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Operators of the α-Bloch space on the open unit ball of a JB*-triple

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple X which may be infinite dimensional. In this paper, we characterize the bounded weighted composition operators from the Hardy space $H^\infty(\mathbb{B}_X)$ into the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$ on $\mathbb{B}_X$. Later, we show the multiplication operator from $H^\infty(\mathbb{B}_X)$ into $\mathcal{B}^\alpha(\mathbb{B}_X)$ is bounded. Also, we give the operator norm of the bounded multiplication operator.

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1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C}; |z| < 1\}$ be the unit disc in $\mathbb{C}$, and let $H(\mathbb{U}, \mathbb{C})$ be the set of all holomorphic function from $\mathbb{U}$ to $\mathbb{C}$. In 1925, Bloch [4] showed that a holomorphic function $f \in H(\mathbb{U}, \mathbb{C})$ with $f'(0) = 1$ biholomorphically maps onto a disc, called a schlicht disc, with the radius $r(f)$ greater than some positive absolute constant. Then, the 'best possible' constant $\mathbf{B}$ for all such functions, that is,

$$\mathbf{B} = \inf\{r(f) : f \in H(\mathbb{U}, \mathbb{C}),\ f'(0) = 1\},$$

is called the Bloch constant.

Let $r_f(z)$ be the radius of the largest schlicht disc around $f(z)$. W. Seidel and J. L. Walsh [33] gave that $r_f(z) < (1 - |z|^2)lf'(z)|$. Then, we considered the space $\mathcal{B}$ of all holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ satisfying

$$\|f\|_{\mathcal{B},s} = \sup_{z\in\mathbb{U}}(1 - |z|^2)|f'(z)| < \infty$$

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endowed with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}.s}$. The Banach space $\mathcal{B} = (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is called the Bloch space $\mathcal{B}(\mathbb{U})$ on $\mathbb{U}$.

In 1975, Hahn [12] introduced the concept of a Bloch mapping on a finite dimensional bounded homogeneous domain, under the name of normal mapping of finite order. Later, Timoney [35] gave several equivalent definitions for Bloch functions on a finite dimensional bounded homogeneous domain Bergman metric plays an essential role in the definition and equivalent conditions for Bloch functions using the Bergman metric. Chu, Hamada, Honda and Kohr [10] generalized to Bloch functions on a bounded symmetric domain realized as the open unit ball of a JB*-triple X, which may be infinite dimensional, using the Kobayashi metric. We remark that the Bergman metric is not available in infinite dimensional bounded domains.

Also, operators of the Bloch space have been studied. Ohno [30] studied the weighted composition operators from the Hardy space $H^\infty(\mathbb{U})$ to the Bloch space $\mathcal{B}(\mathbb{U})$ on $\mathbb{U}$ in $\mathbb{C}$. Li and Stević [25, 26], Zhang and Chen [37] investigated weighted composition operators from $H^\infty(\mathbb{U})$ to the α-Bloch space. Allen and Colonna [2] gave a characterization of the bounded weighted composition operators from $H^\infty(\Omega)$ to the Bloch space on a bounded homogeneous domain Ω, and gave some estimates for the operator norm. Xiong [36] proved that the composition operator is bounded on the Bloch space and gave estimates for its operator norm. Furthermore, Xiong [36] obtained several necessary conditions for the composition operator to be an isometry. Colonna, Easley and Singman [11] obtained sharper estimates for the operator norm of the multiplication operators from $H^\infty(\mathbb{U})$ to the Bloch space on a bounded symmetric domain.

One of the main purposes of this paper is to generalize the above results for α-Bloch mappings to any bounded symmetric domain realized as the unit ball $\mathbb{B}_X$ of a JB*-triple X which may be infinite dimensional. Kaup [23] showed that the bounded symmetric domains in complex Banach spaces are exactly the open unit balls of JB*-triples which are complex Banach spaces equipped with a Jordan triple structure. Hamada and Kohr [19] gave a definition of α-Bloch mappings on $\mathbb{B}_X$ which is a generalization of α-Bloch functions on the unit disc in $\mathbb{C}$ by using the Bergman operator of the underlying finite dimensional JB*-triple X. When $\alpha = 1$, it is equivalent to the definition of Bloch mappings on $\mathbb{B}^n$ by Liu [27]. Honda [21, 22] gave a characterization of the bounded weighted composition operators from $H^\infty(\mathbb{B}_X)$ into the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$ when $\mathbb{B}_X$ is the open unit ball of a finite dimensional JB*-triple X.

