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Linear invariance and extension operators of Pfaltzgraff–Suffridge type

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. We consider the image of a linear-invariant family $\mathcal{F}$ of normalized locally biholomorphic mappings defined in the Euclidean unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ under the extension operator

$$\Phi_{n,m,\beta}[f](z,w) = \Big(f(z), [Jf(z)]^{\beta} w\Big), \quad (z,w) \in \mathbb{B}_{n+m} \subseteq \mathbb{C}^n \times \mathbb{C}^m,$$

where $\beta \in \mathbb{C}$, Jf denotes the Jacobian determinant of f, and the branch of the power function taking 0 to 1 is used. When $\beta = 1/(n+1)$ and $m = 1$, this is the Pfaltzgraff–Suffridge extension operator. In particular, we determine the order of the linear-invariant family on $\mathbb{B}_{n+m}$ generated by the image in terms of the order of $\mathcal{F}$, taking note that the resulting family has minimum order if and only if either $\beta \in (-1/m, 1/(n+1)]$ and the family $\mathcal{F}$ has minimum order or $\beta = -1/m$. We will also see that order is preserved when generating a linear-invariant family from the family obtained by composing $\mathcal{F}$ with a certain type of automorphism of $\mathbb{C}^n$, leading to consequences for various extension operators including the modified Roper–Suffridge extension operator introduced by the author.

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1. Introduction

In this note, we generate linear-invariant families on the Euclidean unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ from other linear-invariant families defined on $\mathbb{B}_n$ or on a lower-dimensional ball (or disk) in a manner that allows us to determine the order of a new family from the order of the family from which it is generated. In many cases, the new linear-invariant families will have minimum order if the families that generate them do.

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The primary mechanism used will involve a perturbation of an extension operator originally presented by Pfaltzgraff and Suffridge [17].

In order to more thoroughly preview our work, we present some basic notation. We reserve $n, m \in \mathbb{N}$ for the dimensions of complex Euclidean spaces. If $a \in \mathbb{C}^n$ and $r > 0$, then $B_n(a; r)$ denotes the ball centered at a of radius r. Thus $\mathbb{B}_n = B_n(0; 1)$, and we write $\mathbb{S}_n = \partial\mathbb{B}_n$ for the unit sphere. When $n = 1$, $\mathbb{D} = \mathbb{B}_1$ is the unit disk in $\mathbb{C}$. For $z \in \mathbb{C}^n$ with $n \geq 2$, we write $z = (z_1, \hat{z})$, where $z_1 \in \mathbb{C}$ and $\hat{z} = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$. When used in matrix-algebra calculations, we treat elements of $\mathbb{C}^n$ as column vectors, although we express them as n-tuples. To avoid confusion, n-tuples are always written within parentheses $(\cdot)$ and matrices are always written within brackets $[\cdot]$. The canonical basis vectors in $\mathbb{C}^n$ are $e_1, \dots, e_n$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$. We write $L(\mathbb{C}^n)$ for the algebra $L(\mathbb{C}^n, \mathbb{C}^n)$ and I_n for its identity. The adjoint (conjugate-transpose) of $A \in L(\mathbb{C}^n, \mathbb{C}^m)$ is $A^* \in L(\mathbb{C}^m, \mathbb{C}^n)$. If $\Omega \subseteq \mathbb{C}^n$ is open, then $H(\Omega, \mathbb{C}^m)$ is the space of all holomorphic mappings from Ω into $\mathbb{C}^m$. If $m = 1$, we use the shorthand $H(\Omega)$. For $f \in H(\Omega, \mathbb{C}^m)$, the Fréchet derivative of f is $Df \colon \Omega \to L(\mathbb{C}^n, \mathbb{C}^m)$. When $m = n$, the Jacobian of f is denoted $Jf = \det Df$. The second Fréchet derivative of f at $z \in \Omega$ is the symmetric bilinear operator $D^2 f(z) \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^m$. It is useful to note that if $a \in \mathbb{C}^n$ is fixed and $g \in H(\Omega, \mathbb{C}^m)$ is given by $g(z) = Df(z)a$ for $z \in \Omega$, then

$$Dg(z) = D^2 f(z)(a, \cdot), \qquad z \in \Omega.$$

In particular, $D^2 f(z)(a, \cdot)$ is an element of $L(\mathbb{C}^n, \mathbb{C}^m)$, and when $m = n$ we may consider its trace. We are broadly interested in the family of normalized locally biholomorphic mappings

$$\mathcal{LS}(\mathbb{B}_n) = \{f \in H(\mathbb{B}_n, \mathbb{C}^n) : f(0) = 0,\ Df(0) = I_n, \text{ and } Jf(z) \neq 0 \text{ for } z \in \mathbb{B}_n\}.$$

The subfamily of biholomorphic mappings is

$$\mathcal{S}(\mathbb{B}_n) = \{f \in \mathcal{LS}(\mathbb{B}_n) : f \text{ is biholomorphic}\}.$$

The group of unitary operators on $\mathbb{C}^n$ is $\mathcal{U}(n) \subseteq L(\mathbb{C}^n)$. The group of biholomorphic automorphisms of a domain $\Omega \subseteq \mathbb{C}^n$ (i.e., biholomorphic mappings of Ω onto Ω) is $\operatorname{Aut}\Omega$. Any $\varphi \in \operatorname{Aut}\mathbb{B}_n$ has the unique decomposition $\varphi = U \circ \varphi_a$ for $U \in \mathcal{U}(n)$ and $a \in \mathbb{B}_n$, where $\varphi_a \in \operatorname{Aut}\mathbb{B}_n$ is given by

$$\varphi_a(z) = T_a\left(\frac{a - z}{1 - \langle z, a\rangle}\right) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}_n. \tag{1.1}$$

Here, P_a is the orthogonal projection of $\mathbb{C}^n$ onto the subspace spanned by a, $Q_a = I_n - P_a$ is the orthogonal projection of $\mathbb{C}^n$ onto the orthogonal complement of a, $s_a = \sqrt{1 - \|a\|^2}$, and $T_a = P_a + s_a Q_a$. (See [8, 23].) Observe that φ_a is an involution exchanging 0 and a. We also note that any $\varphi \in \operatorname{Aut}\mathbb{B}_n$ can likewise be written as $\varphi = \varphi_b \circ V|_{\mathbb{B}_n}$ for $b \in \mathbb{B}_n$ and $V \in \mathcal{U}(n)$. (See [19].)

