the Taylor expansion of f be

$$f(x) = x + \frac{1}{2!}D^2 f(0)[x^2] + \frac{1}{3!}D^3 f(0)[x^3] + o(\|x\|^3), \qquad (4.3)$$

so that

$$Df(x)[w] = w + D^2 f(0)[x, w] + \frac{1}{2}D^3 f(0)[x^2, w] + o(\|x\|^2). \qquad (4.4)$$

Take the Taylor expansion $h(z) = Ax + \frac{1}{2}D^2 h(0)[x^2] + \frac{1}{6}D^3 h(0)[x^3] + o(\|x\|^3)$ and substitute it together with (4.3)–(4.4) into the equality

$$Df(x)[h(x)] = Af(x).$$

This gives us

$$\begin{aligned}
& Ax + \frac{1}{2}D^2 h(0)[x^2] + \frac{1}{6}D^3 h(0)[x^3] + D^2 f(0)[x, Ax] \\
+ \;& \frac{1}{2}D^2 f(0)[x, D^2 h(0)x^2] + \frac{1}{2}D^3 f(0)[x^2, Ax] + o(\|x\|^3) \\
= \;& Ax + \frac{1}{2}AD^2 f(0)[x^2] + \frac{1}{6}AD^3 f(0)[x^3] + o(\|x\|^3).
\end{aligned}$$

Equating terms of the same order leads to

$$\frac{1}{2}D^2h(0)[x^2] + D^2f(0)[x, Ax] = \frac{1}{2}AD^2f(0)[x^2]$$

and

$$\frac{1}{6}D^3h(0)[x^3] + \frac{1}{2}D^2f(0)[x, D^2h(0)x^2] + \frac{1}{2}D^3f(0)[x^2, Ax] = \frac{1}{6}AD^3f(0)[x^3].$$

In turn, these equalities imply

$$D^2h(0)[x^2] = AD^2f(0)[x^2] - 2D^2f(0)[x, Ax]$$

and

$$\begin{aligned}&D^3h(0)[x^3] = AD^3f(0)[x^3] - 3D^2f(0)[x, D^2h(0)x^2] - 3D^3f(0)[x^2, Ax]\\&= AD^3f(0)[x^3] - 3D^3f(0)[x^2, Ax]\\&-3D^2f(0)\left[x, AD^2f(0)[x^2]\right] + 6D^2f(0)\left[x, D^2f(0)[x, Ax]\right].\end{aligned}$$

Recall that $\ell_x(Ax) \in V(A) \subset g(\mathbb{D})$, so $\tau \in \Delta$ is well-defined. Similarly to the proof of the Theorem 3.1 in [19], denote

$$\varphi(t) = \begin{cases} \frac{\ell_x(h(tx))}{t}, & t \in \mathbb{D} \setminus \{0\},\\ \ell_x(Ax), & t = 0.\end{cases}$$

Then $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ by assertion (i) of Lemma 3.2 with $B = \mathrm{Id}$,

$$b_1 = \frac{1}{2!}\ell_x\left(D^2h(0)[x^2]\right) \quad \text{and} \quad b_2 = \frac{1}{3!}\ell_x\left(D^3h(0)[x^3]\right).$$

Using a_2, $\widetilde{a}_2^2$ and a_3 defined in (4.1) we get

$$b_1 = -a_2 \quad \text{and} \quad b_2 = 2\widetilde{a}_2^2 - 2a_3.$$

Therefore,

$$\left|a_3 - \widetilde{a}_2^2 - (\nu-1)a_2^2\right| = \frac{1}{2}\left|b_2 - 2(1-\nu)b_1^2\right|.$$

Also, by assertion (ii) of the same Lemma 3.2, $\varphi \prec \widehat{g}$, $\widehat{g}(t) = g(\frac{\tau - t}{1-\overline{\tau}t})$.
To this end we apply Lemma 3.1 with $p = \widehat{g}$ and $\mu = 2(1-\nu)$ and obtain estimate (4.2). □

There are two ways to make the above result more explicit: to fix some concrete forms of the function g, or to put additional restrictions on the mapping f. We start with some concrete choices of g.