In this paper, when X is a JB*-triple which may be infinite dimensional, we will characterize the bounded weighted composition operators from $H^\infty(\mathbb{B}_X)$ into the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$. We also give estimates for the operator norm. Later, we show that the multiplication operators from $H^\infty(\mathbb{B}_X)$ into $\mathcal{B}^\alpha(\mathbb{B}_X)$ is bounded. Finally, we show that there exist no isometric multiplication operators.

2. Preliminaries

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X. Let Y be a complex Banach space and let $H(\mathbb{B}_X, Y)$ denote the set of all holomorphic mappings from $\mathbb{B}_X$ into Y.

Let $L(X, Y)$ denote the set of all continuous linear operators from X into Y. Let I_X be the identity in $L(X) = L(X, X)$. For a linear operator $A \in L(X, Y)$, we denote the operator norm $\|A\|_{X,Y}$ of A by

$$\|A\|_{X,Y} = \sup \{\|Az\|_Y : \|z\|_X = 1\},$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y, respectively. Let $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{C}^n$. For $A \in L(X)$, we write $\|A\| = \|A\|_{X,X}$. In the case $Y = \mathbb{C}^n$ is the Euclidean space, we write $\|A\|_{X,e}$ for $A \in L(X, \mathbb{C}^n)$.

A complex Banach space X is called a JB*-triple if it is a complex Banach space equipped with a continuous Jordan triple product

$$X \times X \times X \ni (x, y, z) \mapsto \{x, y, z\} \in X$$

satisfying

1. $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,
2. $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
3. $x \Box x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,
4. $\|\{x, x, x\}\|_X = \|x\|_X^3$

for $a, b, x, y, z \in X$, where the *box operator* $x \Box y : X \to X$ is defined by

$$x \Box y(\cdot) = \{x, y, \cdot\}.$$

Example 2.1. A complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ is a JB*-triple with the triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$$

for $x, y, z \in H$.

For every $x, y \in X$, let $B(x, y)$ be the Bergman operator defined by

$$B(x, y) = I_X - 2x \Box y + Q_x Q_y \in L(X),$$

where $Q_a : X \to X$ is the conjugate linear operator defined by $Q_a(x) = \{a, x, a\}$. When $\|x \Box y\| < 1$, the fractional power $B(x, y)^r \in L(X)$ exists for every $r \in \mathbb{R}$, since the spectrum of $B(x, y)$ lies in $\{\zeta \in \mathbb{C} : |\zeta - 1| < 1\}$ (cf. [23, p.517]).
Let $\mathbb{B}_X$ be the unit ball of a JB*-triple X. we have

$$\|B(a, a)\| \le \|B(a, a)^{1/2}\|^2 \le 1 \tag{2.1}$$

for $a \in \mathbb{B}_X$ (cf. [6, p194]). For each $a \in \mathbb{B}_X$, let g_a be the Möbius transformation on $\mathbb{B}_X$ defined by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x \Box a)^{-1} x.$$

Then g_a is a biholomorphic mapping of $\mathbb{B}_X$ onto itself with $g_a(0) = a$, $g_a(-a) = 0$, $g_{-a} = g_a^{-1}$, $Dg_a(0) = B(a, a)^{1/2}$ and $Dg_{-a}(a) = B(a, a)^{-1/2}$. Moreover, we have

$$\|B(a, a)^{-1/2}\| = \frac{1}{1 - \|a\|_X^2} \tag{2.2}$$

by [24, Corollary 3.6] (see also [6, Proposition 3.2.13]).

Let $z \in \mathbb{B}_X, x \in X$. We define the infinitesimal Kobayashi metric $\kappa(z,x)$ on $\mathbb{B}_X$ by

$$\kappa(z,x) = \inf\{|\eta| > 0 : \exists \phi \in H(U, \mathbb{B}_X), \phi(0) = z, D\phi(0)\eta = x\},$$

where U is the unit disc in $\mathbb{C}$. Then $\kappa(0,x) = \|x\|$, $\kappa(z, \alpha x) = |\alpha|\kappa(z,x)$ for all $\alpha \in \mathbb{C}$ and

$$\kappa(z,x) = \|Dg_{-z}(z)x\|_X = \|B(z,z)^{-1/2}x\|_X \leq \frac{\|x\|_X}{1 - \|z\|_X^2} \quad (z \in \mathbb{B}_X, x \in X). \quad (2.3)$$

