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On a pure traction problem for the nonlinear elasticity system in Sobolev spaces with variable exponents

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Abstract. The paper deals with a nonlinear elasticity system with nonconstant coefficients. The existence and uniqueness of the solution of Neumann's problem is proved using Galerkin techniques and monotone operator theory, in Sobolev spaces with variable exponents.

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1. Introduction

The study of PDE problems with variable exponents is a novel and quite interesting topic. It comes from the theory of nonlinear elasticity, elastic mechanics, fluid dynamics, electrorheological fluids, and image processing, etc. (see [1], [15], [16]). First, we introduce the notations needed in this article. Let Ω an connected open bounded domain of $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{N}=3$) with Lipschitz boundary Γ. To a given field of displacement u, we associate a nonlinear deformation tensor E defined by

$$E\left(\nabla u(x)\right)=\frac{1}{2}\left(\nabla u^{T}+\nabla u+\nabla u^{T}\nabla u\right),$$

whose components are:

$$E_{ij}\left(\nabla u(x)\right)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+\sum_{m=1}^{3}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right),\ 1\leq i,j\leq 3. \tag{1.1}$$

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The corresponding nonlinear constraints tensor $\sigma(u) = (\sigma_{ij}(u(x)))_{1\leq i,j\leq 3}$ is then given by:

$$\sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)),\ 1 \leq i,j \leq 3, \tag{1.2}$$

which describes a nonlinear relation between the stress tensor $(\sigma_{ij})_{i,j=1,2,3}$ and the deformation tensor $(E_{ij})_{i,j=1,2,3}$. The coefficients of elasticity a_{ijkh} satisfy the following symmetry properties:

$$a_{ijkh} = a_{jikh} = a_{ijhk},\ \text{for all } 1 \leq i,j,k,h \leq 3. \tag{1.3}$$

The aim of this paper is to prove the existence and uniqueness of weak solutions for the following nonlinear elliptic problem, encountered in the theory of nonlinear elasticity:

$$\begin{cases} -\sum_{j=1}^{3} \dfrac{\partial}{\partial x_j}\sigma_{ij}(u(x)) = f_i(x,u(x)) \text{ in } \Omega,\ 1 \leq i \leq 3, \\ \sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)) \text{ in } \Omega,\ 1 \leq i,j \leq 3, \\ E_{ij}\left(\nabla u(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \text{ in } \Omega,\ 1 \leq i,j \leq 3, \\ \sum_{j=1}^{3} \sigma_{ij}\left(u\left(x\right)\right)\eta_j = 0 \text{ on } \Gamma,\ 1 \leq i \leq 3. \end{cases} \tag{P}$$

Problem (P) models the behavior of a heterogeneous material with Neumann's condition on the boundary. The consideration of this general material is in no way restrictive. Indeed, we can applied this study to the most particular elastic materials, but this particular case makes it easy, to describe the different stages of this work. The tensor of the constraints considered here is nonlinear and grouped, as special cases, some models used in Ciarlet [2], Dautry-Lions [4] and Lions [10]. Let us cite by way of example (see [2] , [8]):

1. The problem of displacement for a homogeneous or heterogeneous material of St Vennan-Kirchhoff where:
   - the applied volumetric forces f are dead (does not depend on u),
   - the tensor of stress is in the form (material of StVennan-Kirchhoff):

$$\begin{cases} \sigma_{ij}(u(x)) = \lambda(trE_{ij}(\nabla u(x))) + 2\mu E_{ij}(\nabla u(x)), \\ \qquad 1 \leq i,j \leq 3,\ \lambda > 0,\ \mu > 0, \end{cases}$$

2. The coefficients of elasticity have the form:

$$a_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}), 1 \leq i,\ j,\ p,\ q \leq 3$$

   with, λ and μ depend on x or not,
3. The applied volumetric forces f have the form $f(\xi) = |\xi|^{p(x)-1}\,\xi$,
4. Some models called "LES" (Large Eddy Simulations) used in fluid mechanics. These problems are:

$$-\operatorname{div}(\psi(x)a(\nabla u(x))) = f(x).$$

For $\psi \equiv 1$ and $a(\xi) = |\xi|^{p(x)-2}\,\xi$, the above equation may be described by:

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\,\nabla u) = f.$$

The operator $\Delta_{p(x)} : u \longrightarrow \Delta_{p(x)}(u) = \operatorname{div}(|\nabla u|^{p(x)-2}\,\nabla u)$ is called the $p\,(x)$-Laplacian.

