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A relaxed version of the gradient projection method for variational inequalities with applications

Nguyen The Vinh and Ngo Thi Thuong

Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper, we propose a relaxed version of the gradient projection method for strongly monotone variational inequalities defined on a level set of a (possibly non-differentiable) convex function. Our algorithm can be implemented easily since it computes on every iteration one projection onto some half-space containing the feasible set and only one value of the underlying mapping. Under mild and standard conditions we establish the strong convergence of the proposed algorithm. Numerical results and comparisons for the image deblurring problem show that our method can outperform related algorithms in the literature.

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1. Introduction

The variational inequality problem (VIP) is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0 \quad \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ with the inner product $\langle ., . \rangle$ and its induced norm $\|.\|$, and $A : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping. Let us denote by $\mathrm{Sol}(C, A)$ the solution set of the problem (1.1), i.e.,

$$\mathrm{Sol}(C, A) = \{x \in C : \langle Ax, y - x \rangle \geq 0 \quad \forall y \in C\}.$$

The variational inequality problem (VIP) has received much attention in the past several decades due to its applications in a large variety of problems arising in economics, optimization, transportation research, game theory, signal and image

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processing, data science, etc., see [1, 4, 8, 14, 15, 18, 19, 22, 20] and the references therein. There are many iterative methods for solving variational inequalities, most of which are based on projection methods. The simplest form is the gradient projection method [5] as follows:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = P_C(x^k - \lambda A x^k), \ k \geq 0, \end{cases}$$

where P_C denotes the metric projection of $\mathcal{H}$ onto the set C, λ is a positive real number. The convergence of this method can be proved under a strong condition that the mapping A is strongly monotone and Lipschitz continuous. In order to relax the strong monotonicity assumption, Korpelevich [15] proposed the extragradient method which requires an additional projection at each iteration. Under the conditions that A is monotone and Lipschitz continuous, this method is shown to be weakly convergent in the setting of Hilbert spaces. Many researchers proposed improvements of the extragradient method, see, e.g., Censor et al. [4], He [6], Iusem-Svaiter [11], Khobotov [13], Malitsky and Semenov [22], Popov [23], Solodov and Svaiter [24], Tinti [25], Tseng [26], Malitsky [20], Maingé [18], Maingé and Gobinddass [19], Malitsky [21] and the references therein. In many real world applications, the feasible set is given in the form of $C = \{x \in \mathcal{H} : c(x) \leq 0\}$, where c is a convex function but not necessarily differentiable. For example, in LASSO problem, the function $c(x) = \|x\|_1 - \tau, \ \tau > 0$ satisfies the above requirement. Very recently, the authors in [2, 7, 9] used the subgradient extragradient method [4] and projection and contraction method [6] to propose relaxed projection algorithms for the variational inequality (1.1). However, the convergence of algorithms in [2, 9, 7] requires that c is a continuously differentiable convex function such that $c'(x)$ is Lipschitz continuous. This makes the real applications of their method very restrictive.

Our concern now is the following: *Can we design a new relaxed projection method to solve the variational inequality (*1.1*) efficiently without demanding differentiability of the convex function c?*

In this paper, we give a positive answer to this question. Motivated by the algorithms in [2, 7, 8, 9], we will introduce an efficient new algorithm for solving the VIP (1.1). The main feature of our method is that it requires only one value of the underlying mapping per iteration with no need for projections onto the feasible set. Theoretical analysis and experimental results show that our algorithm is more efficient than the previous ones for variational inequality problems.

The rest of the paper is organized as follows. After collecting some definitions and basic results in Section 2, we prove in Section 3 the strong convergence of the proposed algorithm. Finally, in Section 4 we provide some numerical results to illustrate the convergence of our algorithm and compare it with the previous algorithms.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. When $\{x^k\}$ is a sequence in $\mathcal{H}$, we denote strong convergence of $\{x^k\}$ to $x \in \mathcal{H}$ by $x^k \to x$ and weak convergence by $x^k \rightharpoonup x$. For a given sequence $\{x^k\} \subset \mathcal{H}$, $\omega_w(x^k)$ denotes the weak ω-limit set of $\{x^k\}$, i.e.,

$$\omega_w(x^k) := \{x \in \mathcal{H} : x^{k_j} \rightharpoonup x \text{ for some subsequence } \{k_j\} \text{ of } \{k\}\}.$$

A useful and simple norm equality is the following

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 \\ - \gamma\beta\|y-z\|^2 - \alpha\gamma\|x-z\|^2,\end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta, \gamma \in [0,1]$ satisfying $\alpha + \beta + \gamma = 1$. Let C be a nonempty closed convex subset of $\mathcal{H}$. For every element $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$P_C x := \underset{y \in C}{\operatorname{argmin}} ||x - y||.$$

P_C is called the metric projection of $\mathcal{H}$ onto C.