Recall that for every strongly accretive operator A and every spirallike mapping f relative to A, the mapping $h := (Df)^{-1}Af$ is holomorphically accretive. Hence one can always choose $g = g_0$, where $g_0(z) = \frac{1+z}{1-z}$ is defined above. Denoting $\ell := \ell_x(Ax)$ and using Remark 4.2, we conclude that every spirallike mapping relative to A satisfies

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le \Re\ell \cdot \max\left(1, |1 + 4(\nu-1)\Re\ell|\right). \tag{4.5}$$

In the one-dimensional case, this inequality coincides with the result of Theorem 1 in [21] for $\lambda = 0$. By choosing other $g \prec g_0$ functions and denoting $\ell := \ell_x(Ax)$ as above, more precise estimates can be obtained.

Assume, for example, that $\ell_x(h(x))$ belongs to some sector of the form $\left\{w : |\arg w| < \frac{\pi\alpha}{2}\right\}$, $\alpha \in (0,1)$, for all $x \in \mathbb{B}$, where $h = (Df)^{-1} Af$. Then one can set $g = g_1^\alpha$ and to get

Corollary 4.3. *Every $f \in \widehat{S}_{g_1^\alpha}(\mathbb{B})$ satisfies*

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le \alpha|\ell| \cos\arg \ell^{\frac{1}{\alpha}} \cdot \max\left\{1, Q_{1,\alpha}\right\},$$

where $Q_{1,\alpha} = \Re \ell^{\frac{1}{\alpha}} \left| 4\alpha(\nu-1)\ell^{\frac{\alpha-1}{\alpha}} + \frac{1}{\ell^{\frac{1}{\alpha}}}\left(\alpha + i\tan\arg \ell^{\frac{1}{\alpha}}\right)\right|$.

Also assuming that $\frac{\ell_x(h(x))}{\|x\|}$ is bounded away from the imaginary axis, namely,

$$\Re\frac{\ell_x(h(x))}{\|x\|} > \alpha,\ \ \alpha \in (0,1),$$

we choose $g = g_2^\alpha$. In this situation, we have

Corollary 4.4. *Every $f \in \widehat{S}_{g_2^\alpha}(\mathbb{B})$ satisfies*

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le \Re\ell \cdot \max\left\{1, Q_{2,\alpha}\right\},$$

where $Q_{2,\alpha} = |1 + 4(\nu-1)(1-\alpha)\Re\ell|$.

In particular, taking $\alpha = 0$, we return to inequality (4.5) for all spirallike mappings relative to the linear operator A.

Another interesting (and, as we mentioned, dual) case occurs when $\frac{\ell_x(h(x))}{\|x\|}$ lies in some circle tangent to the imaginary axis. We can then set $g = g_3^\alpha$.

Corollary 4.5. *Every $f \in \widehat{S}_{g_3^\alpha}(\mathbb{B})$ satisfies*

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le (\Re\ell - |\ell|^2\alpha) \cdot \max\left\{1, Q_{3,\alpha}\right\},$$

where $Q_{3,\alpha} = \left|1 - 2\overline{\ell}\alpha + 4(\nu-1)(\Re\ell - |\ell|^2\alpha)\right|$.

Recall that for $A = e^{i\beta}\,\mathrm{Id}$, the class $\widehat{S}_{g_3^\alpha}(\mathbb{B})$ consists of so-called spirallike mappings of type β of order α.

Remark 4.6. It is worth mentioning that even for the the case in which A is a scalar operator, the estimates above (starting from (4.5)) are new. Since the class of spirallike mappings contains the class of starlike mappings, these estimates generalize Corollary 3.4 (i)–(iv) in [19] for starlike mappings.

In the rest of this section we deal with mappings f that satisfy:

Assumption 2. There exists a function $\kappa : \partial\mathbb{B} \to \mathbb{C}$ such that

$$D^2 f(0)[x^2] = \kappa(x)x, \quad x \in \partial\mathbb{B}. \tag{4.6}$$

The Fréchet derivatives of f of second and third order $D^2 f(0)$ and $D^3 f(0)$ commute with the linear operator A in the sense that

$$D^k f(0)[x^{k-1}, Ax] = AD^k f(0)[x^k], \quad k = 2, 3. \tag{4.7}$$

Condition (4.6) holds automatically for one-dimensional type mappings (spirallike mappings of one-dimensional type were studied, for instance, in [10, 22, 7]), while condition (4.7) holds automatically whenever A is a scalar operator.