Several authors studied the system of elasticity with laws of particular behavior and using various techniques in constant exposants Sobolev spaces for example in [2] Ciarlet used the implicit function theorem to show the existence and uniqueness of a solution, in [4] Dautry-Lions studied the linear problem in a regular boundary domain, in [11], [12], [13] Merouani studied the Lamé (elasticity) system in a polygonal boundary domain.

The bibliography quoted here does not claim to be exhaustive and the deficiencies it certainly entails must be attributed to the author's ignorance and not to the author's ill will.

To solve our problem, we will consider an operator: $u \to A(u) = -\sum\limits_{j=1}^{3} \dfrac{\partial}{\partial x_j}\sigma_{ij}(u(x))$ as operator of Leray-Lions [9], with Neumann's condition on Γ, and we prove a theorem of existence and uniqueness of solution using Galerkin techniques and monotone operator theory.

This paper is organized as follows:

- Notations and properties of variable exponent Lebesgue-Sobolev spaces,
- Hypotheses and main result,
- Proof of theorem,
- Conclusion and bibliography.

2. Properties of variable exponent Lebesgue-Sobolev spaces

In this section, we recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, when Ω is a bounded open set of $\mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 1)$ with a smooth boundary.

Let $p : \overline{\Omega} \to [1, +\infty)$ be a continuous, real-valued function.

Denote by $p_- = \min\limits_{x\in\overline{\Omega}} p(x)$ and $p_+ = \max\limits_{x\in\overline{\Omega}} p(x)$.

We introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; u \text{ is measurable with } \int_\Omega |u\,(x)|^{p(x)}\,dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u\,(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

The following inequality will be used later

$$\min\left\{ \|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)} \right\} \leq \int_\Omega |u\,(x)|^{p(x)}\,dx \leq \max\left\{ \|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)} \right\}$$

for any $u \in L^{p(x)}(\Omega)$.

Lemma 2.1. [3], [5], [6], [7]

- *The space* $\left(L^{p(x)}(\Omega), \|.\|_{L^{p(x)}(\Omega)}\right)$ *is a Banach space.*
- *If* $p_- > 1$, *then* $L^{p(x)}(\Omega)$ *is reflexive and its conjugate space can be identified with* $L^{p'(x)}(\Omega)$ *where,* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. *Moreover, for any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$, *we have the Hölder inequality*

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \leq 2 \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

- *If* $p_+ < +\infty$, *then* $L^{p(x)}(\Omega)$ *is separable.*
- *Some embedding stay true, for example, if* $0 < |\Omega| < \infty$ *and* p_1, p_2 *are variable exponent so that* $p_1(x) \leq p_2(x)$ *almost everywhere in* Ω, *then we have* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, we define also the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega);\ \ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

endowed with the following norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Definition 2.2. The variable exponent $p : \overline{\Omega} \to [1, +\infty)$ is said to satisfy the log-Hölder continuous condition if

$$\forall x, y \in \overline{\Omega},\ \ |x - y| < 1,\ \ |p(x) - p(y)| < w(|x - y|),$$

where $w : (0, \infty) \to \mathbb{R}$ is a nondecreasing function with $\lim_{\alpha \to 0} \sup w(\alpha) \ln\left(\frac{1}{\alpha}\right) < \infty$.

Lemma 2.3. [3], [5], [6], [7]

- *If* $1 < p_- \leq p_+ < \infty$, *then the space* $\left(W^{1,p(x)}(\Omega), \|.\|_{W^{1,p(x)}(\Omega)}\right)$ *is a separable and reflexive Banach space.*
- *If* $p(x)$ *satisfies the log-Hölder continuous condition, then* $C^\infty(\Omega)$ *is dense in* $W^{1,p(x)}(\Omega)$. *Moreover, we can define the Sobolev space with zero boundary values,* $W_0^{1,p(x)}(\Omega)$ *as the completion of* $C_0^\infty(\Omega)$ *with respect to the norm* $\|.\|_{W^{1,p(x)}(\Omega)}$.
- *For all* $u \in W_0^{1,p(x)}(\Omega)$, *the Poincaré inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)},$$

holds. Moreover, $\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}$ *is a norm in* $W_0^{1,p(x)}(\Omega)$.