Let $\mathrm{Aut}(\mathbb{B}_X)$ denote the family of holomorphic automorphisms of $\mathbb{B}_X$.
Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ were first defined by Hahn [12]. In [34, Theorem 3.4], Timoney gave some equivalent definitions. Chu, Hamada, Honda and Kohr [10] generalized them to Bloch functions on a bounded symmetric domain realized as the open unit ball of a JB*-triple X. Recently, Hamada and Kohr [19] gave the definition of α-Bloch mappings on the open unit ball $\mathbb{B}_X$ in a finite dimensional JB*-triple X using Bergman operator $B(a,a)$. We will define α-Bloch mappings on the open unit ball $\mathbb{B}_X$ in a JB*-triple X which may be infinite dimensional, using the Kobayashi metric $\kappa(z,u)$ as the following.

Definition 2.2. Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple X, and let $\alpha > 0$. A holomorphic mapping $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is called an α-Bloch mapping if

$$\|f\|_\alpha + \|f(0)\|_e < +\infty,$$

where $\|f\|_\alpha$ denotes the α-Bloch semi-norm of f defined by

$$\|f\|_\alpha = \sup\{Q_f^\alpha(z) : z \in \mathbb{B}_X\} < +\infty,$$

$$Q_f^\alpha(z) = \sup\left\{\frac{\|Df(z)u\|_e}{\kappa(z,u)^\alpha} : u \in X \setminus \{0\}, \|u\|_X = 1\right\}.$$

Let $\mathcal{B}_{X,n}^\alpha(\mathbb{B}_X)$ be the space of α-Bloch mappings $f : \mathbb{B}_X \to \mathbb{C}^n$. For $f \in \mathcal{B}_{X,n}^\alpha(\mathbb{B}_X)$ the norm $\|\cdot\|_{\mathcal{B}_n^\alpha}$ is given by $\|f\|_{\mathcal{B}_n^\alpha} = \|f\|_\alpha + \|f(0)\|_e$. Then, the space $(\mathcal{B}_{X,n}^\alpha(\mathbb{B}_X), \|\cdot\|_{\mathcal{B}_n^\alpha})$ is a complex Banach space. We write $\mathcal{B}^\alpha(\mathbb{B}_X) = \mathcal{B}_{X,1}^\alpha(\mathbb{B}_X)$ and $\|f\|_{\mathcal{B}^\alpha} = \|f\|_{\mathcal{B}_X^\alpha}$ for $f \in \mathcal{B}^\alpha(\mathbb{B}_X)$.
Let $f : \mathbb{B}_X \to \mathbb{C}$ be a holomorphic function and let $\alpha > 0$. We say that f is an α-Bloch function on $\mathbb{B}_X$ if $f \in \mathcal{B}^\alpha(\mathbb{B}_X)$. Then, by the definitions of $Q_f^\alpha(z)$ and $\|f\|_{\mathcal{B}^\alpha}$,

$$Q_f^\alpha(z) = \sup\left\{\frac{|Df(z)u|}{\kappa(z,u)^\alpha} : u \in X \setminus \{0\}, \|u\|_X = 1\right\}, \|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

Moreover, we have

$$\begin{aligned} & \sup\left\{\frac{|Df(z)x|}{\kappa(z,x)} : x \in X \setminus \{0\}\right\} \\ = & \sup\left\{\frac{\|x\|_X |Df(z)\frac{x}{\|x\|_X}|}{\|x\|_X \kappa(z, \frac{x}{\|x\|_X})} : x \in X \setminus \{0\}\right\} \\ = & \sup\left\{\frac{|Df(z)u|}{\kappa(z,u)} : u \in X \setminus \{0\}, \|u\|_X = 1\right\} \\ = & Q_f^1(z). \end{aligned}$$

In the case that $\alpha = 1$, 1-Bloch mappings are equivalent to Bloch mappings in the sense of Chu, Hamada, Honda and Kohr [10].

Remark 2.3. Let $\alpha > 0$, $f = (f_1, \dots, f_n) \in H(\mathbb{B}_X, \mathbb{C}^n)$. Then f is an α-Bloch mapping if and only if each f_j is an α-Bloch function. If $\alpha = 1$, then f_j is a Bloch function on $\mathbb{B}_X$ if and only if $\|D(f_j \circ g)(0)\|_{X,e}$ is uniformly bounded for $g \in \mathrm{Aut}(\mathbb{B}_X)$.

