Stud. Univ. Babeş-Bolyai Math. 67(2022), No. 2, 309–316
DOI: 10.24193/subbmath.2022.2.07

Polynomial convexity properties of closure of domains biholomorphic to balls

Cezar Joiţa

Dedicated to the memory of Professor Gabriela Kohr

Abstract. We discuss the connections between the polynomial convexity properties of a domain biholomorphic to ball and its closure.

Mathematics Subject Classification (2010): 32H02, 32E15.

Keywords: Biholomorphic image of ball, Runge pair.

1. Introduction

A classical theorem of Runge states that for every simply connected open subset U of $\mathbb{C}$, the restriction morphism $\mathcal{O}(\mathbb{C}) \to \mathcal{O}(U)$ has dense image. As usual, the topology on the space of holomorphic functions is the topology of uniform convergence on compacts. We say then that U is Runge in $\mathbb{C}$. This is not longer true in $\mathbb{C}^n$ for $n \geq 2$. It was shown in [13], [14], [15] that there are open subsets of $\mathbb{C}^n$ that are biholomorphic to a polydisc and are not Runge in $\mathbb{C}^n$. E. F. Wold proved in [16] that there are Fatou-Bieberbach domains that are not Runge and hence any open subset of $\mathbb{C}^n$, $n \geq 2$, is biholomorphic to a non-Runge open subset of $\mathbb{C}^n$. In [5] it was given an example of a bounded open subset of $\mathbb{C}^3$ which is biholomorphic to a ball and it is not Runge in any strictly larger open subset of $\mathbb{C}^3$.

In this short paper, motivated by [9], which in turn is based on [7], we want to discuss the possible connections between the polynomial convexity properties of $f(B^n)$ and $\overline{f(B^n)}$ where $f : B^n \to \mathbb{C}^n$ is biholomorphic map onto its image. More precisely we will show that, in general, there is no such connection.

Received 10 January 2022; Accepted 18 January 2022.

2. Results

We start be recalling a few basic notions.

Definition 2.1. Let M be a complex manifold. By $\mathcal{O}(M)$ we will denote the set of holomorphic functions defined on M. If $K \subset M$ is a compact subset we denote by $\widehat{K}^M$ the holomorphically convex hull of K,

$$\widehat{K}^M = \{z \in M : |f(z)| \leq \sup_{x \in K} |f(x)|, \ \forall f \in \mathcal{O}(M)\}.$$

K is called holomorphically convex in M if $\widehat{K}^M = K$.

If $M = \mathbb{C}^n$, then $\widehat{K}^{\mathbb{C}^n}$ is the same as the polynomially covex hull of K,

$$\{z \in M : |f(z)| \leq \sup_{x \in K} |f(x)|, \ \forall \text{ polynomial function } f\}.$$

Definition 2.2. If M is a Stein manifold and U is a Stein open subset then U is called Runge in M if the restriction morphism $\mathcal{O}(M) \to \mathcal{O}(U)$ has dense image

It is well-known, see e.g. [8], that, in the above setting, the following statements are equivalent:

1. U is Runge in M.
2. For every compact set $K \subset U$ we have $\widehat{K}^U = \widehat{K}^M$.
3. For every compact set $K \subset U$ we have $\widehat{K}^M \subset U$.

We recall that a Fatou-Bieberbach domain is a proper open subset of $\mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$. We will need the precise statement of the main theorem of [16] mentioned in the introduction. This is the following.

Theorem 2.3. *There exits a Fatou-Bieberbach domain $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ which is Runge in $\mathbb{C} \times \mathbb{C}^*$ but not in $\mathbb{C}^2$.*

We will move now to our discussion of the closure of domains in $\mathbb{C}^n$ that are biholomorphic to a ball. We denote by B^n the unit ball in $\mathbb{C}^n$ centered at the origin. We will begin with some remarks.

Remark 2.4.