Throughout this paper, we shall assume that the variable exponent $p(x)$ satisfy the log-Hölder condition, and $\mathbb{N} < p_- \leq p_+ < \infty$ because if $p(x) > \mathbb{N}$ then $W^{1,p(x)}(\Omega) \subset C(\Omega)$ for every $x \in \Omega$.

3. Hypotheses and main result

We consider the following problem:

$$\begin{cases} -\sum\limits_{j=1}^{3} \dfrac{\partial}{\partial x_j} \sigma_{ij}(u(x)) = f_i(x, u(x)) \text{ in } \Omega,\ 1 \le i \le 3, \\ \sigma_{ij}(u(x)) = \sum\limits_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)) \text{ in } \Omega,\ 1 \le i, j \le 3, \\ E_{ij}(\nabla u(x)) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum\limits_{m=1}^{3} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \text{ in } \Omega,\ 1 \le i, j \le 3, \\ \sum\limits_{j=1}^{3} \sigma_{ij}(u(x))\, \eta_j = 0 \text{ on } \Gamma,\ 1 \le i \le 3. \end{cases} \tag{3.1}$$

This problem being that of Neumann, we must impose the necessary conditions of existence namely the condition of compatibility:

$$\int_\Omega f dx = 0.$$

This is the hypotheses which concern E_{kh} and f :

$$\begin{cases} \forall i, j, k, h = 1 \text{ to } 3 : \\ 1)\ E_{kh} \text{ is a continuous function,} \\ 2)\ \text{(Coercivity) } \exists \alpha > 0;\ \text{such that } E_{kh}(\xi)\, \xi_{ij} \ge \alpha\, |\xi|^{p(x)}, \\ \quad \forall \xi \in \mathbb{R}^{3\times 3} \text{ and, } \xi_{ij} \in \mathbb{R}, \\ 3)\ \text{(Increase) } \exists C \in \mathbb{R};\ |E_{kh}(\xi)| \le C\left(1 + |\xi|^{p(x)-1}\right), \\ 4)\ (E_{kh}(\xi) - E_{kh}(\eta))\,(\xi_{ij} - \eta_{ij}) \ge 0, \forall \xi, \eta \in \mathbb{R}^{3\times 3},\ \text{and} \\ \quad \xi_{ij}, \eta_{ij} \in \mathbb{R}, \\ 5)\ a_{ijkh} \in L^\infty(\Omega);\ \exists \alpha_0 > 0;\ a_{ijkh} \ge \alpha_0 \text{ a.e. in } \Omega, \\ 6)\ f = (f_1, f_2, f_3) \text{ is a Caratheodory function and,} \\ \quad f \in \left(L^{\frac{p(x)}{p(x)-1}}(\Omega) \right)^3. \end{cases} \tag{3.2}$$

Let us look for an adequate weak form of (3.1). Note that if $w \in \left(L^{p(x)}(\Omega)\right)^9$, then the growth condition on E_{kh} gives

$$\begin{aligned} |E_{kh}(w)| &\le C\left(1 + |w|^{p(x)-1}\right) \\ &\le \left(C + C\,|w|^{p(x)-1}\right) \in L^{\frac{p(x)}{p(x)-1}}(\Omega),\ 1 \le k, h \le 3. \end{aligned}$$