To prove the classical Koebe distortion theorem for $\mathcal{S}(\mathbb{D})$, a standard technique is to apply the bound on the second coefficient of the Taylor series expansion of a function in $\mathcal{S}(\mathbb{D})$ to the function formed by composing an element of $\mathcal{S}(\mathbb{D})$ with a member of $\operatorname{Aut}\mathbb{D}$ and renormalizing, an operation now known as a Koebe transform. Pommerenke [20] coined the term "linear-invariant" for families in $\mathcal{LS}(\mathbb{D})$ that are invariant under all Koebe transforms and defined the order of a linear-invariant family to be the supremum of the moduli of the second coefficients in the Taylor series expansions of its elements. Pfaltzgraff [16] generalized this notion to several complex variables as follows. For $\varphi \in \operatorname{Aut}\mathbb{B}_n$, the Koebe transform with respect to φ is $\Lambda_\varphi\colon \mathcal{LS}(\mathbb{B}_n) \to \mathcal{LS}(\mathbb{B}_n)$ given by

$$\Lambda_\varphi[f](z) = D\varphi(0)^{-1}Df(\varphi(0))^{-1}[f(\varphi(z)) - f(\varphi(0))], \qquad f \in \mathcal{LS}(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

For $\varphi, \psi \in \operatorname{Aut}\mathbb{B}_n$, we have $\Lambda_{\varphi\circ\psi} = \Lambda_\psi \circ \Lambda_\varphi$. It follows that $\Lambda_{\varphi^{-1}} = \Lambda_\varphi^{-1}$. A linear-invariant family is a set $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ such that $\Lambda_\varphi[f] \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $\varphi \in \operatorname{Aut}\mathbb{B}_n$. If $\mathcal{G} \subseteq \mathcal{LS}(\mathbb{B}_n)$, then the linear-invariant family generated by $\mathcal{G}$ is

$$\Lambda[\mathcal{G}] = \{\Lambda_\varphi[g] : \varphi \in \operatorname{Aut}\mathbb{B}_n,\ g \in \mathcal{G}\}.$$

The complexity inherent in the generalization to higher dimensions manifests when defining the order of a linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$. In [16], Pfaltzgraff defined the order of $\mathcal{F}$ to be

$$\operatorname{ord}\mathcal{F} = \frac{1}{2}\sup_{u\in\mathbb{S}_n}\sup_{f\in\mathcal{F}}|\operatorname{tr} D^2f(0)(u,\cdot)| \in \left[\frac{n+1}{2}, \infty\right], \tag{1.2}$$

and proved the sharp lower bound is as given. (Any $f \in \mathcal{LS}(\mathbb{B}_n)$ has a Taylor series expansion of the form

$$f(z) = z + \frac{1}{2}D^2f(0)(z,z) + o(\|z\|^2), \qquad z \in \mathbb{B}_n,$$

and hence the expression (1.2) reduces to the definition of order given by Pommerenke when $n = 1$.) He then proved the following volume-distortion theorem.

Theorem 1.1. *Let $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ be a linear-invariant family such that $\alpha = \operatorname{ord}\mathcal{F} < \infty$. For all $f \in \mathcal{F}$,*

$$\frac{(1-\|z\|)^{\alpha-(n+1)/2}}{(1+\|z\|)^{\alpha+(n+1)/2}} \le |Jf(z)| \le \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}, \qquad z \in \mathbb{B}_n.$$

When $n = 1$, this becomes Pommerenke's distortion theorem for linear-invariant families on $\mathbb{D}$. In particular, $\mathcal{S}(\mathbb{D})$ is a linear-invariant family such that $\operatorname{ord}\mathcal{S}(\mathbb{D}) = 2$ (due to the classical Bieberbach estimate [1]), and Theorem 1.1 reduces to the aforementioned Koebe distortion theorem. When $n \ge 2$, the linear-invariant family $\mathcal{S}(\mathbb{B}_n)$ has infinite order, but other families of interest have finite order. One such linear-invariant family is the family of convex mappings

$$\mathcal{K}(\mathbb{B}_n) = \{f \in \mathcal{S}(\mathbb{B}_n) : f(\mathbb{B}_n) \text{ is convex}\}$$

(it is compact; see [8]). When $n = 1$, $\operatorname{ord}\mathcal{K}(\mathbb{D}) = 1$ (the minimum possible order) and any linear-invariant family of order 1 on $\mathbb{D}$ must be a subset of $\mathcal{K}(\mathbb{D})$. For $n \ge 2$, things are not so nice. Pfaltzgraff and Suffridge [18] showed that $\operatorname{ord}\mathcal{K}(\mathbb{B}_n) > (n+1)/2$ (that is, $\mathcal{K}(\mathbb{B}_n)$ does not have minimum order), and the problem of determining the exact value of $\operatorname{ord}\mathcal{K}(\mathbb{B}_n)$ remains open. Partly motivated by this, Pfaltzgraff and Suffridge introduced [19] a second notion of order based on the operator norm of $D^2f(0)/2$ for f in a linear-invariant family. This notion of order has some advantages. For instance, the minimum norm order of a linear-invariant family on $\mathbb{B}_n$ is 1 regardless of n and $\mathcal{K}(\mathbb{B}_n)$ has this norm order. Furthermore, having finite norm order implies a linear-invariant family is a normal family and growth estimates can be obtained. Nevertheless, the norm order is, in general, much more difficult to use and does not seem to be as compatible with the extension operators we will study.

The following two identities from [16] used to prove Theorem 1.1 will be useful in our work.

Lemma 1.2. *Let $f \in \mathcal{LS}(\mathbb{B}_n)$ and $\varphi \in \operatorname{Aut}\mathbb{B}_n$. If $g = \Lambda_\varphi[f]$, then*

$$\operatorname{tr} D^2 g(0)(w,\cdot) = \operatorname{tr} Df(\varphi(0))^{-1} D^2 f(\varphi(0))(D\varphi(0)w,\cdot) + \operatorname{tr} D\varphi(0)^{-1} D^2\varphi(0)(w,\cdot)$$

for all $w \in \mathbb{C}^n$.

Lemma 1.3. *For $a \in \mathbb{B}_n$, we have*

$$D\varphi_a(0)^{-1} D^2\varphi_a(0)(w,\cdot) = \langle w, a\rangle I_n + wa^*, \qquad w \in \mathbb{C}^n.$$

In addition, Godula, Liczberski, and Starkov [5] obtained the following converse to Theorem 1.1.