- If U is a bounded Runge open subset of $\mathbb{C}$ then it is simply connected and hence biholomorphic to a disc. In general $\overline{U}$ might not be holomorphically convex. It is easy to give such an example. However, if U has smooth boundary, then $\overline{U}$ is holomorphically convex.
- If $n \geq 2$ on can construct a bounded Runge open subset of $\mathbb{C}^n$ biholomorphic to a ball and with smooth boundary such that $\overline{U}$ is not holomorphically convex. One possible construction is the following: start with $F : B^2 \to \mathbb{C}^2$ biholomorphic onto its image such that $F(B^2)$ is not Runge in $\mathbb{C}^2$. Let $B(0,r) \subset \mathbb{C}^2$ be the ball centered at the origin and of radius r. It is easy to see that if r is small enough then $F(B(0,r))$ is Runge. Let $r_0 = \sup\{r : F(B(0,r)) \text{ is Runge}\}$. Because an increasing union of Runge domains is Runge as well we have that $r_0 < 1$ and $F(B(0,r_0))$ is Runge. It was noticed in [10] that $\overline{F(B(0,r_0))}$ is not polynomially convex.

- The interior of a polynomially convex compact set is Runge. Hence if one is trying to find $F : B^2 \to \mathbb{C}^2$ which is a biholomorpism onto its image such that $F(B^2)$ is not Runge and $\overline{F(B^2)}$ is polynomially convex then one must have that the interior of $\overline{F(B^2)}$ is strictly larger then $F(B^2)$.

Proposition 2.5. *Suppose that M is a connected complex manifold, $\overline{\Gamma}$ and $\overline{\Delta}$ two closed sets, U and V two open sets such that $\overline{\Gamma} \subset U \subset \overline{\Delta} \subset V$. Moreover, we assume that there exist an open set $\tilde{U} \subset \mathbb{C}^n$ containing a closed ball $\overline{B}$, a biholomorphism $F : \tilde{U} \to U$ such that $F(\overline{B}) = \overline{\Gamma}$, an open set $\tilde{V} \subset \mathbb{C}^n$ containing a closed polydisc $\overline{P}$, and a biholomorphism $G : \tilde{V} \to V$ such that $G(\overline{P}) = \overline{\Gamma}$. Then there exists an open and dense subset of M which is biholomorphic to a ball and contains $\overline{\Gamma}$.*

Proof. This proposition is simply a consequence of some of the results and the proofs given in [3], [4] and [2]. For the reader's convenience, we we will recall the main steps needed to prove the proposition. Actually in [3] and [2] the authors prove more than density results: they obtain full-measure embeddings.

We recall that a complex manifold M is called taut if for every complex manifold N (in fact it suffices to work with the unit disc in $\mathbb{C}$, see [1]) the space of holomorphic maps from N to M is a normal family.

• It was noticed in [3] that in any complex manifold M there exists $M_1 \subset M$ a Stein, dense, open subset.

• Another remark from [3] is that for any Stein manifold, M_1, there exists $M_2 \subset M_1$ a taut dense open subset.

• It was proved in [3] that in a taut manifold an increasing union of open sets each one biholomorphic to a polydisc is biholomorphic to a polydisc. A similar statement holds for an increasing union of balls instead of polydiscs.

• A consequence of Theorem II.4 in [4] is the following: if $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset U of a complex manifold M, $\overline{\Delta} = F(\overline{P})$ and x is any point in M then there exists an open subset Δ_1 of M, biholomorphic to a polydisc, such that $\overline{\Delta} \cup \{x\} \subset \Delta_1$.

• This last statement implies easily that if $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset U of a complex manifold M and $\overline{\Delta} = F(\overline{P})$ then there exists an increasing sequence of open subsets biholomorphic to polydiscs in M, $\Delta_1 = \Delta \Subset \Delta_2 \Subset \cdots$ such that $\bigcup \Delta_j$ is dense in M. Indeed, it suffices to consider a dense sequence $\{x_k\}_{k\geq 1} \subset M$ and to construct inductively the polydiscs such that $\{x_1, \ldots, x_k\} \subset \overline{\Delta}_k$.

It follows then from the previous statements that:

• If M is any complex manifold, $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset U of M and $\overline{\Delta} = F(\overline{P})$ then there exists a dense open subset of M biholomorphic to polydisc that contains $\overline{\Delta}$.

• Lemma 2.1 in [2] implies the following statement: suppose that P is a polydisc in $\mathbb{C}^n$, U is an open subset of P such that there exists $\tilde{U} \subset \mathbb{C}^n$ an open neighborhood of a closed ball $\overline{B}$ and a biholomorphism $F : \tilde{U} \to U$. If $\overline{\Gamma} = F(\overline{B})$ and x is any point in P then there exists an open subset Γ_1 of P, biholomorphic to a ball, such that

$\overline{\Delta} \cup \{x\} \subset \Gamma_1$. As before we deduce that there exists an open and dense subset of P that contains $\overline{\Gamma}$.