In turn, relations (4.7) in Assumption 2 imply that formulae (4.1) become

$$\begin{aligned} a_2 &= \frac{1}{2!}\ell_x\left(AD^2f(0)[x^2]\right), \\ \widetilde{a}_2^2 &= \frac{1}{4}\ell_x\left(AD^2f(0)[x,D^2f(0)[x^2]]\right), \\ a_3 &= \frac{1}{3!}\ell_x\left(AD^3f(0)[x^3]\right). \end{aligned} \tag{4.8}$$

Corollary 4.7. *If $f \in \widehat{S}_A(\mathbb{B})$ satisfies Assumption 2, then for any $\nu \in \mathbb{C}$,*

$$\left|a_3 - \left(\nu - 1 + \frac{1}{\ell_x\left(Ax\right)}\right)a_2^2\right| \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\nu-1)q_1\right|\right\}. \tag{4.9}$$

Proof. Indeed, denote $\alpha = \ell_x\left(Ax\right)$.Then $a_2 = \frac{1}{2}\kappa(x)\alpha$ and

$$\begin{aligned} \widetilde{a}_2^2 &= \frac{1}{4}\ell_x\left(AD^2f(0)[x,\kappa(x)x]\right) = \frac{1}{4}\cdot\kappa(x)\ell_x\left(AD^2f(0)[x^2]\right) \\ &= \frac{1}{4}\cdot\kappa(x)\ell_x\left(A\kappa(x)x\right) = \frac{\alpha}{4}\cdot(\kappa(x))^2. \end{aligned}$$

Thus $\widetilde{a}_2^2 = \frac{1}{\alpha}a_2^2$ and hence

$$|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2| = \left|a_3 - \left(\nu - 1 + \frac{1}{\alpha}\right)a_2^2\right|.$$

So, estimate (4.9) follows from Theorem 4.1. $\square$

Let A be a scalar operator. Without loss of generality, we assume $A = e^{i\beta}\,\mathrm{Id},\ |\beta| < \frac{\pi}{2}$. Then it follows from Assumption 2 that formulae (4.1) (or (4.8)) become

$$a_2 = \frac{1}{2!}\kappa(x)e^{i\beta} \quad \widetilde{a}_2^2 = \left(\frac{1}{2!}\kappa(x)\right)^2 e^{i\beta}, \quad a_3 = \frac{1}{3!}\ell_x\left(D^3f(0)[x^3]\right)e^{i\beta}.$$

These relations and Lemma 3.3 imply immediately

Corollary 4.8. *If $f \in \mathrm{Hol}(\mathbb{B}, X)$ is a spirallike mapping of type β, that satisfies Assumption 2. Then for any $\mu \in \mathbb{C}$ we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\mu - e^{-i\beta})q_1\right|\right\}.$$

If, in addition, f is of one-dimensional type, then for any $x \in \partial\mathbb{B}$ we have

$$\begin{aligned} &\left\|\frac{1}{3!}D^3f(0)[x^3] - \mu\cdot\frac{1}{2!}D^2f(0)\left[x, \frac{1}{2!}D^2f(0)[x^2]\right]\right\| \\ &\qquad \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\mu - e^{-i\beta})q_1\right|\right\}. \end{aligned}$$

The last estimate coincides with Theorem 2 in [7].

5. Fekete–Szegö inequalities for normalized non-linear resolvents

As above, we suppose that $A \in L(X)$ and $g \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfy Assumption 1 and $h \in \mathcal{N}_A(g)$. In this section we concentrate on the non-linear resolvent

$$J_r := (\mathrm{Id} + rh)^{-1}, \ r > 0,$$

that is well-defined self-mappings of the open unit ball $\mathbb{B}$ that solves the functional equation

$$J_r(x) + rh(J_r(x)) = x \in \mathbb{B}, \quad r > 0. \tag{5.1}$$

Lemma 5.1.