Theorem 1.4. *If $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ is a linear-invariant family with* $\operatorname{ord}\mathcal{F} < \infty$*, then*

$$\operatorname{ord}\mathcal{F} = \inf\left\{\alpha \geq \frac{n+1}{2} : |Jf(z)| \leq \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}} \text{ for } z \in \mathbb{B}_n \text{ and } f \in \mathcal{F}\right\}.$$

Generally speaking, an extension operator is a function $\Phi\colon \mathcal{F} \to \mathcal{LS}(\mathbb{B}_{n+m})$, where $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$, such that $\Phi[f](z,0) = (f(z),0)$ for all $f \in \mathcal{F}$ and $z \in \mathbb{B}_n$. (We will consistently write points in $\mathbb{C}^{n+m}$ as ordered pairs in $\mathbb{C}^n \times \mathbb{C}^m$.) The study of extension operators focuses on those for which $\Phi[f]$ inherits a useful characteristic of f such as a geometric property of the mapping's range or the ability to embed the mapping in a Loewner chain. In this work, we will focus primarily on the operator

$$\Phi_{n,m,\beta}[f](z,w) = \big(f(z), [Jf(z)]^\beta w\big), \qquad f \in \mathcal{LS}(\mathbb{B}_n),\ (z,w) \in \mathbb{B}_{n+m},$$

for $\beta \in \mathbb{C}$. The branch of the power function taking 0 to 1 is used. The operator $\Phi_{1,n-1,1/2}$, $n \geq 2$, is the Roper–Suffridge extension operator, the first such operator studied. It was introduced in [21] where the authors showed $\Phi_{1,n-1,1/2}[\mathcal{K}(\mathbb{D})] \subseteq \mathcal{K}(\mathbb{B}_n)$. Graham and Kohr [9] showed $\Phi_{1,n-1,1/2}[\mathcal{S}^*(\mathbb{D})] \subseteq \mathcal{S}^*(\mathbb{B}_n)$, where

$$\mathcal{S}^*(\mathbb{B}_n) = \{f \in \mathcal{S}(\mathbb{B}_n) : f(\mathbb{B}_n) \text{ is starlike with respect to } 0\}.$$

Pfaltzgraff and Suffridge introduced the operator $\Phi_{n,1,1/(n+1)}$ in [17] in their study of linear-invariant families. Of note, Chirilă [3] first considered the perturbation $\Phi_{n,1,\beta}$ for $\beta \in [0, 1/(n+1)]$ in connection with Loewner theory and showed $\Phi_{n,1,\beta}[\mathcal{S}^*(\mathbb{B}_n)] \subseteq \mathcal{S}^*(\mathbb{B}_{n+1})$ for such β, generalizing the same inclusion for the original Pfaltzgraff–Suffridge extension operator given in [11] by Graham, Kohr, and Pfaltzgraff.

The image of a linear-invariant family under an extension operator will generally not be linear-invariant, but we may consider the linear-invariant family generated by the image. That approach was taken by Pfaltzgraff and Suffridge in [17] where they showed that if $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ is a linear-invariant family of finite order, then

$$\operatorname{ord}\Lambda[\Phi_{n,1,1/(n+1)}[\mathcal{F}]] = \frac{n+2}{n+1}\operatorname{ord}\mathcal{F}.$$

In particular, a family of minimum order on $\mathbb{B}_n$ is extended in this manner to a family of minimum order on $\mathbb{B}_{n+1}$. Graham, Hamada, Kohr, and Suffridge [7] used a different approach for a perturbation of the Roper–Suffridge extension operator, showing

$$\operatorname{ord}\Lambda[\Phi_{1,n-1,\beta}[\mathcal{F}]] = (1+(n-1)\beta)\operatorname{ord}\mathcal{F} + \frac{(n-1)(1-2\beta)}{2}$$

for a linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{D})$ of finite order, $\beta \in [0, 1/2]$, and $n \geq 2$. Again, a family of minimum order is extended to a family of minimum order for such β.

In what follows, we generalize the work of Graham, Hamada, Kohr, and Suffridge to produce an order result of this type for the general operator $\Phi_{n,m,\beta}$, $\beta \in \mathbb{C}$, that will imply both of the above results and maintain the characteristic that families of minimum order on $\mathbb{B}_n$ extend to families of minimum order on $\mathbb{B}_{n+m}$ for $\beta \in [-1/m, 1/(n+1)]$. We will also observe that any linear-invariant family on $\mathbb{B}_n$ is extended to a family of minimum order on $\mathbb{B}_{n+m}$ when $\beta = -1/m$. In no other cases will a family of minimum order be produced. We will follow with some results showing how the generation of a linear-invariant family from the composition of a linear-invariant family on $\mathbb{B}_n$ with a member of a particular subgroup of $\operatorname{Aut}\mathbb{C}^n$ preserves order, allowing us to generate large linear-invariant families of minimum order and to observe results as above for other commonly studied extension operators.

2. The order of $\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]$

The following lemma is a generalization of [7, Lemma 4.1], and its proof uses a technique from [19].

Lemma 2.1. *Let $\mathcal{A} \subseteq \operatorname{Aut}\mathbb{B}_n$ be such that $\{\varphi(0) : \varphi \in \mathcal{A}\} = \mathbb{B}_n$. If $\mathcal{G} \subseteq \mathcal{LS}(\mathbb{B}_n)$, then*

$$\operatorname{ord}\Lambda[\mathcal{G}] = \frac{1}{2}\sup_{f\in\mathcal{G}}\sup_{\varphi\in\mathcal{A}}\sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2\Lambda_\varphi[f](0)(u,\cdot)|.$$

Proof. Let $g \in \Lambda[\mathcal{G}]$. There are $\psi \in \operatorname{Aut}\mathbb{B}_n$ and $f \in \mathcal{G}$ such that $g = \Lambda_\psi[f]$. Choose $\varphi \in \mathcal{A}$ such that $\varphi(0) = \psi(0)$. Then there is a $U \in \mathcal{U}(n)$ such that $\psi = \varphi \circ U|_{\mathbb{B}_n}$. Let $h = \Lambda_\varphi[f] \in \Lambda[\mathcal{G}]$. It follows that $g = \Lambda_U[h]$. That is, $g(z) = U^*h(Uz)$ for all $z \in \mathbb{B}_n$. Then $Dg(z)u = U^*Dh(Uz)Uu$ for $z \in \mathbb{B}_n$ and $u \in \mathbb{S}_n$. Differentiation of both sides with respect to z yields

$$D^2g(z)(u,\cdot) = U^*D^2h(Uz)(Uu,\cdot)U, \qquad z \in \mathbb{B}_n,\ u \in \mathbb{S}_n.$$

It follows that

$$\sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2g(0)(u,\cdot)| = \sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2h(0)(u,\cdot)|$$

because the trace is a similarity invariant. This implies the result. □

The following lemma is a generalization of [7, Lemma 4.2], although our proof uses a somewhat different technique (see also [8]).