The conclusion of the proposition is now straightforward. □

Corollary 2.6. *There exists $F : B^2 \to \mathbb{C}^2$ wich is biholomorphic onto its image and such that $F(B^2)$ is not Runge in $\mathbb{C}^2$, and that $\overline{F(B^2)}$ is a holomorphically convex compact subset of $\mathbb{C}^2$.*

Proof. Let $\Omega \subset \mathbb{C}^2$ be a Fatou-Bieberbach domain which is not Runge in $\mathbb{C}^2$. Such a domain exists by Theorem 2.3. Let also $F : \mathbb{C}^2 \to \Omega$ be a biholomorphism.

As Ω is not Runge in $\mathbb{C}^2$, there exists a compact $K \subset \Omega$ such that $\widehat{K}^{\mathbb{C}^2} \not\subset \Omega$. Choose a point $a \in \widehat{K}^{\mathbb{C}^2} \setminus \Omega$. Choose also a ball B and a polydisc P in $\mathbb{C}^2$ such that

$$F^{-1}(K) \subset B \subset \overline{B} \subset P,$$

and an open ball $U \subset \mathbb{C}^2$ such that $\{a\} \cup F(\overline{P}) \subset U$.

We apply now Proposition 2.5 for $M = U \setminus \{a\}$ and we deduce that there exists a dense open subset Γ of $U \setminus \{a\}$ which is biholomorphic to a ball and contains $F(\overline{B})$. In particular it contains K while it does not contain a. This implies that Γ is not Runge in $\mathbb{C}^2$. The closure of Γ is, of course, $\overline{U}$ which is polynomially convex. □

Proposition 2.5 and Corollary 2.6 are geometric in nature in the sense that they are not concerned with the behaviour of the map $F : B^2 \to \mathbb{C}^2$ (except that it is biholomorphic onto its image). Our next theorem exhibits a somehow stranger behaviour of the map.

Theorem 2.7. *There exists $F : B^2 \to \mathbb{C}^2$ biholomorphic onto its image such that $F(B^2)$ is not Runge in $\mathbb{C}^2$ and for every open set $V \in \mathbb{C}^2$ with $V \cap \partial B^2 \neq \emptyset$ we have $\overline{F(B^2 \cap V)} \supset (\mathbb{C}^2 \setminus F(B))$.*

Before we prove the theorem, we need some preliminaries.

For the following definition, see [11].

Definition 2.8. A complex manifold M has the density property if every holomorphic vector field on M can be approximated locally uniformly by Lie combinations of complete vector fields.

Manifolds with the density property have been studied in [11] and [12]. In particular one has:

Proposition 2.9. $\mathbb{C} \times \mathbb{C}^*$ *has the density property.*

The following theorem is a particular case of Theorem 0.2 in [12]. If $M = \mathbb{C}^n$, it is Corollary 2.2 in [6].

Theorem 2.10. *Suppose that M is a conected Stein manifold that satisfies the density property. Let K be a holomorphically convex compact subset of M and g a metric on M. Suppose also given: ε a positive number, A a finite subset of K, and $\{x_1, \ldots, x_s\}$, $\{y_1, \ldots, y_s\}$ two finite subsets of $M \setminus K$ of same cardinality. Then there exists an automorphism $F : M \to M$ such that:*

1. $\sup_{x\in K} d_g(F(x),x) < \varepsilon$ *where* d_g *is the distance induced by* g,
2. $F(a) = a$ *and* $\mathrm{d}F(a) = Id$ *for every* $a \in A$,
3. $F(x_j) = y_j$ *for every* $j = 1,\dots,s$.

We need also the following elementary lemma.

Lemma 2.11. *Suppose that* U, V, Ω *are connected open subsets of* $\mathbb{C}^n$ *with* $V \Subset U \Subset \Omega$. *Let* $r > 0$ *be such that there exists a ball* $B(x_0, r)$ *of radius* r *with* $B(x_0, r) \subset V$ *and let* δ *be the distance between* $\overline{V}$ *and* ∂U. *If* $F : \Omega \to F(\Omega) \subset \mathbb{C}^n$ *is a biholomorphism onto its image and* $\sup_{x\in\overline{U}} \|F(x) - x\| < \min\{\delta, r\}$ *then* $\overline{V} \subset F(U)$.