Lemma 2.4. *Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the unit ball of a JB*-triple X. If $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is an α-Bloch mapping, then we have*

$$\|Df(z)\|_{X,e} \le \frac{Q_f^\alpha(z)}{(1-\|z\|_X^2)^\alpha} \le \frac{\|f\|_\alpha}{(1-\|z\|_X^2)^\alpha}, \quad z \in \mathbb{B}_X.$$

Proof. Since $(1-\|z\|_X^2)\kappa(z,u) \le 1$ from (2.3), we have

$$\begin{aligned}
\|Df(z)\|_{X,e} &= \sup_{\|u\|_X=1} \|Df(z)u\|_e \\
&= \sup_{\|u\|_X=1} \left(\frac{1-\|z\|_X^2}{\kappa(z,u)} \frac{\kappa(z,u)}{1-\|z\|_X^2} \right)^\alpha \|Df(z)u\|_e \\
&\le \frac{1}{(1-\|z\|_X^2)^\alpha} \sup_{\|u\|_X=1} \frac{\|Df(z)u\|_e}{\kappa(z,u))^\alpha} \\
&= \frac{Q_f^\alpha(z)}{(1-\|z\|_X^2)^\alpha} \le \frac{\|f\|_\alpha}{(1-\|z\|_X^2)^\alpha}.
\end{aligned}$$

This completes the proof. □

Remark 2.5. By Lemma 2.4, α-Bloch mappings are bounded on B_X for $\alpha \in (0,1)$.

Lemma 2.6. *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples X and Y, respectively. Let $\alpha \ge 1$, $f \in H(\mathbb{B}_Y, \mathbb{C}^n)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_Y)$. Then*

$$\frac{\|D(f\circ\varphi)(z)u\|_e}{\kappa_X(z,u)^\alpha} \le Q_f^\alpha(\varphi(z)) \quad (u \in X, \|u\|_X = 1).$$

Proof. Let $z \in \mathbb{B}_X$ and $u \in X$ with $\|u\|_X = 1$ be fixed.
If $D\varphi(z)u = 0$, then $D(f\circ\varphi)(z)u = Df(\varphi(z))D\varphi(z)u = 0$. So the above estimate holds.
If $D\varphi(z)u \ne 0$, then we have

$$\frac{\|D(f\circ\varphi)(z)u\|_e}{\kappa_X(z,u)^\alpha} \le \frac{\|Df(\varphi(z))D\varphi(z)u\|_e}{\kappa_Y(\varphi(z), D\varphi(z)u)^\alpha} \le Q_f^\alpha(\varphi(z)).$$

This completes the proof. □

By Lemma 2.6 and the definition of $Q_f^\alpha(z)$ that we have the following.

Proposition 2.7. *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples X and Y, respectively. Let $\alpha \ge 1$, $f \in H(\mathbb{B}_Y, \mathbb{C}^n)$.*

(i) *If $\varphi \in H(\mathbb{B}_X, \mathbb{B}_Y)$, then $Q_{f\circ\varphi}^\alpha(z) \le Q_f^\alpha(\varphi(z))$ for each $z \in \mathbb{B}_X$.*

(ii) *If φ is a biholomorphic mapping from $\mathbb{B}_X$ onto $\mathbb{B}_Y$, then $Q_{f\circ\varphi}^\alpha(z) = Q_f^\alpha(\varphi(z))$ for each $z \in \mathbb{B}_X$.*

Using the above lemmas, we have the following useful lemma.

Lemma 2.8. *Let $\mathbb{B}_X$ be the unit ball of JB*-triples X. Let $f \in H(\mathbb{B}_X, \mathbb{C}^n)$.*

(i) $Q_f^{\beta}(z) \le Q_f^{\alpha}(z)$ *holds for* $z \in \mathbb{B}_X$, $\alpha \le \beta$.

(ii) $Q_f^{\alpha}(0) = Q_f^{1}(0)$ *holds for* $\alpha \ge 1$.