Lemma 2.2. *Let $f \in \mathcal{LS}(\mathbb{B}_n)$ and $g \in H(\mathbb{B}_n)$ such that $g(0) = 1$ and $g(z) \neq 0$ for all $z \in \mathbb{B}_n$. If $F \in H(\mathbb{B}_{n+m}, \mathbb{C}^{n+m})$ is defined by $F(z,w) = (f(z), g(z)w)$, then $F \in \mathcal{LS}(\mathbb{B}_{n+m})$ and*

$$\sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(0,b)}}[F](0,0)((u,v),\cdot)| = \max\left\{n+m+1, \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2 F(0,0)((u,v),\cdot)|\right\}.$$

Proof. Let $b \in \mathbb{B}_m$, and set $G = \Lambda_{\varphi_{(0,b)}}[F]$. In block matrix form, we have

$$DF(z,w) = \begin{bmatrix} Df(z) & 0 \\ wDg(z) & g(z)I_m \end{bmatrix}, \qquad (z,w) \in \mathbb{B}_{n+m}. \tag{2.1}$$

Clearly, $F \in \mathcal{LS}(\mathbb{B}_{n+m})$. For all $(z,w) \in \mathbb{B}_{n+m}$ and $(u,v) \in \mathbb{C}^{n+m}$, differentiation of $DF(z,w)(u,v) = (Df(z)u, wDg(z)u + g(z)v)$ with respect to (z,w) gives

$$D^2F(z,w)((u,v),\cdot) = \begin{bmatrix} D^2f(z)(u,\cdot) & 0 \\ wD^2g(z)(u,\cdot) + vDg(z) & (Dg(z)u)I_m \end{bmatrix}. \tag{2.2}$$

Taking advantage of the fact that the operators in (2.1) and (2.2) are both lower block-triangular, we obtain

$$\operatorname{tr} DF(0,b)^{-1}D^2F(0,b)((u,v),\cdot) = \operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u, \qquad (u,v) \in \mathbb{C}^{n+m}.$$

A direct calculation using (1.1) reveals $D\varphi_{(0,b)}(0,0) = (0,b)(0,b)^* - T_{(0,b)}$.

If $P \in L(\mathbb{C}^{n+m}, \mathbb{C}^n)$ is given by $P(z,w) = z$ for $(z,w) \in \mathbb{C}^{n+m}$, then $PD\varphi_{(0,b)}(0,0)(u,v) = -s_b u$ for $(u,v) \in \mathbb{C}^{n+m}$. For such (u,v), we obtain

$$\operatorname{tr} DF(0,b)^{-1}D^2F(0,b)(D\varphi_{(0,b)}(0,0)(u,v),\cdot) = -s_b(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u).$$

We note that $\operatorname{tr}(u,v)(0,b)^* = \langle v,b\rangle$ for $(u,v) \in \mathbb{C}^{n+m}$. For such (u,v), Lemma 1.3 gives

$$\operatorname{tr} D\varphi_{(0,b)}(0,0)^{-1}D^2\varphi_{(0,b)}(0,0)((u,v),\cdot) = (n+m+1)\langle v,b\rangle,$$

and by Lemma 1.2, we find

$$\operatorname{tr} D^2G(0,0)((u,v),\cdot) = -s_b(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u) + (n+m+1)\langle v,b\rangle$$

for the given b.

If we write $s_x = \sqrt{1-x^2}$ for $x \in [0,1]$, we now have

$$\begin{aligned}
&\sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(0,b)}}[F](0,0)((u,v),\cdot)| \\
&\quad= \sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |-s_b(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u) + (n+m+1)\langle v,b\rangle| \\
&\quad= \sup_{b\in\mathbb{B}_m} \sup_{x\in[0,1]} \sup_{u\in\mathbb{S}_n} \sup_{v\in\mathbb{S}_m} |-s_b x(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u) \\
&\qquad\qquad\qquad + (n+m+1)s_x\langle v,b\rangle| \\
&\quad= \sup_{b\in\mathbb{B}_m} \sup_{x\in[0,1]} \left(xs_b \sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u| + (n+m+1)\|b\|s_x \right).
\end{aligned}$$

It is a simple exercise in calculus to see that the function $h\colon [0,1]\times[0,1]\to\mathbb{R}$ given by $h(x,y)=\alpha x\sqrt{1-y^2}+\beta y\sqrt{1-x^2}$, where $\alpha,\beta\geq 0$, attains its maximum value at (at least) one of the points $(1,0)$ or $(0,1)$. From (2.2), we see that

$$\sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2F(0,0)((u,v),\cdot)| = \sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2f(0)(u,\cdot)+mDg(0)u|,$$

giving the result. $\square$

We now present our main result.