(a) *For any $r > 0$, the operator $B_r := DJ_r(0) = (\mathrm{Id} + rA)^{-1}$ is strongly contractive, that is, $\rho_r := \|B_r\| < 1$.*

(b) *If h is of one-dimensional type, then A is a scalar operator and J_r, $r > 0$, is of one-dimensional type too.*

Proof. Assertion (a) follows from the strong accretivity of A.
Since h is of one-dimensional type, it has the form $h(x) = s(x)x$, where $s \in \mathrm{Hol}(\mathbb{B}, \mathbb{C})$. Therefore $A = Dh(0) = s(0)\,\mathrm{Id}$. In addition, (5.1) implies

$$x = J_r(x) + rs(J_r(x))J_r(x) = (1 + rs(J_r(x)))J_r(x),$$

that is, $J_r(x)$ is collinear to x. □

Further, it is natural to consider the family of normalized resolvents $(\mathrm{Id} + rA)J_r$ and to study the Fekete–Szegö problem for these mappings.

We now present the main result of this section.

Theorem 5.2. *Let $h \in \mathcal{N}_A(g)$ and J_r be the nonlinear resolvent of h for some $r > 0$. For $x \in \partial\mathbb{B}$ and $\ell_r := \ell_{B_r x} \in T(B_r x)$, let*

$$\begin{aligned}
\widetilde{a}_2^2 \ &:= \ \ell_r \left((\mathrm{Id} + rA)\frac{1}{2!}D^2 J_r(0) \left[x, (\mathrm{Id} + rA)\frac{1}{2!}D^2 J_r(0)[x^2] \right] \right), \\
a_2 \ &:= \ \ell_r \left((\mathrm{Id} + rA)\frac{1}{2!}D^2 J_r(0)[x^2] \right), \\
a_3 \ &:= \ \ell_r \left((\mathrm{Id} + rA)\frac{1}{3!}D^3 J_r(0)[x^3] \right).
\end{aligned} \tag{5.2}$$

Then for any $\nu \in \mathbb{C}$ we have

$$\left| a_3 - 2\widetilde{a}_2^2 - (\nu - 2)a_2^2 \right| \le r|q_1|\|B_r x\|\rho_r^2 \max\left(1, Q_r(x)\right), \tag{5.3}$$

where

$$Q_r(x) := \left| \frac{q_2}{q_1} - (2 - \nu) r q_1 \|B_r x\| \right| \tag{5.4}$$

and q_1, q_2 are the Taylor coefficients of $\widehat{g}(t) = g\left(\frac{\tau - t}{1 - t\overline{\tau}}\right)$ with $\tau = g^{-1}\left(\frac{\ell_r(AB_r x)}{\|B_r x\|}\right)$.

Proof. Denote $x_r := B_r x$. Using the functional equation (5.1), one finds

$$(I + rA)\, D^2 J_r(0)[x, y] = -rD^2 h(0)\,[x_r, B_r y]$$

and

$$\begin{aligned}(\mathrm{Id} + rA)\frac{1}{2!}D^2 J_r(0)[x^2] &= -r\frac{1}{2!}D^2 h(0)\left[(x_r)^2\right],\\ (\mathrm{Id} + rA)\frac{1}{3!}D^3 J_r(0)[x^3] &= -r\frac{1}{3!}B_r D^3 h(0)\left[(x_r)^3\right]\\ &+2r^2\cdot\frac{1}{2!}B_r D^2 h(0)\left[x_r, B_r\frac{1}{2!}D^2 h(0)\left[(x_r)^2\right]\right].\end{aligned}$$

Thus the quantities $a_2, \widetilde{a}_2^2$ and a_3 defined by (5.2) can be expressed by the Fréchet derivatives of h:

$$\begin{aligned}\widetilde{a}_2^2 &= r^2\frac{1}{2!}\ell_r\left(D^2 h(0)\left[x_r, \frac{1}{2!}B_r D^2 h(0)\left[(x_r)^2\right]\right]\right)\\ a_2 &= -r\frac{1}{2!}\ell_r\left(D^2 h(0)\left[(x_r)^2\right]\right)\\ a_3 &= -r\frac{1}{3!}\ell_r\left(D^3 h(0)\left[(x_r)^3\right]\right)\\ &+ 2r^2\ell_r\left(\frac{1}{2!}D^2 h(0)\left[x_r, \frac{1}{2!}B_r D^2 h(0)\left[(x_r)^2\right]\right]\right).\end{aligned} \tag{5.5}$$

Denote

$$\varphi(t) = \begin{cases}\frac{\ell_r(h(tx_r))}{t}, & t \in \mathbb{D}\setminus\{0\},\\ \ell_r(Ax_r), & t = 0.\end{cases}$$