So, if $u \in H$, we have $E_{kh}(\nabla u) \in L^{p'(x)}(\Omega)$. Or

$$H = \left\{ u \in \left(W^{1,p(x)}(\Omega)\right)^3,\ \frac{1}{mes\,(\Omega)} \int_\Omega u(x)\, dx = 0 \right\},$$

is a closed vector subspace of $\left(W^{1,p(x)}(\Omega)\right)^3$, provided with the norm

$$\|u\|_H = \|\nabla u\|_{L^{p(x)}(\Omega)},$$

which is equivalent to the norm of $\left(W^{1,p(x)}(\Omega)\right)^3$. We note that:

$$\left(W^{1,p(x)}(\Omega)\right)^3 = H \oplus F,$$

where F is the space of constants. Let's take then $v \in H$, we have $\nabla v \in \left(L^{p(x)}(\Omega)\right)^9$. So we obtain from the inequality of Hölder:

$$E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j} \in L^1(\Omega), \forall\, i,j,k,h = 1 \text{ to } 3.$$

It is therefore natural to look $u \in H$ and take the test functions in H. We also recall that if $f(.,s) \in \left(L^{p'(x)}(\Omega)\right)^3$, the mapping $v \to \int_\Omega f(x,u(x))v(x)\,dx$ acting from H to $\mathbb{R}$, is an element of H'. We denote by f this element, that is to say for $f \in \left(L^{p'(x)}(\Omega)\right)^3$, we have

$$\langle f, v\rangle_{H',H} = \int_\Omega f(x,u(x))\,v(x)\,dx,\ \forall v \in H.$$

The weak form of (3.1) is thus:

$$\begin{cases} u \in H, \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx = \langle f, v\rangle_{H',H},\ \forall v \in H. \end{cases} \tag{3.3}$$

Theorem 3.1. *Under the hypotheses* (3.2)*, there exist* $u \in H$ *solution of* (3.3)*. If, moreover,* $(E_{kh}(\xi) - E_{kh}(\eta))(\xi_{ij} - \eta_{ij}) > 0$*, for all* $\xi, \eta \in \mathbb{R}^{3\times 3}, \xi_{ij}, \eta_{ij} \in \mathbb{R}, \xi_{ij} \neq \eta_{ij}$ *then there exist a unique solution* u *of* (3.3).

For the proof of this theorem, we will need the following (classical) integration lemmas:

Lemma 3.2. *Let* $p : \Omega \to]1,+\infty[$. *If* $f_n \to f$ *in* $L^{p(x)}(\Omega)$ *and* $g_n \to g$ *weakly in* $L^{p'(x)}(\Omega)$. *So*

$$\int_\Omega f_n g_n dx \to \int_\Omega f g dx \text{ when } n \to \infty.$$

Demonstration of lemma (3.2). We have:

$$\begin{aligned}\left|\int_\Omega (f_n\, g_n - f\, g)\,dx\right| &= \left|\int_\Omega (f_n\, g_n - f\, g - f\, g_n + f\, g_n)\,dx\right| \\ &= \left|\int_\Omega [(f_n - f)\, g_n + f\, (g_n - g)]\,dx\right| \\ &\le \int_\Omega |f_n - f|\ |g_n|\,dx + \left|\int_\Omega f\, (g_n - g)dx\right| \\ &\le 2.\|f_n - f\|_{L^{p(x)}(\Omega)}\|g_n\|_{L^{p'(x)}(\Omega)} + |\langle g_n - g, f\rangle| \to 0.\end{aligned}$$

Lemma 3.3. *If* $E_{kh} \in C\left(\mathbb{R}^{3\times 3}, \mathbb{R}\right)$, $|E_{kh}(\xi)| \le C\left(1 + |\xi|^{p(x)-1}\right)$, $k,h = 1$ *to* 3, *for all* $\xi \in \mathbb{R}^{3\times 3}$ *and if* $u_n \to u$ *in* $\left(W^{1,p(x)}(\Omega)\right)^3$ *then* $E_{kh}(\nabla u_n) \to E_{kh}(\nabla u)$, $k,h = 1$ *to* 3, *in* $L^{p'(x)}(\Omega)$.

The lemma (3.3) is proved by Lebesgue's dominated convergence theorem.