(iii) *Let g_{-z} be the Möbius transformation on $\mathbb{B}_X$ for $z \in \mathbb{B}_X$. Then, for $\alpha \ge 1$,*

$$Q^{\alpha}_{f\circ g_{-z}}(z) = Q^{\alpha}_f(0).$$

Proof. (i) Since $\|u\|_X = \|B(z,z)^{1/2}B(z,z)^{-1/2}u\|_X \le \|B(z,z)^{1/2}\|\,\|B(z,z)^{-1/2}u\|_X$ for $u \in X$ with $\|u\|_X = 1$, we have, from (2.1),

$$\frac{1}{\|B(z,z)^{-1/2}u\|_X} \le \|B(z,z)^{1/2}\| \le 1.$$

Hence, it follows from (2.3) that

$$\begin{aligned}\frac{\|Df(z)u\|_e}{\kappa(z,u)^{\beta}} &= \frac{\|Df(z)u\|_e}{\kappa(z,u)^{\beta-\alpha}\kappa(z,u)^{\alpha}} \\ &= \frac{\|Df(z)u\|_e}{\kappa(z,u)^{\alpha}}\left(\frac{1}{\|B(z,z)^{-1/2}u\|_X}\right)^{\beta-\alpha} \\ &\le \frac{\|Df(z)u\|_e}{\kappa(z,u)^{\alpha}} \le Q_f^{\alpha}(z).\end{aligned}$$

(ii) Since $\kappa(0,u) = \|u\| = 1$, we have

$$\frac{\|Df(0)u\|_e}{\kappa(0,u)^{\alpha}} = \|Df(0)u\|_e = \frac{\|Df(0)u\|_e}{\kappa(0,u)^{1}}.$$

(iii) By Proposition 2.7 (ii), we have $Q^{\alpha}_{f\circ g_{-z}}(z) = Q^{\alpha}_f(g_{-z}(z)) = Q^{\alpha}_f(0)$. □

Remark 2.9.

(1) Any α-Bloch mapping on $\mathbb{B}_X$ is also a β-Bloch mapping on $\mathbb{B}_X$ for $\alpha \le \beta$.
(2) Since 1-Bloch functions are equivalent to Bloch functions in the sense of Chu, Hamada, Honda and Kohr [10], it follows that any Bloch function is also an α-Bloch function, for $\alpha \ge 1$.

For $f \in H(\mathbb{B}_X, \mathbb{C})$, we denote the operator norm by

$$\|f\|_{\infty} = \sup\{|f(z)| : z \in \mathbb{B}_X\}.$$

Let

$$H^{\infty}(\mathbb{B}_X) = \{f \in H(\mathbb{B}_X, \mathbb{C}) : \|f\|_{\infty} < +\infty\}$$

be the space, called the Hardy space, of bounded holomorphic functions on $\mathbb{B}_X$. When the target is the unit disc in $\mathbb{C}$, Chu, Hamada, Honda and Kohr [10, Lemma 3.12] obtained the following Schwarz Pick lemma.

Lemma 2.10. *Let $f \in H^{\infty}(\mathbb{B}_X)$ be such that $\|f\|_{\infty} \le 1$. Then we have*

$$\|Df(z)\|_{X,e} \le \frac{1-|f(z)|^2}{1-\|z\|_X^2}, \quad z \in \mathbb{B}_X.$$

Using the above lemma, we obtain the following.

Lemma 2.11. *For $\alpha \geq 1$, $H^\infty(\mathbb{B}_X) \subset \mathcal{B}^\alpha(\mathbb{B}_X)$ and the inclusion mapping*

$$i : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$$

is a linear operator satisfying

$$\|f\|_\alpha \leq \|f\|_\infty.$$

Proof. Let $f \in H^\infty(\mathbb{B}_X)$. We may assume $\|f\|_\infty = 1$. Since $Q_f^\alpha(z) \leq Q_f^1(z)$ for $z \in \mathbb{B}_X$ by Lemma 2.8, we have

$$\|f\|_\alpha \leq \|f\|_1 = \sup\{Q_f^1(z) : z \in \mathbb{B}_X\}.$$

Let g_a be the Möbius transformation on $\mathbb{B}_X$ for $a \in \mathbb{B}_X$. Then, we have $f \circ g_a \in H^\infty(\mathbb{B}_X)$ and $\|f \circ g_a\|_\infty \leq 1$. By Lemmma 2.10, we have for $\|u\|_X = 1$,

$$\begin{aligned}
\frac{\|Df(a)u\|_e}{\kappa(a,u)^1} &= \frac{\|Df(g_a(g_{-a}(a))Dg_a(g_{-a}(a))Dg_{-a}(a)u\|_e}{\|Dg_{-a}(a)u\|_X} \\
&\leq \|Df(a) \circ Dg_a(0)\|_{X,e} \\
&= \|D(f \circ g_a)(0)\|_{X,e} \\
&\leq 1 - |f \circ g_a(0)|^2 \\
&\leq 1 = \|f\|_\infty.
\end{aligned}$$