Theorem 2.3. *Let $\mathcal{F}\subseteq\mathcal{LS}(\mathbb{B}_n)$ be a linear-invariant family and $\beta\in\mathbb{C}$. If $\operatorname{ord}\mathcal{F}<\infty$, then*

$$\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]] = |1+m\beta|\operatorname{ord}\mathcal{F}+\frac{m|1-\beta(n+1)|}{2}.$$

If $\operatorname{ord}\mathcal{F}=\infty$, then

$$\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]] = \begin{cases} \infty & \text{if } \beta\neq -1/m, \\ (n+m+1)/2 & \text{if } \beta=-1/m. \end{cases}$$

Proof. Let $f\in\mathcal{F}$ and $F=\Phi_{n,m,\beta}[f]$. Now choose $a\in\mathbb{B}_n$ and put $G=\Lambda_{\varphi_{(a,0)}}[F]$. For clarity, we write $\psi_a\in\operatorname{Aut}\mathbb{B}_n$ for the involution described in (1.1) to distinguish it from members of $\operatorname{Aut}\mathbb{B}_{n+m}$. Using (1.1), we obtain

$$\varphi_{(a,0)}(z,w)=\left(\psi_a(z),\frac{-s_aw}{1-\langle z,a\rangle}\right),\qquad (z,w)\in\mathbb{B}_{n+m}.$$

We calculate

$$D\varphi_{(a,0)}(0,0)=\begin{bmatrix} D\psi_a(0) & 0 \\ 0 & -s_aI_m \end{bmatrix},\qquad DF(a,0)=\begin{bmatrix} Df(a) & 0 \\ 0 & [Jf(a)]^\beta I_m \end{bmatrix}.$$

It follows that G can be written in terms of $\Lambda_{\psi_a}[f]$ and a function $g\in H(\mathbb{B}_n)$ as follows:

$$G(z,w)=(\Lambda_{\psi_a}[f](z),g(z)w),\qquad g(z)=\frac{[Jf(\psi_a(z))]^\beta}{[Jf(a)]^\beta(1-\langle z,a\rangle)},\qquad (z,w)\in\mathbb{B}_{n+m}.$$

Now G has the general form considered in Lemma 2.2, allowing us to take advantage of the second-derivative expression (2.2) to write

$$\operatorname{tr} D^2G(0,0)((u,v),\cdot)=\operatorname{tr} D^2\Lambda_{\psi_a}[f](0)(u,\cdot)+mDg(0)u,\qquad (u,v)\in\mathbb{S}_{n+m}.$$

If $\Omega\subseteq\mathbb{C}$ is open and $A\colon\Omega\to L(\mathbb{C}^n)$ is analytic and such that $A(\zeta)$ is invertible for all $\zeta\in\Omega$, then Jacobi's formula for differentiation of the determinant of A (see [6]) is

$$\frac{d}{d\zeta}\det A(\zeta)=\det A(\zeta)\operatorname{tr}[A(\zeta)^{-1}A'(\zeta)],\qquad \zeta\in\Omega.$$

We apply this for $z \in \mathbb{B}_n$ and $u \in \mathbb{C}^n$ to obtain

$$\begin{aligned} D[Jf](z)u &= \sum_{k=1}^{n} u_k \frac{\partial}{\partial z_k} Jf(z) \\ &= \sum_{k=1}^{n} u_k Jf(z) \operatorname{tr}\left(Df(z)^{-1} \frac{\partial}{\partial z_k} Df(z) \right) \\ &= \sum_{k=1}^{n} u_k Jf(z) \operatorname{tr} Df(z)^{-1} D^2 f(z)(e_k, \cdot) \\ &= Jf(z) \operatorname{tr} Df(z)^{-1} D^2 f(z)(u, \cdot). \end{aligned}$$

Observe that Lemmas 1.2 and 1.3 show

$$\begin{aligned} \operatorname{tr} Df(a)^{-1} D^2 f(a)(D\psi_a(0)u, \cdot) &= \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) - \operatorname{tr}(\langle u, a\rangle I_n + ua^*) \\ &= \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) - (n+1)\langle u, a\rangle \end{aligned}$$

for $u \in \mathbb{C}^n$. For such u, we now can compute

$$\begin{aligned} Dg(0)u &= \frac{\beta D[Jf](a) D\psi_a(0)u}{Jf(a)} + \langle u, a\rangle \\ &= \beta \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) + (1 - \beta(n+1))\langle u, a\rangle. \end{aligned}$$

It follows that

$$\operatorname{tr} D^2 G(0,0)((u,v), \cdot) = (1 + m\beta) \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) + m(1 - \beta(n+1))\langle u, a\rangle$$

for all $(u, v) \in \mathbb{S}_{n+m}$.

From the above, we now have

$$\begin{aligned} &\sup_{f \in \mathcal{F}} \sup_{a \in \mathbb{B}_n} \sup_{(u,v) \in \mathbb{S}_{n+m}} |\operatorname{tr} D^2 \Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v), \cdot)| \\ &\quad = \sup_{f \in \mathcal{F}} \sup_{a \in \mathbb{B}_n} \sup_{u \in \mathbb{S}_n} |(1 + m\beta) \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) + m(1 - \beta(n+1))\langle u, a\rangle|. \end{aligned} \tag{2.3}$$

In the case that $\alpha = \operatorname{ord} \mathcal{F} < \infty$, we use $\Lambda_{\psi_a}[f] \in \mathcal{F}$ for all $f \in \mathcal{F}$, (1.2), and (2.3) to see that

$$\begin{aligned} &\sup_{f \in \mathcal{F}} \sup_{a \in \mathbb{B}_n} \sup_{(u,v) \in \mathbb{S}_{n+m}} |\operatorname{tr} D^2 \Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v), \cdot)| \\ &\quad \le 2\alpha|1 + m\beta| + m|1 - \beta(n+1)|. \end{aligned} \tag{2.4}$$

Let $\varepsilon \in (0, 2\alpha)$, and set $\theta = \arg(1 + m\beta)$ and $\eta = \arg(1 - \beta(n+1))$. There exist $g_0 \in \mathcal{F}$ and $u_0 \in \mathbb{S}_n$ such that

$$e^{i\theta} \operatorname{tr} D^2 g_0(0)(u_0, \cdot) \ge 2\alpha - \varepsilon.$$

Let $a_0 = te^{i\eta} u_0 \in \mathbb{B}_n$ for $t \in (0,1)$ and $f_0 = \Lambda_{\psi_{a_0}}[g_0]$. Since ψ_{a_0} is an involution, we have $g_0 = \Lambda_{\psi_{a_0}}[f_0]$, and hence

$$\begin{aligned} &|(1 + m\beta) \operatorname{tr} D^2 \Lambda_{\psi_{a_0}}[f_0](0)(u_0, \cdot) + m(1 - \beta(n+1))\langle u_0, a_0\rangle| \\ &\quad \ge (2\alpha - \varepsilon)|1 + m\beta| + mt|1 - \beta(n+1)|. \end{aligned}$$

The arbitrary choices of ε and t and (2.3) show that equality is attained in (2.4).