Proof. Because $\sup_{x\in\overline{U}} \|F(x) - x\| < \delta$, we get that $F(\partial U) \cap \overline{V} = \emptyset$. In particular $V \subset F(U) \cup (\mathbb{C}^n \setminus \overline{U})$. At the same time $\sup_{x\in\overline{U}} \|F(x) - x\| < r$ implies that $F(x_0) \in B(x_0, r)$ and hence $F(U) \cap V \neq \emptyset$. As V is connected, we deduce that $V \subset F(U)$. Finally, $F(\partial U) \cap \overline{V} = \emptyset$ implies that $\overline{V} \subset F(U)$. □

Proof of Theorem 2.7. We consider the Fatou-Beiberbach domain $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ given by Theorem 2.3 which is Runge in $\mathbb{C} \times \mathbb{C}^*$ but not in $\mathbb{C}^2$. Let K be a compact subset of Ω such that $\widehat{K}^{\mathbb{C}^2} \not\subset \mathbb{C} \times \mathbb{C}^*$. Let $F_0 : \mathbb{C}^2 \to \Omega$ be a Fatou-Beiberbach map. Of course we may assume that $F_0(B^2) \supset K$. We fix also a point $a \in K$.

We choose a strictly increasing sequence of open balls, $\{B_s\}_{s\geq -1}$, centered at the origin, such that $\bigcup_s B_s = B^2$ and such that $B_{-1} \supset F_0^{-1}(K)$.

We will construct inductively a sequence of automorphisms $\{H_s\}_{s\geq 0}$ of $\mathbb{C} \times \mathbb{C}^*$ such that, if we set $F_s = H_s \circ \cdots \circ H_0 \circ F_0 \in \mathcal{S}(B^2)$, then the map we are looking for will be $F = \lim_s F_s$. Note that $F(B^2)$ will be also a subset of $\mathbb{C} \times \mathbb{C}^*$ because $\mathbb{C} \times \mathbb{C}^*$ is Stein.

We have to make sure that the sequence converges to a nondegenerate map on B^2. At the same time we would like to have $F_0(B_{-1}) \subset F(B^2)$. If this is the case, we will have $K \subset F(B^2)$ and this will imply that $F(B^2)$ is not Runge in $\mathbb{C}^2$. In fact we will need more that that, namely we would like to have $F_s(\overline{B}_{s-1}) \subset F(B^2)$ for every s. To force this inclusion we will apply Lemma 2.11. Hence we will introduce a sequence of positive real numbers $\{\varepsilon_s\}_{s\geq 0}$ that will act as the bounds needed in that lemma.

For the remaining property, we will need to introduce an increasing sequence of of finite subsets of B^2, $\{A_s\}_s \in \mathbb{N}$, $A_s \subset A_{s+1}$ that will help "spreading" the image of F.

• We consider $\{x_n\}_{n\geq 1} \subset \partial B^2$ a dense sequence. For each $n \in \mathbb{N}$ we consider $\{x_n^p\}_{p\in\mathbb{N}} \subset B^2$ a sequence that converges to x_n. Moreover we assume that $x_n \neq x_m$ for $n \neq m$ and $x_n^p \neq x_m^q$ for $(n,p) \neq (m,q)$.

• We set H_0 to be the identity and $A_0 = \{a\}$, $\varepsilon_0 = 1$.

• We assume that we have constructed $H_0,\dots,H_s$, $A_0,\dots,A_s$, $\varepsilon_0,\dots,\varepsilon_s$ and that $H_j(a) = a$ for $j \leq s$ and we will construct H_{s+1}, A_{s+1}, and ε_{s+1}.

We choose $T_1^{s+1},\dots,T_{s+1}^{s+1}$ pairwise disjoint, finite, subsets of $\mathbb{C} \times \mathbb{C}^*$, , such that for every $j = 1,\dots,s+1$ we have

⋄ $T_j^{s+1} \cap (F_s(\overline{B}_s) \cup F_s(A_s)) = \emptyset$ and

$\diamond \bigcup_{z\in T_j^{s+1}} B(z,\frac{1}{s}) \supset \{z \in \mathbb{C}^2 \setminus F_s(B_s) : d(z, F_s(\overline{B}_s)) \le s\}$.