□

3. Weighted composition operators

Let $\psi \in H(\mathbb{B}_X, \mathbb{C})$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. The weighted composition operator $W_{\psi,\varphi} : H(\mathbb{B}_X, \mathbb{C}) \to H(\mathbb{B}_X, \mathbb{C})$ is defined by

$$W_{\psi,\varphi}(f)(z) = \psi(z)(f \circ \varphi)(z), \quad f \in H(\mathbb{B}_X, \mathbb{C}), z \in \mathbb{B}_X.$$

We set

$$\begin{aligned}
\theta_\varphi^\alpha(z) &= \sup\{Q_{f\circ\varphi}^\alpha(z) : f \in H^\infty(\mathbb{B}_X), \|f\|_\infty \leq 1\}, \\
\theta_{\psi,\varphi}^\alpha &= \sup_{z \in \mathbb{B}_X} |\psi(z)| \theta_\varphi^\alpha(z) \qquad .
\end{aligned}$$

Theorem 3.1. *Let $W_{\psi,\varphi} : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ be the weighted composition operator for $\alpha \geq 1$, $\psi \in H(\mathbb{B}_X, \mathbb{C})$, $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. Then,*

(1) *$W_{\psi,\varphi}$ is bounded if and only if $\theta_{\psi,\varphi}^\alpha$ is finite and $\psi \in \mathcal{B}^\alpha(\mathbb{B}_X)$,*
(2) *if $W_{\psi,\varphi}$ is bounded, then the following inequalities hold.*

$$\max\{\|\psi\|_{\mathcal{B}^\alpha}, \theta_{\psi,\varphi}^\alpha\} \leq \|W_{\psi,\varphi}\| \leq \|\psi\|_{\mathcal{B}^\alpha} + \theta_{\psi,\varphi}^\alpha.$$

Proof. We will prove the following inequality :

$$\theta_{\psi,\varphi}^\alpha \leq \|W_{\psi,\varphi}\|. \tag{3.1}$$

We may assume that $\theta^\alpha_{\psi,\varphi} > 0$. Let $\varepsilon \in (0, \theta^\alpha_{\psi,\varphi})$ and fixed. Then, there exist $a \in \mathbb{B}_X$ and $f \in H^\infty(\mathbb{B}_X)$ with $\|f\|_\infty \leq 1$ such that

$$|\psi(a)| Q^\alpha_{f\circ\varphi}(a) > \theta^\alpha_{\psi,\varphi} - \varepsilon.$$

Now, by the maximum principle for holomorphic functions, $|f(z)| < 1$ for all $z \in \mathbb{B}_X$. Let $g : U \to U$ be a biholomorphic function defined by

$$g(\zeta) = \frac{\zeta - f(\varphi(a))}{1 - \overline{f(\varphi(a))}\zeta}, \quad \zeta \in U.$$

Then, we obtain $\|g \circ f\|_\infty \leq 1$, $g(f(\varphi(a))) = 0$ and $g'(f(\varphi(a))) = 1/(1 - |f(\varphi(a))|^2)$. Hence

$$\begin{aligned} \|W_{\psi,\varphi}\| &\geq \|W_{\psi,\varphi}(g \circ f)\|_{\mathcal{B}^\alpha} \\ &\geq Q^\alpha_{\psi(g\circ f\circ\varphi)}(a) \\ &= \frac{|\psi(a)|}{1 - |f(\varphi(a))|^2} Q^\alpha_{f\circ\varphi}(a) \\ &> \theta^\alpha_{\psi,\varphi} - \varepsilon. \end{aligned}$$

Since $\varepsilon \in (0, \theta^\alpha_{\psi,\varphi})$ is arbitrary, we obtain (3.1).

(1) Now, we assume $W_{\psi,\varphi}$ is bounded. Then, from (3.1), we have $\theta^\alpha_{\psi,\varphi}$ is finite. Also, using the constant function $\mathbf{1} \in H^\infty(\mathbb{B}_X)$,

$$\|\psi\|_{\mathcal{B}^\alpha} = \|\psi(\mathbf{1} \circ \varphi)\|_{\mathcal{B}^\alpha} = \|W_{\psi,\varphi}(\mathbf{1})\|_{\mathcal{B}^\alpha} \leq \|W_{\psi,\varphi}\|. \tag{3.2}$$

Hence $\psi \in \mathcal{B}^\alpha(\mathbb{B}_X)$.