Lemma 2.1. *The metric projection P_C has the following basic properties:*

(1) $\langle x - P_C x, y - P_C x \rangle \le 0$ *for all* $x \in \mathcal{H}$ *and* $y \in C$*;*

(2) $\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2$ *for all* $x \in \mathcal{H}$, $y \in C$*;*

(3) $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y \rangle$ *for every* $x, y \in \mathcal{H}$*;*

(4) $\|P_C(x) - P_C(y)\| \le \|x - y\|$ *for all* $x, y \in \mathcal{H}$.

We will focus on solving the problem (1.1) governed by Lipschitz continuous and strongly monotone A, i.e., there exist two positive constants L and η such that

$$\|Ax - Ay\| \le L\|x - y\| \quad \forall x, y \in \mathcal{H},$$

and

$$\langle Ax - Ay, x - y \rangle \ge \eta\|x - y\|^2 \quad \forall x, y \in \mathcal{H},$$

respectively. In this case, we also say that A is L-Lipschitz continuous and η-strongly monotone.

Let $g : \mathcal{H} \to (-\infty, \infty]$, $\mathrm{dom} g := \{x \in \mathcal{H} : g(x) < +\infty\}$. We recall that the subdifferential of g at $x \in \mathcal{H}$ is defined as the set of all subgradients of g at x:

$$\partial g(x) := \{w \in \mathcal{H} : g(y) - g(x) \ge \langle w, y - x \rangle \quad \forall y \in \mathcal{H}\}. \tag{2.2}$$

g is strongly convex with constant $m > 0$ if and only if $g(x) - \frac{m}{2}\|x\|^2$ is convex. We already know that if g is lower semicontinuous convex at $x \in \mathrm{int}(\mathrm{dom} g)$, then $\partial g(x)$ is nonempty and bounded. The next lemmas are essential for our analysis in the sequel.

Lemma 2.2. (Cegielski and Zalas [3], Theorem 5) *Assume that A is a L-Lipschtz continuous and η-strongly monotone operator and μ is a constant such that $\mu \in \left(0, \dfrac{2\eta}{L^2}\right)$. Let $T^\mu = P_C(I - \mu A)$ (or $I - \mu A$), where I is the identity operator on $\mathcal{H}$. Then T^μ is a strict contraction with coefficient $1 - \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu L^2)$.*

Lemma 2.3. (Maingé [16], Lemma 3.1; Xu [27], Lemma 2.5) *Let* $\{a_k\}, \{b_k\}$ *and* $\{c_k\}$ *be sequences of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \delta_k)a_k + b_k + c_k, \quad k \geq 1,$$

where $\{\delta_k\}$ *is a sequence in* $(0, 1)$ *and* $\{b_k\}$ *is a real sequence. Assume that*

$$\sum_{k=1}^{\infty} c_k < \infty.$$

Then the following results hold:

(1) *If* $b_k \leq \delta_k M$ *for some* $M \geq 0$ *and for all* $k \geq 1$ *then* $\{a_k\}$ *is a bounded sequence.*

(2) *If* $\sum_{k=1}^{\infty} \delta_k = \infty$ *and* $\limsup_{k\to\infty} b_k/\delta_k \leq 0$, *then* $\lim_{k\to\infty} a_k = 0$.

Lemma 2.4. (Maingé [17], Lemma 3.1) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}$ *of* $\{\Gamma_n\}$ *such that* $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j \geq 0$. *Also consider the sequence of integers* $\{\tau(n)\}_{n\geq n_0}$ *defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty} \tau(n) = \infty$ *and, for all* $n \geq n_0$,

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

3. A relaxed gradient projection algorithm

In this section, we consider VIP (1.1) in which C is given by

$$C = \{x \in \mathcal{H} : c(x) \leq 0\}.$$

where $c : \mathcal{H} \to \mathbb{R}$ is a convex and lower semicontinuous function.

We need the following basic assumptions for VIP (1.1):

(C1) $\mathrm{Sol}(C, A) \neq \emptyset$;

(C2) The mapping A is strongly monotone and L-Lipschitz continuous;

(C3) ∂c is a bounded operator (i.e., bounded on bounded sets).

3.1. The algorithm

The algorithm is designed as follows.

Algorithm 3.1 (Relaxed gradient projection algorithm)

Step 0 (Initialization): Select initial $x^0, x^1 \in C$, $\theta \in [0,1)$ and two positive real number sequences $\{\beta_k\}$, $\{\epsilon_k\}$ satisfying

$$\lim_{k\to\infty} \beta_k = 0, \quad \sum_{k=0}^{\infty} \beta_k = +\infty, \quad \epsilon_k = o(\beta_k), \tag{3.1}$$

where $\epsilon_k = o(\beta_k)$ means that the sequence $\{\epsilon_k\}$ is an infinitesimal of higher order than $\{\beta_k\}$. Set $k := 1$.