Remark 3.4. [14] Let $p \in L_{+}^{\infty}(\Omega) = \{p \in L^{\infty}(\Omega), p_{-} \geq 1\}, (u_n) \subset L^{p(x)}(\Omega)$ and $u \in L^{p(x)}(\Omega)$. If $\lim\limits_{n\to\infty} \|u_n - u\|_{L^{p(x)}(\Omega)} = 0$. Then there exist a subsequence $(u_{nj}) \subset (u_n)$ and a function $g \in L^{p(x)}(\Omega)$ such that:
(i) $u_{nj} \to u$ a.e. in Ω,
(ii) $|u_{nj}| \leq g(x)$ a.e. in Ω.

Demonstration of lemma (3.3). $u_n \to u$ in $\left(W^{1,p(x)}(\Omega)\right)^3$ involves: $u_n \to u$ in $\left(L^{p(x)}(\Omega)\right)^3$ and $\nabla u_n \to \nabla u$ in $\left(L^{p(x)}(\Omega)\right)^9$.
$\nabla u_n \to \nabla u$ in $\left(L^{p(x)}(\Omega)\right)^9$ involves $\nabla u_n \to \nabla u$ a.e. in Ω, and as E_{kh} is continuous then:

$$E_{kh}(\nabla u_n) \to E_{kh}(\nabla u) \text{ a.e., } k,h = 1 \text{ to } 3$$

we have also

$$|E_{kh}(\nabla u_n)| \leq (C + C|\nabla u_n|^{p(x)-1}) \in L^{\frac{p(x)}{p(x)-1}}(\Omega), \; k,h = 1 \text{ to } 3.$$

So we deduce that

$$E_{kh}(\nabla u_n) \to E_{kh}(\nabla u) \text{ in } L^{\frac{p(x)}{p(x)-1}}(\Omega).$$

We will also need for the proof the following lemma:

Lemma 3.5. *(Finite-dimensional coercive operator) Let V be a finite-dimensional space, and $T : V \to V'$ continuous. We suppose that T is coercive, namely:*

$$\frac{\langle T(v).v\rangle_{V',V}}{\|v\|_V} \to +\infty \text{ when } \|v\|_V \to +\infty.$$

Then, for every $b \in V'$ there exist $v \in V$ such that $T(v) = b$.

4. Proof of theorem

Study of finite dimension problem

Since H is separable, (because H is a closed vector subspace of $\left(W^{1,p(x)}(\Omega)\right)^3$, and $\left(W^{1,p(x)}(\Omega)\right)^3$ is a Banach space separable) then there exist a countable family $(f_n)_{n\in\mathbb{N}^*}$ dense in H. Let $V_n = Vect\{f_i, \; i = 1,...,n\}$ be the vector space generated by the first n functions of this family. So we have $\dim V_n \leq n$, $V_n \subset V_{n+1}$ for all $n \in \mathbb{N}^*$ and we have $\overline{\bigcup\limits_{n\in\mathbb{N}} V_n} = H$. We deduce that for all $v \in H$ there exist a sequence $v_n \in V_n$, such that $v_n \to v$ in H when $n \to +\infty$.
In the first step, we fix $n \in \mathbb{N}^*$ and look for u_n solution of the following problem, posed in finite dimension:

$$\left\{ \begin{array}{c} u_n \in V_n, \\ \int\limits_{\Omega} \sum\limits_{i,j=1}^{3} \sum\limits_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \dfrac{\partial v_i}{\partial x_j} dx = \langle f, v\rangle_{H',H}, \; \forall v \in V_n. \end{array} \right. \tag{3.4}$$

The application $v \to \langle f, v\rangle_{H',H}$ is a linear mapping of V_n to $\mathbb{R}$ (it is also continuous because $\dim V_n < +\infty$). We denote by b_n this application. So $b_n \in V_n'$ and

$$\langle b_n, v\rangle_{V_n', V_n} = \langle f, v\rangle_{H',H}.$$

Let $u \in V_n$. We denote by $T_n(u)$ the mapping of V_n into V_n' which has $v \in V_n$ associated

$$\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u(x)) \frac{\partial v_i}{\partial x_j} dx.$$

This application is linear, so it is also an element of $V_n^{'}$ and we have

$$\langle T_n(u), v\rangle_{V_n', V_n} = \int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u(x)) \frac{\partial v_i}{\partial x_j} dx.$$

We have thus defined an application T of V_n to V_n'. We shall show that T is continuous and coercive. We can thus deduce by the lemma (3.5), that T is surjective, and therefore that there exist $u_n \in V_n$ satisfying $T(u_n) = b_n$, that is to say u_n is the solution of the problem (3.4).