In the case that $\operatorname{ord}\mathcal{F}=\infty$ and $\beta=-1/m$, we use (2.3) to see that

$$\begin{aligned}\sup_{f\in\mathcal{F}}\sup_{a\in\mathbb{B}_n}\sup_{(u,v)\in\mathbb{S}_{n+m}}&|\operatorname{tr}D^2\Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)|\\&=m|1-\beta(n+1)|\\&=n+m+1.\end{aligned}\tag{2.5}$$

If $\operatorname{ord}\mathcal{F}=\infty$ and $\beta\neq-1/m$, letting $a=0$ in the supremum in (2.3) shows

$$\begin{aligned}\sup_{f\in\mathcal{F}}\sup_{a\in\mathbb{B}_n}\sup_{(u,v)\in\mathbb{S}_{n+m}}&|\operatorname{tr}D^2\Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)|\\&\geq\sup_{f\in\mathcal{F}}\sup_{u\in\mathbb{S}_n}|1+m\beta||\operatorname{tr}D^2f(0)(u,\cdot)|\\&=\infty\end{aligned}\tag{2.6}$$

Now let $\mathcal{A}=\{\varphi_{(a,0)}\circ\varphi_{(0,b)}:a\in\mathbb{B}_n,\ b\in\mathbb{B}_m\}\subseteq\operatorname{Aut}\mathbb{B}_{n+m}$. For any $a\in\mathbb{B}_n$ and $b\in\mathbb{B}_m$, we have

$$(\varphi_{(a,0)}\circ\varphi_{(0,b)})(0,0)=\varphi_{(a,0)}(0,b)=(a,-s_ab).$$

As noted above, $\Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]]$ has the form in Lemma 2.2 for all $a\in\mathbb{B}_n$. Since $\{(a,-s_ab):a\in\mathbb{B}_n,\ b\in\mathbb{B}_m\}=\mathbb{B}_{n+m}$, we use Lemma 2.1 along with Lemma 2.2 to find that

$$\begin{aligned}&\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]\\&\quad=\frac{1}{2}\sup_{f\in\mathcal{F}}\sup_{a\in\mathbb{B}_n}\sup_{b\in\mathbb{B}_m}\sup_{(u,v)\in\mathbb{S}_{n+m}}|\operatorname{tr}D^2(\Lambda_{\varphi_{(0,b)}}\circ\Lambda_{\varphi_{(a,0)}})[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)|\\&\quad=\frac{1}{2}\sup_{f\in\mathcal{F}}\sup_{a\in\mathbb{B}_n}\max\Bigg\{n+m+1,\\&\qquad\qquad\sup_{(u,v)\in\mathbb{S}_{n+m}}|\operatorname{tr}D^2\Lambda_{\varphi_{(0,a)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)|\Bigg\}.\end{aligned}$$

The result for the case $\operatorname{ord}\mathcal{F}=\infty$ immediately follows from (2.5) and (2.6).

If $\alpha=\operatorname{ord}\mathcal{F}<\infty$, the equality in (2.4) observed earlier implies

$$\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]=\frac{1}{2}\max\{n+m+1,2\alpha|1+m\beta|+m|1-\beta(n+1)|\}.$$

Using $\alpha\geq(n+1)/2$, we have

$$\begin{aligned}2\alpha|1+m\beta|+m|1-\beta(n+1)|&\geq|n+1+m\beta(n+1)|+|m-m\beta(n+1)|\\&\geq|n+1+m|.\end{aligned}\tag{2.7}$$

This gives the result in this case. □

Example 2.4. To utilize Theorem 2.3 in a concrete manner, one must possess knowledge of the order of a linear-invariant family $\mathcal{F}\subseteq\mathcal{LS}(\mathbb{B}_n)$. One may consult [5, 8, 17] among other references for examples of linear-invariant families of various orders. We will simply take note of the well-known results that when $n = 1$ and $\alpha \geq 1$, we have $\operatorname{ord} \Lambda[\{k_\alpha\}] = \alpha$, where $k_\alpha \in \mathcal{LS}(\mathbb{D})$ is the generalized Koebe function

$$k_\alpha(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right], \qquad z \in \mathbb{D},$$

where the principal branch of the power is used. (See [20].) In addition (see [2], for instance), if $g \in \mathcal{LS}(\mathbb{D})$ is given by

$$g(z) = \frac{1}{2}\left(1 - \exp\frac{-2z}{1-z}\right), \qquad z \in \mathbb{D},$$

then $\operatorname{ord} \Lambda[\{g\}] = \infty$.

We now observe an immediate consequence.

Corollary 2.5. *Let $\mathcal{F}$ be a linear-invariant family of minimum order on $\mathbb{B}_n$ and $\beta \in \mathbb{C}$. Then $\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]$ is a linear-invariant family of minimum order on $\mathbb{B}_{n+m}$ if and only if $\beta \in [-1/m, 1/(n+1)]$.*

Proof. Clearly, $\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]$ has minimum order if and only if both inequalities in (2.7) are equalities. This holds for the first inequality because $\alpha = (n+1)/2$.

Equality in the second inequality occurs if and only if either the complex numbers $n+1+m\beta(n+1)$ and $m - m\beta(n+1)$ have the same argument or one of them is equal to 0. This clearly occurs if and only if both numbers are nonnegative real numbers, which coincides with $\beta \in [-1/m, 1/(n+1)]$. □

Remark 2.6. It is worth considering the special case where $\beta = -1/m$ in Theorem 2.3. Indeed, for any linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$, the family $\Lambda[\Phi_{n,m,-1/m}[\mathcal{F}]]$ has minimum order on $\mathbb{B}_{n+m}$. This includes the family $\Lambda[\Phi_{n,m,-1/m}[\mathcal{LS}(\mathbb{B}_n)]]$. This is not as surprising as it may seem, for if $F \in \Phi_{n,m,-1/m}[\mathcal{LS}(\mathbb{B}_n)]$, we can calculate $JF(z,w) = 1$ for all $(z,w) \in \mathbb{B}_{n+m}$. It is known [5, 17] that the linear-invariant family generated by

$$\mathcal{G} = \{f \in \mathcal{LS}(\mathbb{B}_n) : Jf(z) = 1 \text{ for all } z \in \mathbb{B}_n\}$$

has minimum order.