By assertion (i) of Lemma 3.2 with $B = B_r$, the function φ is analytic in the disk of radius $\frac{1}{\rho_r}$ and

$$b_1 = \frac{1}{2!}\cdot\ell_r\left(D^2 h(0)[(x_r)^2]\right) \quad\text{and}\quad b_2 = \frac{1}{3!}\cdot\ell_r\left(D^3 h(0)[(x_r)^3]\right). \tag{5.6}$$

Comparing formulae (5.6) and (5.5) we see that

$$b_1 = -\frac{1}{r}a_2 \qquad\text{and}\qquad b_2 = -\frac{1}{r}(a_3 - 2\widetilde{a}_2^2).$$

Therefore,

$$\left|a_3 - \widetilde{a}_2^2 - (\nu - 2)a_2^2\right| = r\left|b_2 - r(2-\nu)b_1^2\right|.$$

Also, by assertion (ii) of Lemma 3.2, $\varphi \prec \|x_r\|\widehat{g}(\rho_r\cdot)$.
To complete the proof we apply Lemma 3.1 with $p = \|x_r\|\widehat{g}(\rho_r\cdot)$ and $\mu = r(2-\nu)$. □

From now on, for any $x \in \partial\mathbb{B}$ we will adopt the notations $x_r = B_r x$ and $\ell_r := \ell_{x_r} \in T(x_r)$. To compare our results with the previous ones we consider some special cases. If, for example, $A = \lambda\,\mathrm{Id}$, $\Re\lambda > 0$, is a scalar operator, then $B_r = \frac{1}{1+\lambda r}\,\mathrm{Id}$, $x_r = \frac{1}{1+\lambda r}x$ and $\rho_r = \|x_r\| = \frac{1}{|1+\lambda r|}$. Thus

$$\tau = g^{-1}\left(\frac{\ell_r(\lambda x_r)}{\|x_r\|}\right) = g^{-1}(\lambda). \tag{5.7}$$

Thus inequality (5.3) takes the form

$$\left|a_3 - 2\widetilde{a}_2^2 - (\nu-2)a_2^2\right| \le \frac{|q_1|r}{|1+\lambda r|^3} \max\left(1, \left|\frac{q_2}{q_1} - \frac{q_1 r}{|1+\lambda r|}(2-\nu)\right|\right), \tag{5.8}$$

where q_1, q_2 are the Taylor coefficients of $\widehat{g}(t) = g\left(\frac{\tau - t}{1-t\overline{\tau}}\right)$ with $\tau = g^{-1}(\lambda)$.

Corollary 5.3. *Assume that $A = \lambda\,\mathrm{Id}$, $\Re\lambda > 0$ and $g = g_0$. Then for any $\nu \in \mathbb{C}$ we have*

$$\left|a_3 - 2\widetilde{a}_2^2 - (\nu-2)a_2^2\right| \le \frac{|1+\lambda^2|r}{|1+\lambda r|^3} \max\left(1, \left|\lambda - (2-\nu)r\frac{1+\lambda^2}{|1+\lambda r|}\right|\right). \tag{5.9}$$

Proof. Since $g = g_0$, formula (5.7) is $\tau = g^{-1}(\lambda) = \frac{\lambda-1}{\lambda+1}$. Thus $q_1 = -(1+\lambda^2)$ and $q_2 = \lambda(1+\lambda^2)$. Then (5.9) follows from (5.8). □

For $A = \mathrm{Id}$, Corollary 5.3 coincides with [19, Theorem 5.6].

Another interesting case occurs when h satisfies Assumption 2.

Corollary 5.4. *If $h \in \mathcal{N}_A(g)$ satisfies Assumption 2, then*

$$\left|a_3 - (\nu - 2 + 2\delta)a_2^2\right| \le r|q_1|\|x_r\|\rho_r^2 \max\left(1, Q_r(x)\right), \tag{5.10}$$

where $Q_r(x)$ is defined by (5.4) and $\delta = \frac{\ell_r(B_r x_r)}{\|x_r\|^2}$.