Continuity of T_n. To ease the writing, we note $V = V_n$ equipped with $\|u\|_V = \|u\|_H$ and note $T = T_n$. Let u, $\overline{u} \in V$, we have:

$$\begin{aligned}
\|T(u) - T(\overline{u})\|_{V'} &= \max_{v\in V,\ \|v\|_V = 1} \langle T(u) - T(\overline{u}), v\rangle_{V',V} \\
&= \max_{v\in V,\ \|v\|_H = 1} \int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)(E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u})) \frac{\partial v_i}{\partial x_j} dx, \\
&\le \max_{v\in H,\ \|v\|_H = 1} \int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)(E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u})) \frac{\partial v_i}{\partial x_j} dx.
\end{aligned}$$

Putting

$$a = \|a_{ijkh}\|_{L^\infty(\Omega)},$$

we obtain by Hölder inequality

$$\begin{aligned}
&\|T(u) - T(\overline{u})\|_{V'} \\
&\le \max_{v\in H, \|v\|_H = 1} 2a \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \|E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u})\|_{L^{p'(x)}} \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^{p(x)}(\Omega)} \\
&\le 2a \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \|E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u})\|_{L^{p'(x)}(\Omega)}.
\end{aligned}$$

Thus if $(u_n)_{n\in\mathbb{N}}$ is a sequence of V such that $u_n \to \overline{u}$ in V, we have

$$\|T(u_n) - T(\overline{u})\|_{V'} \le 2a \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \|E_{kh}(\nabla u_n) - E_{kh}(\nabla \overline{u})\|_{L^{p'(x)}(\Omega)}.$$

As the norm in H equivalent to the norm in $\left(W^{1,p(x)}(\Omega)\right)^3$, then $u_n \to \overline{u}$ in V involves $u_n \to \overline{u}$ in $\left(W^{1,p(x)}(\Omega)\right)^3$.

In view of lemma (3.3), we obtain $E_{kh}(\nabla u_n) \to E_{kh}(\nabla \overline{u})$ in $L^{p'(x)}(\Omega)$, $\forall k, h = 1$

to 3. We have thus shown that $T(u_n) \to T(\overline{u})$ in V', so T is continuous.
Coercivity of T_n. Taking into account, definition and assumptions (3.2), we obtain:

$$\begin{aligned}
\langle T(u).u\rangle_{V',V} &= \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial u_i}{\partial x_j}dx,\\
&\geq \alpha_0 \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} E_{kh}(\nabla u(x))\frac{\partial u_i}{\partial x_j}dx,\\
&\geq \alpha_0\alpha C_1 \int_\Omega |\nabla u|^{p(x)}\,dx,\\
&\geq \alpha_0\alpha C_1 \min\left\{\|\nabla u\|_{L^{p(x)}(\Omega)}^{p-}, \|\nabla u\|_{L^{p(x)}(\Omega)}^{p+}\right\}\\
&\geq \alpha_0\alpha C_1 \min\left\{\|u\|_V^{p-}, \|u\|_V^{p+}\right\}.
\end{aligned}$$

Consequently, the operator T is coercive. This yields the existence of solution for problem (3.4).

Study of infinite dimension problem

The solution of the problem (3.4) is obtained.
So to show the existence of u a solution of (3.3), we will estimate u_n the solution of (3.4) and then by crossing to the limit when $n \to +\infty$ we will have the solution u of our problem (3.3).
Therefore that technique used to show that the limit of the nonlinear term is the desired term.
a. Estimation on u_n
In view of coercivity, if we substitute v by u_n in (3.4), we obtain:

$$\alpha_0\alpha C_1 \int_\Omega |\nabla u_n|^{p(x)}\,dx \leq \|f\|_{H'}\|u_n\|_H,$$