Here $d(z, F_s(\overline{B}_s))$ stands for the distance between z and the compact set $F_s(\overline{B}_s)$.

After we chose these finite sets T_j^{s+1}, we choose, for each $j = 1, \dots, s+1$, a finite subset, A_j^{s+1}, of $\{x_j^p : p \in \mathbb{N}\}$ such that:

$\diamond \#A_j^{s+1} = \#T_j^{s+1}$,

$\diamond A_j^{s+1} \cap (\overline{B}_s \cup A_s) = \emptyset$,

$\diamond \|x_j - x\| < \frac{1}{s}$ for every $x \in A_j^{s+1}$.

We set

$$A_{s+1} = A_s \cup \left(\bigcup_{j=1}^{s+1} A_j^{s+1}\right).$$

Let δ_s denote the distance between $F_s(\overline{B}_{s-1})$ and $\partial F_s(\overline{B}_s)$. $F_s(B_{s-1})$ is an open subset of $\mathbb{C} \times \mathbb{C}^*$. Let $r_s > 0$ be such that there exists a ball of radius r_s included in $F_s(B_{s-1})$.

We define

$$\varepsilon_{s+1} := \frac{1}{2^{s+1}} \min\{\delta_s, r_s, \varepsilon_0 \dots, \varepsilon_s\}.$$

Because H_j, $j \le s$, are automorphisms of $\mathbb{C} \times \mathbb{C}^*$ we have that $F_s(B^2)$ is Runge in $\mathbb{C} \times \mathbb{C}^*$ and hence $F_s(\overline{B}_s)$ is holomorphically convex in $\mathbb{C} \times \mathbb{C}^*$. As A_s is a finite set, $F_s(\overline{B}_s \cup A_s)$ is holomorphically convex in $\mathbb{C} \times \mathbb{C}^*$.

We apply Theorem 2.10 and we deduce that there exists an automorphism H_{s+1} of $\mathbb{C} \times \mathbb{C}^*$ such that

1. $\|H_{s+1}(z) - z\| < \varepsilon_{s+1}$ for every $z \in F_s(\overline{B}_s)$,
2. $H_{s+1}(z) = z$ for every $z \in F_s(A_s)$ (in particular $H_{s+1}(a) = a$),
3. $\mathrm{d}H_{s+1}(a) = I_2$,
4. $H_{s+1}(F_s(A_j^{s+1})) = T_j^{s+1}$ for every $j = 1, \dots s+1$.

Note now that property 1 implies that $F = \lim_s F_s$ (where $F_s = H_s \circ \cdots \circ H_0 \circ F_0$) is holomorphic and property 3 that it is nondegenerate. Hence F is biholomorphic on B^2. Also property 2, together with Lemma 2.11, imply that $F_s(\overline{B}_{s-1}) \subset F(B^2)$ (in fact it implies that $F_s(\overline{B}_{s-1}) \subset F(B_s)$) for every s. In particular $K \subset F(B^2)$ and therefore $F(B^2)$ is not Runge in $\mathbb{C}^2$.

It remains to check that for every $V \in \mathbb{C}^2$ with $V \cap \partial B^2 \ne \emptyset$ we have $\overline{F(B^2 \cap V)} \supset (\mathbb{C}^2 \setminus F(B))$. Fix then such an open set V and a point $p \in \mathbb{C}^2 \setminus F(B^2)$. We recall that the sequence $\{x_n\}$ was chose to be dense in ∂B^2. Let $x_j \in V \cap \partial B^2$. Let $m \in \mathbb{N}$ be large enough such that $m > j$, $\|p - a\| < m$, and $B(x_j, \frac{1}{m}) \subset V$.