Conversely, we assume that $\theta^\alpha_{\psi,\varphi}$ is finite and $\psi \in \mathcal{B}^\alpha(\mathbb{B}_X)$. For $f \in H^\infty(\mathbb{B}_X)$ with $\|f\|_\infty \leq 1$,

$$Q^\alpha_{\psi(f\circ\varphi)}(z) \leq |\psi(z)| Q^\alpha_{f\circ\varphi}(z) + |f \circ \varphi(z)| Q^\alpha_\psi(z) \leq \theta^\alpha_{\psi,\varphi} + \|\psi\|_\alpha.$$

So, we have

$$\|\psi(f \circ \varphi)\|_\alpha = \sup_{z\in\mathbb{B}_X} Q^\alpha_{\psi(f\circ\varphi)}(z) \leq \theta^\alpha_{\psi,\varphi} + \|\psi\|_\alpha.$$

It follows from this that

$$\|W_{\psi,\varphi}(f)\|_{\mathcal{B}^\alpha} = \|\psi(f \circ \varphi)\|_\alpha + |\psi(0) f(\varphi(0))| \leq \theta^\alpha_{\psi,\varphi} + \|\psi\|_{\mathcal{B}^\alpha}. \tag{3.3}$$

(2) This follows from (3.1), (3.2), and (3.3). □

Remark 3.2. Honda [22] obtained when X is a finite dimensional JB*-triple. The case $\alpha = 1$ was obtained by Hamada in [13].

4. Multiplication operators

For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* : \ l_x(x) = \|x\|, \ \|l_x\| = 1\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $\psi \in H(\mathbb{B}_X, \mathbb{C})$. The multiplication operator $M_\psi : H(\mathbb{B}_X, \mathbb{C}) \to H(\mathbb{B}_X, \mathbb{C})$ is defined by

$$M_\psi(f)(z) = \psi(z) f(z)$$

for $f \in H(\mathbb{B}_X, \mathbb{C}), z \in \mathbb{B}_X$. We can give the operator norm of the bounded multiplication operator M_ψ from the Hardy space $H^\infty(\mathbb{B}_X)$ to the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$. The following theorem gives an answer to [11, Conjecture, p.628]. The case $\alpha = 1$ was obtained by Hamada [13].

Theorem 4.1. *Let $\psi \in H(\mathbb{B}_X, \mathbb{C})$, $\alpha \geq 1$. Then,*

(1) *$M_\psi : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ is bounded if and only if $\psi \in H^\infty(\mathbb{B}_X)$,*
(2) *if M_ψ is bounded, then the following equality holds.*

$$\|M_\psi\| = |\psi(0)| + \|\psi\|_\infty,$$

where $\|M_\psi\| = \sup\left\{\|M_\psi(f)\|_{\mathcal{B}^\alpha} : f \in H^\infty(\mathbb{B}_X), \|f\|_\infty = 1\right\}.$

Proof. (1) If $\psi \in H^\infty(\mathbb{B}_X)$, then, by Lemma 2.11, we have, for $f \in H^\infty(\mathbb{B}_X)$ with $\|f\|_\infty = 1$,

$$\begin{aligned} \|M_\psi(f)\|_{\mathcal{B}^\alpha} &= |M_\psi(f)(0)| + \|M_\psi(f)\|_\alpha \\ &= |\psi(0)f(0)| + \|\psi f\|_\alpha \\ &\leq |\psi(0)| + \|\psi\|_\infty. \end{aligned}$$

This implies that

$$\|M_\psi\| \leq |\psi(0)| + \|\psi\|_\infty. \tag{4.1}$$

Hence, M_ψ is bounded.