Proof. Since h satisfies condition (4.6), there exists a function $\kappa : \partial\mathbb{B} \to \mathbb{C}$ such that $D^2h(0)[x^2] = \kappa(x)x$, $x \in \partial\mathbb{B}$. Thus,

$$\begin{aligned} a_2^2 &= \frac{r^2}{4}\left(\ell_r\left(D^2h(0)\left[(x_r)^2\right]\right)\right)^2 = \frac{r^2}{4}\left(\ell_r\left(\kappa(x_r)x_r\right)\right)^2 \\ &= \left(\frac{r}{2}\kappa(x_r)\right)^2 \|x_r\|^2. \end{aligned}$$

At the same time,

$$\begin{aligned} \widetilde{a}_2^2 &= r^2\frac{1}{2!}\ell_r\left(D^2h(0)\left[x_r, \frac{1}{2!}B_r D^2h(0)\left[(x_r)^2\right]\right]\right) \\ &= r^2\frac{1}{4}\ell_r\left(D^2h(0)\left[x_r, B_r\kappa(x_r)x_r\right]\right) \\ &= r^2\frac{1}{4}\kappa(x_r)\ell_r\left(D^2h(0)\left[x_r, B_r x_r\right]\right). \end{aligned}$$

The mapping h also satisfies condition (4.7), then

$$\begin{aligned} \widetilde{a}_2^2 &= r^2\frac{1}{4}\kappa(x_r)\ell_r\left(B_r D^2h(0)\left[(x_r)^2\right]\right) \\ &= r^2\frac{1}{4}\kappa(x_r)\ell_r\left(B_r\kappa(x_r)x_r\right) = \left(\frac{r}{2}\kappa(x_r)\right)^2 \ell_r\left(B_r x_r\right). \end{aligned}$$

Now estimate (5.10) follows from the relation $\widetilde{a}_2^2 = \delta a_2^2$ with $\delta = \frac{\ell_r(B_r x_r)}{\|x_r\|^2}$. □

If h is of a one-dimensional type, then $A = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$ by Lemma 5.1. In this case formula (5.10) gets a simpler form.

Corollary 5.5. *If $h \in \mathcal{N}_A(g)$ is one-dimensional type with $A = \lambda \operatorname{Id}$, then for any $\nu \in \mathbb{C}$ we have*

$$\left\| (\operatorname{Id} + rA)\frac{1}{3!}D^3 J_r(0)[x^3] - \mu(\operatorname{Id} + rA)\frac{1}{2!}D^2 J_r(0)\left[x, (\operatorname{Id} + rA)\frac{1}{2!}D^2 J_r(0)[x^2]\right] \right\|$$
$$= \left|a_3 - \mu a_2^2\right| \le \frac{r|q_1|}{|1+\lambda r|^3} \cdot \max\left(1, \left|\frac{q_2}{q_1} - (2\delta - \mu)\frac{rq_1}{|1+\lambda r|}\right|\right)$$

with $\delta = \frac{|1+\lambda r|}{1+\lambda r}$.

In particular, if $A = \operatorname{Id}$ and $g = g_0$, this coincides with [19, Corollary 5.7].

Proof. By Lemma 3.3, there is a function κ such that $\frac{1+\lambda r}{2!}D^2 J_r(0)[x^2] = \kappa(x)x$. Then the left-hand term equals to

$$\left\| \frac{1+r\lambda}{3!}D^3 J_r(0)[x^3] - \mu\frac{1+r\lambda}{2!}\kappa(x)D^2 J_r(0)[x^2] \right\|.$$

Lemma 3.3 states that this is equal to

$$\begin{aligned} & \left| \ell_x\left(\frac{1+r\lambda}{3!}D^3 J_r(0)[x^3] - \mu\frac{1+r\lambda}{2!}\kappa(x)D^2 J_r(0)[x^2]\right) \right| \\ = \; & \left|a_3 - \mu a_2 \kappa(x)\right| = \left|a_3 - \mu a_2^2\right|. \end{aligned}$$

Set $\mu = \nu - 2 + 2\delta$. Then we proceed by Corollary 5.4:

$$\begin{aligned} \le \; & \frac{r|q_1|}{|1+\lambda r|^3} \cdot \max\left(1, \left|\frac{q_2}{q_1} - (2-\nu)\frac{rq_1}{|1+\lambda r|}\right|\right) \\ = \; & \frac{r|q_1|}{|1+\lambda r|^3} \cdot \max\left(1, \left|\frac{q_2}{q_1} - (2\delta-\mu)\frac{rq_1}{|1+\lambda r|}\right|\right). \end{aligned}$$

□

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