on the other hand

$$\alpha_0\alpha C_1 \min\left\{\|u_n\|_H^{p-}, \|u_n\|_H^{p+}\right\} \leq \|f\|_{H'}\|u_n\|_H.$$

b. Passage to the limit
Since $(u_n)_{n\in\mathbb{N}}$ is bounded in H, which is reflexive (because H is a closed vector subspace of $\left(W^{1,p(x)}(\Omega)\right)^3$, and $\left(W^{1,p(x)}(\Omega)\right)^3$ is a reflexive Banach space), we deduce that there exist a subsequence denoted again $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ weakly in H.
By hypothesis (3), the sequence $(E_{kh}(\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $L^{p'(x)}(\Omega)$, hence there exist $\rho \in L^{p'(x)}(\Omega)$ such that, with a close subsequence,

$$E_{kh}(\nabla u_n) \to \rho \text{ weakly in } L^{p'(x)}(\Omega).$$

Let $v \in H$, then there exist $v_n \in V_n$, $n \in \mathbb{N}^*$ such that

$$v_n \to v \text{ in } H,$$

$$\nabla v_n \to \nabla v \text{ in } \left(L^{p(x)}(\Omega)\right)^9.$$

We substitute v by v_n in (3.4), we obtain:

$$\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \frac{\partial v_{ni}}{\partial x_j} dx = \langle f, v_n \rangle_{H',H}, \ \forall v \in V_n.$$

Since $\langle f, v_n \rangle \to \langle f, v \rangle$, $E_{kh}(\nabla u_n) \to \rho$ weakly in $L^{p'(x)}(\Omega)$ and $\frac{\partial v_{ni}}{\partial x_j} \to \frac{\partial v_i}{\partial x_j}$ for $i = 1$ to 3 strongly in $L^{p(x)}(\Omega)$ (because $\nabla v_n \to \nabla v$ in $\left(L^{p(x)}(\Omega)\right)^9$strongly), using the lemma (3.2), we obtain

$$\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial v_i}{\partial x_j} dx = \langle f, v \rangle_{H',H}, \ \forall v \in H. \tag{3.5}$$

We tend to conclude that ρ is equal to $E_{kh}(\nabla u)$. Unfortunately, this is not obvious because the E_{kh} are nonlinear.

c. Limit of nonlinear term

Finally, it remains to prove that

$$\begin{cases} \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial v_i}{\partial x_j} dx = \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u(x)) \frac{\partial v_i}{\partial x_j} dx, \ \forall v \in H. \end{cases} \tag{3.6}$$

(I) First, we have

$$\lim_{n\to\infty} \int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \frac{\partial u_{ni}}{\partial x_j} dx = \int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial u_i}{\partial x_j} dx.$$

Indeed

$$\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \frac{\partial u_{ni}}{\partial x_j} dx = \langle f, u_n \rangle \to \langle f, u \rangle$$

(II) Proof of (3.6)

Let $v \in H$, there exist $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in V_n$ for all $n \in \mathbb{N}$ and $v_n \to v$ in H when $n \to +\infty$. We will pass to the limit in the term

$$\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \frac{\partial v_{ni}}{\partial x_j} dx,$$

thanks to the hypothesis (4) of (3.2).
Indeed,

$$
\begin{aligned}
0 &\le \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)(E_{kh}(\nabla u_n) - E_{kh}(\nabla v_n))\left(\frac{\partial u_{ni}}{\partial x_j} - \frac{\partial v_{ni}}{\partial x_j}\right)dx \\
&= \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u_n)\frac{\partial u_{ni}}{\partial x_j}dx - \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u_n)\frac{\partial v_{ni}}{\partial x_j}dx \\
&- \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla v_n)\frac{\partial u_{ni}}{\partial x_j}dx + \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla v_n)\frac{\partial v_{ni}}{\partial x_j}dx \\
&= T_{1,n} - T_{2,n} - T_{3,n} + T_{4,n}.
\end{aligned}
$$