Conversely, assume that M_ψ is bounded. Fix $a \in \mathbb{B}_X \setminus \{0\}$. We set $f(z) = l_a(z)$, where $l_a \in T(a)$. Let g_a be the Möbius transformation on $\mathbb{B}_X$. Then $\|f \circ g_a\|_\infty = 1$, $(f \circ g_a)(-a) = 0$. By Lemma 2.8, we have

$$\begin{aligned} \|M_\psi(f \circ g_a)\|_\alpha &= \|\psi(f \circ g_a)\|_\alpha \\ &\geq Q^\alpha_{\psi(f \circ g_a)}(-a) \\ &= |\psi(-a)| Q^\alpha_{f \circ g_a}(-a) \\ &= |\psi(-a)| Q^\alpha_f(0) \\ &= |\psi(-a)| Q^1_f(0) \\ &= |\psi(-a)|. \end{aligned}$$

Therefore, we have

$$\|M_\psi(f \circ g_a)\|_{\mathcal{B}^\alpha} = \|\psi(f \circ g_a)\|_{\mathcal{B}^\alpha} \geq |\psi(0)|\|a\| + |\psi(-a)|.$$

It follows from this that

$$\|M_\psi\| \geq |\psi(0)| + \|\psi\|_\infty. \tag{4.2}$$

Thus, $\psi \in H^\infty(\mathbb{B}_X)$.

(2) This follows from (4.1) and (4.2). $\square$

Allen and Colonna [2, Theorem 6.2] proved the following theorem when $\mathbb{B}_X$ is the unit disc in $\mathbb{C}$. Colonna, Easley and Singman [11, Theorem 5.1] generalized it when $\mathbb{B}_X$ is a finite dimensional bounded symmetric domain which satisfies some assumption. By using the Bloch norm introduced in section 2, we can generalize to any bounded symmetric domain. This result gives a positive answer to [11, Conjecture, p.629]. The case $\alpha = 1$ was obtained by Hamada [13].

Theorem 4.2. *Let $\mathbb{B}_X$ be the unit ball of a JB*-triple X. Then, there exist no isometric multiplication operators from $H^\infty(\mathbb{B}_X)$ to $\mathcal{B}^\alpha(\mathbb{B}_X)$ for $\alpha \geq 1$.*

Proof. Assume that $\psi \in H(\mathbb{B}_X, \mathbb{C})$, and M_ψ is an isometric multiplication operator from $H^\infty(\mathbb{B}_X)$ to $\mathcal{B}^\alpha(\mathbb{B}_X)$. It follows from this that

$$\|M_\psi(f)\|_{\mathcal{B}^\alpha} = \|f\|_\infty \quad \text{for } f \in H^\infty(\mathbb{B}_X). \tag{4.3}$$

We set $f_a(z) = l_a(z) \in T(a)$ for some $a \in X \setminus \{0\}$. Since $f_a \in H^\infty(\mathbb{B}_X)$, by Lemma 2.11 and (4.3), we obtain

$$1 = \|f_a\|_\infty = \|M_\psi(f_a)\|_{\mathcal{B}^\alpha} = \|\psi f_a\|_\alpha \leq \|\psi f_a\|_\infty \leq \|\psi\|_\infty.$$

Moreover, by Theorem 4.1, $|\psi(0)| + \|\psi\|_\infty = \|M_\psi\| = 1$. So, we have $\psi(0) = 0$ and $\|\psi\|_\infty = 1$. It follows this and (4.3) that

$$\|\psi^2\|_{\mathcal{B}^\alpha} = \|\psi\psi\|_{\mathcal{B}^\alpha} = \|M_\psi(\psi)\|_{\mathcal{B}^\alpha} = \|\psi\|_\infty = 1.$$

On the other hand, we set $F = \psi^2$. Let g_z, g_{-z} be the Möbius transformations on $\mathbb{B}_X$ for $z \in \mathbb{B}_X$. Then, by Lemma 2.8, 2.10,

$$\begin{aligned}
Q_F^\alpha(z) &= Q^\alpha_{F\circ g_z \circ g_{-z}}(z) = Q^\alpha_{F\circ g_z}(0) \\
&= \sup\left\{\frac{\|D(F\circ g_z)(0)u\|_e}{\kappa(0,u)^\alpha} : u \in X\setminus\{0\}, \|u\|_X = 1\right\}. \\
&\leq \sup\left\{\frac{2|\psi(g_z(0))|\|D(\psi\circ g_z)(0)u\|_e}{\|u\|_X^\alpha} : u \in X\setminus\{0\}, \|u\|_X = 1\right\}. \\
&\leq 2|\psi(g_z(0))|(1 - |(\psi(g_z(0))|^2) \\
&\leq 2\sup_{x\in[0,1]} \max(x - x^3) = \frac{4}{3\sqrt{3}}.
\end{aligned}$$

It follows from this that $\|\psi^2\|_{\mathcal{B}^\alpha} = \|F\|_{\mathcal{B}^\alpha} \leq \frac{4}{3\sqrt{3}} < 1$. This is a contradiction. $\square$

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