3. Compositions with automorphisms of $\mathbb{C}^n$

Consider the following subgroup of $\operatorname{Aut}\mathbb{C}^n$ and subspace of $H(\mathbb{C}^n)$:

$$\begin{aligned} \operatorname{Aut}_1 \mathbb{C}^n &= \{\Psi \in \operatorname{Aut}\mathbb{C}^n : \Psi(0) = 0,\ D\Psi(0) = I_n, \text{ and } J\Psi(z) = 1 \text{ for } z \in \mathbb{C}^n\}, \\ H_0(\mathbb{C}^n) &= \{G \in H(\mathbb{C}^n) : G(0) = 0 \text{ and } DG(0) = 0\}. \end{aligned}$$

Notable members of $\operatorname{Aut}_1 \mathbb{C}^n$ for $n \geq 2$ are the normalized shears (see [22]) given by

$$\Psi_G(z) = (z_1 + G(\hat{z}), \hat{z}), \qquad z \in \mathbb{C}^n, \tag{3.1}$$

where $G \in H_0(\mathbb{C}^{n-1})$.

As a consequence of the following simple result, we see that the linear-invariant family generated by the composition a linear-invariant family with a member of

$\mathrm{Aut}_1\,\mathbb{C}^n$ has the same order as the original family. For notational convenience, if $\mathcal{F} \subseteq H(\mathbb{B}_n, \mathbb{C}^n)$ and $\Psi \in H(\mathbb{C}^n, \mathbb{C}^n)$, then we write

$$\Psi \circ \mathcal{F} = \{\Psi \circ f : f \in \mathcal{F}\}.$$

The proof relies on the following observation.

Remark 3.1. From the work done in [5] leading to the conclusion of Theorem 1.4 above, the following deduction can be drawn: *If $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ is a linear-invariant family and there exist $\alpha \geq (n+1)/2$ and $r \in (0,1)$ such that*

$$|Jf(z)| \leq \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}, \qquad f \in \mathcal{F},\ z \in B_n(0;r),$$

then $\operatorname{ord}\mathcal{F} \leq \alpha$. Clearly, the analogous result also holds using the lower estimate of the Jacobian from Theorem 1.1.

Proposition 3.2. *If $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ and $\Psi \in \mathrm{Aut}_1\,\mathbb{C}^n$, then* $\operatorname{ord}\Lambda[\Psi \circ \mathcal{F}] = \operatorname{ord}\Lambda[\mathcal{F}]$.

Proof. Let $g = \Lambda_\varphi[\Psi \circ f]$ for some $\varphi \in \mathrm{Aut}\,\mathbb{B}_n$ and $f \in \mathcal{F}$. For all $z \in \mathbb{B}_n$,

$$\begin{aligned} Jg(z) &= \frac{J\Psi(f(\varphi(z)))Jf(\varphi(z))J\varphi(z)}{J\Psi(f(\varphi(0)))Jf(\varphi(0))J\varphi(0)} \\ &= \frac{Jf(\varphi(z))J\varphi(z)}{Jf(\varphi(0))J\varphi(0)} \\ &= J\Lambda_\varphi[f](z). \end{aligned} \tag{3.2}$$

Suppose $\alpha = \operatorname{ord}\Lambda[\mathcal{F}] < \infty$. Any $g \in \Lambda[\Psi \circ \mathcal{F}]$ has the form above, and, by Theorem 1.1, we have

$$|Jg(z)| = |J\Lambda_\varphi[f](z)| \leq \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}, \qquad z \in \mathbb{B}_n,$$

using (3.2). By Remark 3.1, we conclude that $\operatorname{ord}\Lambda[\Psi\circ\mathcal{F}] \leq \alpha$. Since $\Psi^{-1} \in \mathrm{Aut}_1\,\mathbb{C}^n$, we apply the same argument as above to obtain

$$\alpha = \operatorname{ord}\Lambda[\mathcal{F}] = \operatorname{ord}\Lambda[\Psi^{-1}\circ(\Psi\circ\mathcal{F})] \leq \operatorname{ord}\Lambda[\Psi\circ\mathcal{F}],$$

as needed in this case.

If $\operatorname{ord}\Lambda[\mathcal{F}] = \infty$, then let $\alpha \geq (n+1)/2$. By Remark 3.1, there exist $f \in \mathcal{F}$, $\varphi \in \mathrm{Aut}\,\mathbb{B}_n$, and $z \in \mathbb{B}_n$ such that

$$|Jg(z)| = |J\Lambda_\varphi[f](z)| > \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}$$

for $g = \Lambda_\varphi[\Psi\circ f] \in \Lambda[\Psi\circ\mathcal{F}]$ by (3.2). It follows from Theorem 1.1 that $\operatorname{ord}\Lambda[\Psi\circ\mathcal{F}] > \alpha$. The arbitrary choice of α implies the result in this case. □

Remark 3.3. Godula, Liczberski, and Starkov [5] used Theorem 1.4 in a similar manner to argue that if $f, g \in \mathcal{LS}(\mathbb{B}_n)$ are such that $Jf(z) = Jg(z)$ for all $z \in \mathbb{B}_n$, then $\operatorname{ord}\Lambda[\{f\}] = \operatorname{ord}\Lambda[\{g\}]$. While their proof depends on the families being assumed to have finite order, the case of infinite order can be addressed as we have in the proof of Proposition 3.2. Were this done, one could also prove Proposition 3.2 using their result.

A notable point of interest in the following corollary is that the family $\mathcal{G}$ is linear-invariant without needing to be generated by a smaller family.

Corollary 3.4. *Let $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ be a linear-invariant family. Then*

$$\mathcal{G} = \bigcup_{\Psi \in \operatorname{Aut}_1 \mathbb{C}^n} \Psi \circ \mathcal{F}$$

is a linear-invariant family and $\operatorname{ord} \mathcal{G} = \operatorname{ord} \mathcal{F}$.

Proof. Once we verify that $\mathcal{G}$ is a linear-invariant family, the order claim follows from Proposition 3.2. Let $f \in \mathcal{F}$, $\Psi \in \operatorname{Aut}_1 \mathbb{C}^n$, and $\varphi \in \operatorname{Aut} \mathbb{B}_n$. For any $a \in \mathbb{C}^n$, let $S_a \in \operatorname{Aut} \mathbb{C}^n$ be the translation given by $S_a(z) = z + a$. Let

$$\begin{aligned}\Psi_0 = {} & D\varphi(0)^{-1} Df(\varphi(0))^{-1} D\Psi(f(\varphi(0)))^{-1} \circ S_{-\Psi(f(\varphi(0)))} \\ & \circ \Psi \circ S_{f(\varphi(0))} \circ Df(\varphi(0)) D\varphi(0).\end{aligned}$$

Then it is elementary to see that $\Psi_0 \in \operatorname{Aut}_1 \mathbb{C}^n$. A direct calculation reveals $\Lambda_\varphi[\Psi \circ f] = \Psi_0 \circ \Lambda_\varphi[f] \in \mathcal{G}$, as needed. □

Remark 3.5. Proposition 3.2 and Corollary 3.4 are only interesting if $n \geq 2$. Indeed, if $n = 1$, then $\operatorname{Aut}_1 \mathbb{C} = \{I_1\}$ and thus $\mathcal{G} = \mathcal{F}$ in Corollary 3.4.