It has been seen that in **(I)**:

$$\lim_{n\to+\infty} T_{1,n} = \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\rho\frac{\partial u_i}{\partial x_j}dx,$$

we have

$$\lim_{n\to+\infty} T_{2,n} = \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\rho\frac{\partial v_i}{\partial x_j}dx,$$

by a product of a strong convergence in $L^{p(x)}(\Omega)$ and a weak convergence in $L^{p'(x)}(\Omega)$ (lemma (3.2)).
The same

$$\lim_{n\to+\infty} T_{3,n} = \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla v)\frac{\partial u_i}{\partial x_j}dx,$$

by a product of a strong convergence in $L^{p'(x)}(\Omega)$ and a weak convergence in $L^{p(x)}(\Omega)$.
Finally, we have

$$\lim_{n\to+\infty} T_{4,n} = \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla v)\frac{\partial v_i}{\partial x_j}dx,$$

by the product of a strong convergence in $L^{p'(x)}(\Omega)$ and a strong convergence in $L^{p(x)}(\Omega)$.
The passage to the limit in inequality thus gives:

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)(\rho - E_{kh}(\nabla v))\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right)dx \ge 0 \text{ for all } v \in H.$$

The function test v is now astutely chosen. We take $v = u + \dfrac{1}{n}w$ with $w \in H$ and $n \in \mathbb{N}^*$. We obtain

$$-\frac{1}{n}\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\left(\rho - E_{kh}\left(\nabla u + \frac{1}{n}\nabla w\right)\right)\frac{\partial w_i}{\partial x_j}dx \ge 0,$$

so

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u + \frac{1}{n} \nabla w \right) \right) \frac{\partial w_i}{\partial x_j} dx \leq 0,$$

but $u + \dfrac{1}{n} w \to u$ in H, thus by the lemma (3.3),

$$E_{kh} \left(\nabla u + \frac{1}{n} \nabla w \right) \to E_{kh} \left(\nabla u \right) \text{ in } L^{p'(x)} \left(\Omega \right).$$

By passing to the limit when $n \to +\infty$, we obtain then

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u \right) \right) \frac{\partial w_i}{\partial x_j} dx \leq 0, \ \forall w \in H.$$

By the linearity (we can change w in $-w$), we get:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u \right) \right) \frac{\partial w_i}{\partial x_j} dx = 0, \ \forall w \in H,$$

we deduce that

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial w_i}{\partial x_j} dx = \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh} \left(\nabla u \right) \frac{\partial w_i}{\partial x_j} dx, \ \forall w \in H.$$

We have thus proved that u is a solution of (3.3).

Uniqueness

We suppose that $\left(E_{kh} \left(\xi \right) - E_{kh} \left(\eta \right) \right) \left(\xi_{ij} - \eta_{ij} \right) > 0$, if $\xi_{ij} \neq \eta_{ij}$, and f does not depend to u. Let u_1 and u_2 be two solutions:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_l(x)) \frac{\partial v_i}{\partial x_j} dx = \langle f, v \rangle_{H',H}, \ l = 1,2; \ \forall v \in H.$$

Subtracting term to term and substituting v by $u_1 - u_2$, we obtain:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) (E_{kh} \left(\nabla u_1 \right) - E_{kh}(\nabla u_2)) (\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{2i}}{\partial x_j}) dx = 0.$$

Since

$$M = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) (E_{kh} \left(\nabla u_1 \right) - E_{kh}(\nabla u_2)) (\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{2i}}{\partial x_j}) \geq 0,$$

and $M > 0$ if $\dfrac{\partial u_{1i}}{\partial x_j} \neq \dfrac{\partial u_{2i}}{\partial x_j}$; we get $\dfrac{\partial u_{1i}}{\partial x_j} = \dfrac{\partial u_{2i}}{\partial x_j}$ a.e. $\forall i, j = 1$ to 3, and thus $u_1 = u_2$ a.e.

5. Conclusion

In this work, we consider the nonlinear elasticity system as Leray–Lions's operators with variable exponents, to study the existence and uniqueness of Neumann's problem solution by Galerkin techniques and monotone operator theory. It has been found that these techniques adapt well to this type of problems with different boundary conditions.
From a perspective of this work, first, we will consider the same problem with the boundary conditions Robin, Tresca, and secondly, the boundary conditions no homogeneous of Dirichlet, Neumann, mixed and Robin.

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