We distinguish now two cases:

a) $p \notin F_m(\overline{B}_m)$. Note that $\|p - a\| < m$ implies, in particular that $d(p, F_m(\overline{B}_m)) < m$. According to our choice of T_j^{m+1}, there exists a point $z \in T_j^{m+1}$ such that $\|p - z\| < \frac{1}{m}$. By property 4 in the construction of $\{H_s\}$, there exists $x \in A_j^{s+1}$ such that $H_{m+1}(F_m(x)) = z$. According to the choice of A_j^{s+1}, we have that

$\|x_j - x\| < \frac{1}{m}$ and hence $x \in V$. Note also that property 2 in the construction of $\{H_s\}$ implies that $F(x) = z$.

b) $p \in F_m(\overline{B}_m)$. Since $F_{m+1}(\overline{B}_m) \subset F(B^2)$ and $p \notin F(B^2)$, we have that $p \notin F_{m+1}(\overline{B}_m)$. Let $q = H_{m+1}(p)$. It follows that $q \in F_{m+1}(\overline{B}_m)$. At the same time, property 1 in the construction of $\{H_s\}$ implies that $\|q - p\| < \frac{1}{2^{m+1}}$. It follows that $d(p, F_{m+1}(\overline{B}_m)) < \frac{1}{2^{m+1}}$ and therefore $d(p, \partial F_{m+1}(B_m)) < \frac{1}{2^{m+1}}$. Let $v \in \partial F_{m+1}(B_m))$ be such that $\|p - v\| < \frac{1}{2^{m+1}}$. However $\partial F_{m+1}(B_m)) = H_{m+1}(\partial F_m(B_m)$ and we let $u \in \partial F_m(B_m)$ such that $H_{m+1}(u) = v$. We have then $\|u - v\| < \frac{1}{2^{m+1}}$. We use again our choice of T_j^{m+1} and we find a point $z \in T_j^{m+1}$ such that $\|u - z\| < \frac{1}{m}$. Hence $\|p - z\| < \frac{1}{m} + \frac{1}{2^m}$. As above we obtain a point $x \in V$ such that $F(x) = z$.

In both cases we found $x \in V$ such that $\|p - F(x)\| < \frac{1}{m} + \frac{1}{2^m}$. As m can be chosen arbitrarily large, this finishes the proof. □

References

[1] Barth, T.J., *Taut and tight complex manifolds*, Proc. Amer. Math. Soc., **24**(1970), 429-431.

[2] Fornæss, J.E., Stensønes, B., *Density of orbits in complex dynamics*, Ergodic Theory Dynam. Systems, **26**(2006), 169-178.

[3] Fornæss, J.E., Stout, E.L., *Polydiscs in complex manifolds*, Math. Ann., **227**(1977), 145-153.

[4] Fornæss, J.E., Stout, E.L., *Spreading polydiscs on complex manifolds*, Amer. J. Math., **99**(1977), 933-960.

[5] Fornæss, J.E., Wold, E.E., *An embedding of the unit ball that does not embed into a Loewner chain*, Math. Z., **296**(2020), 73-78.

[6] Forstnerič, F., *Interpolation by holomorphic automorphisms and embeddings in* $\mathbb{C}^n$, J. Geom. Anal., **9**(1999), 93-117.

[7] Hamada, H., Iancu, M., Kohr, G., *On certain polynomially convex sets in* $\mathbb{C}^n$, (in preparation).

[8] Hörmander, L., *An Introduction to Complex Analysis in Several Variables*, Third edition, North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990.

[9] Iancu, M., *On certain polynomially convex sets in* $\mathbb{C}^n$, Geometric Function Theory in Several Complex Variables and Complex Banach Spaces – Workshop dedicated to the memory of Professor Gabriela Kohr, Cluj-Napoca, December 1-3, 2021.

[10] Joiţa, C., *On a problem of Bremermann concerning Runge domains*, Math. Ann., **337**(2007), 395-400.

[11] Varolin, D., *The density property for complex manifolds and geometric structures*, J. Geom. Anal., **11**(2001), 135-160.

[12] Varolin, D., *The density property for complex manifolds and geometric structures II*, Int. J. Math., **11**(2000), 837-847.

[13] Wermer, J., *An example concerning polynomial convexity*, Math. Ann., **139**(1959), 147-150.

[14] Wermer, J., *Addendum to "An example concerning polynomial convexity"*, Math. Ann., **140**(1960), 322-323.

[15] Wermer, J., *On a domain equivalent to the bidisk*, Math. Ann., **248**(1980), 193-194.

[16] Wold, E.F., *A Fatou-Bieberbach domain in* $\mathbb{C}^2$ *which is not Runge*, Math. Ann., **340**(2008), 775-780.

Cezar Joiţa
Simion Stoilow Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest 014700, Romania
e-mail: `Cezar.Joita@imar.ro`