When $n \geq 2$, Corollary 3.4 makes clear that linear-invariant families of finite order on $\mathbb{B}_n$ are not, in general, normal families no matter the order. (This has previously been established; see [18], for instance.) Indeed, for any function $G \in H_0(\mathbb{C}^{n-1})$, consider the shear Ψ_G as in (3.1). Let $\hat{P} \in L(\mathbb{C}^n, \mathbb{C}^{n-1})$ be given by $\hat{P}z = \hat{z}$ for $z \in \mathbb{C}^n$. For any element f of the linear-invariant family $\mathcal{G}$ in Corollary 3.4 and G as described here such that $G(\hat{P}f(z)) \neq 0$ for some $z \in \mathbb{B}_n$, we have that $\Psi_{tG} \circ f \in \mathcal{G}$ for all $t > 0$. Clearly, $\{\Psi_{tG} \circ f : t > 0\}$ is not locally uniformly bounded.

Note that this is in contrast to linear-invariant families with finite norm order, which must be normal families as noted in Section 1. Furthermore, since the norm order agrees with the order when $n = 1$, the above observation only holds for $n \geq 2$.

Many extension operators studied in conjunction with the theories of geometric mappings and Loewner chains are of the form $\Phi[f] = \Psi_G \circ \Phi_{1,n-1,\beta}[f]$ for $f \in \mathcal{LS}(\mathbb{D})$ where $G \in H_0(\mathbb{C}^{n-1})$ and $\beta \in [0, 1/2]$. (See [13] for instance.) We now see from Theorem 2.3 and Proposition 3.2 that the extension operator $\Phi_{n,m,\beta,G} \colon \mathcal{LS}(\mathbb{B}_n) \to \mathcal{LS}(\mathbb{B}_{n+m})$ given by

$$\Phi_{n,m,\beta,G}[f](z,w) = (f(z) + G([Jf(z)]^\beta w), [Jf(z)]^\beta w), \qquad (z,w) \in \mathbb{B}_{n+m},$$

where $\beta \in \mathbb{C}$ and $G \in H_0(\mathbb{C}^m)$, is such that for any linear-invariant family $\mathcal{F}$ on $\mathbb{B}_n$, we have

$$\operatorname{ord} \Lambda[\Phi_{n,m,\beta,G}[\mathcal{F}]] = \operatorname{ord} \Lambda[\Psi_G \circ \Phi_{n,m,\beta}[\mathcal{F}]] = \alpha|1 + m\beta| + \frac{m|1 - \beta(n+1)|}{2}$$

if $\alpha = \operatorname{ord} \mathcal{F} < \infty$ and, likewise,

$$\operatorname{ord} \Lambda[\Phi_{n,m,\beta,G}[\mathcal{F}]] = \begin{cases} \infty & \text{if } \beta \neq -1/m, \\ (n+m+1)/2 & \text{if } \beta = -1/m, \end{cases}$$

if $\operatorname{ord} \mathcal{F} = \infty$. In particular, this process produces families of minimum order on $\mathbb{B}_{n+m}$ when beginning with a linear-invariant family of minimum order on $\mathbb{B}_n$ for all

$G \in H_0(\mathbb{C}^m)$ and $\beta \in [-1/m, 1/(n+1)]$ (see Corollary 2.5) or when beginning with any linear-invariant family for all $G \in H_0(\mathbb{C}^m)$ and $\beta = -1/m$.

This is of note even when extending linear-invariant families on $\mathbb{D}$ (in which case Theorem 2.3 reduces to a result in [7] if $\beta \in [0, 1/2]$, as noted previously). Recall from Section 1 that $\mathcal{K}(\mathbb{B}_n)$ is a linear-invariant family that has minimum order if and only if $n = 1$. We see that $\Lambda[\Phi_{1,n-1,\beta,G}[\mathcal{K}(\mathbb{D})]]$ is a linear-invariant family of minimum order on $\mathbb{B}_n$ for any $\beta \in [-1/(n-1), 1/2]$ and $G \in H_0(\mathbb{C}^{n-1})$ if $n \geq 2$. For $G = 0$ and $\beta \geq 0$, this family is a subset of $\mathcal{K}(\mathbb{B}_n)$ if and only if $\beta = 1/2$, as shown in [7] (see also [10]). For $\beta = 1/2$, it was shown in [15] (see also [14]) that this family is a subset of $\mathcal{K}(\mathbb{B}_n)$ if and only if $G = Q$, where $Q\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 such that

$$\|Q\| = \sup_{u \in \mathbb{S}_{n-1}} |Q(u)| \leq \frac{1}{2}.$$

Thus, for many choices of β and G, $\Lambda[\Phi_{1,n-1,\beta,G}[\mathcal{K}(\mathbb{D})]]$ is a linear-invariant family of minimum order not lying in $\mathcal{K}(\mathbb{B}_n)$, while $\Lambda[\Phi_{1,n-1,1/2,Q}[\mathcal{K}(\mathbb{D})]]$ is a linear-invariant family of minimum order lying within $\mathcal{K}(\mathbb{B}_n)$ when $\|Q\| \leq 1/2$. We note that $\Lambda[\Phi_{1,n-1,1/2,Q}[\mathcal{K}(\mathbb{D})]]$ was noted by Kohr to have minimum order (without proof) for $\|Q\|$ of any size in [12].

In view of Corollary 3.4, we also see that

$$\bigcup_{\Psi \in \mathrm{Aut}_1 \mathbb{C}^n} \ \bigcup_{\beta \in [-1/(n-1), 1/2]} \Psi \circ \Lambda[\Phi_{1,n-1,\beta}[\mathcal{K}(\mathbb{D})]]$$

is a linear-invariant family of rather substantial size of minimum order on $\mathbb{B}_n$.

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