Step 1: Given x^{k-1} and x^k $(k \geq 1)$, choose α_k such that

$$\alpha_k = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \neq x^{k-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{3.2}$$

Compute $w^k = x^k + \alpha_k(x^k - x^{k-1})$ and take $\xi^k \in \partial c(w^k)$. Construct the half-space

$$C_k = \{x \in \mathcal{H} : c(w^k) + \langle \xi^k, x - w^k \rangle \leq 0\},$$

and calculate

$$x^{k+1} = P_{C_k}(w^k - \beta_k A w^k). \tag{3.3}$$

Step 2: If $x^{k+1} = w^k$ then stop. Otherwise set $k := k+1$ and return to **Step 1**.

Remark 3.1. We have $C \subseteq C_k$ for every $k \geq 0$. Indeed, we obtain by (2.2) and $\xi^k \in \partial c(w^k)$ that

$$c(x) - c(w^k) \geq \langle \xi^k, x - w^k \rangle \quad \forall x \in \mathcal{H}.$$

If $x \in C$ then we get $c(w^k) + \langle \xi^k, x - w^k \rangle \leq 0$, i.e., $x \in C_k$. Hence, the statement is true.

3.2. Convergence analysis

We first show that the stopping criterion Algorithm 3.1 is valid.

Lemma 3.2. *If $w^k = x^{k+1}$ then $w^k \in \mathrm{Sol}(C, A)$.*

Proof. If $w^k = x^{k+1}$ then by (3.3) and Lemma 2.1 (1), we have

$$\langle w^k - \lambda_k A w^k - w^k, y - w^k \rangle \leq 0 \quad \forall y \in C_k,$$

or equivalently,

$$\langle A w^k, y - w^k \rangle \geq 0 \quad \forall y \in C_k.$$

Therefore, we get

$$\langle A w^k, y - w^k \rangle \geq 0 \quad \forall y \in C.$$

Hence $w^k \in \mathrm{Sol}(C, A)$. □

A key lemma for our convergence theorem is presented next.

Lemma 3.3. *Assume that the conditions (C1)-(C3) hold. Then the sequence $\{x^k\}$ generated by Algorithm* 3.1 *is bounded.*

Proof. We have

$$\begin{aligned}\|x^{k+1}-z\| &= \|P_{C_k}(w^k-\beta_k Aw^k)-z\| \\ &\le \|(I-\beta_k A)w^k-(I-\beta_k A)z-\beta_k Az\| \\ &= (1-\gamma_k)\|w^k-z\|+\beta_k\|Az\|. \end{aligned} \tag{3.4}$$

Moreover, we have

$$\begin{aligned}\|w^k-z\| &= \|x^k-z+\alpha_k(x^k-x^{k-1})\| \\ &\le \|x^k-z\|+\alpha_k\|x^k-x^{k-1}\|. \end{aligned} \tag{3.5}$$

Combining (3.5) and (3.4), we immediately get

$$\|x^{k+1}-z\| \le (1-\gamma_k)\|x^k-z\|+(1-\gamma_k)\alpha_k\|x^k-x^{k-1}\|+\beta_k\|Az\|.$$

By (3.1) and (3.2), we see that

$$\begin{aligned}\lim_{k\to\infty}\frac{b_k}{\gamma_k} &= \lim_{k\to\infty}\frac{(1-\gamma_k)\alpha_k\|x^k-x^{k-1}\|+\beta_k\|Az\|}{\gamma_k}, \\ &= \lim_{k\to\infty}\left[\frac{2(1-\gamma_k)}{2\eta-\beta_k L^2}\frac{\alpha_k}{\beta_k}\|x^k-x^{k-1}\|+\frac{2}{2\eta-\beta_k L^2}\|Az\|\right]=\frac{\|Az\|}{\eta},\end{aligned}$$

where $b_k=(1-\gamma_k)\alpha_k\|x^k-x^{k-1}\|+\beta_k\|Az\|$.
This implies that the sequence $\left\{\frac{b_k}{\gamma_k}\right\}$ is bounded. Using Lemma 2.3 (1), we conclude that the sequence $\{\|x^k-z\|\}$ is bounded. This shows that the sequence $\{x^k\}$ is bounded and so is $\{w^k\}$. □

Lemma 3.4. *Assume that the conditions (C1)-(C3) hold and let $\{x^k\}$ be the sequence generated by Algorithm* 3.1*. Then, for each $z\in C$, we have*

$$\begin{aligned}\|x^{k+1}-z\|^2 &\le (1-\gamma_k)(\|x^k-z\|^2+2\alpha_k\|x^k-x^{k-1}\|\|x^k-z\|+\alpha_k^2\|x^k-x^{k-1}\|^2) \\ &\quad+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az,w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right].\end{aligned}$$

Proof. Let $\gamma_k=\frac{1}{2}\beta_k(2\eta-\beta_k L^2)$. Since $\beta_k\to 0$ as $k\to\infty$, there exists some positive integer k_0 such that

$$0<\beta_k<\frac{\eta}{L^2} \tag{3.6}$$

for all $k\ge k_0$. In view of Lemma 2.6, we obtain from (3.6) that $P_{C_k}(I-\beta_k A)$ (so is $I-\beta_k A$) is a strict contraction with coefficient $1-\gamma_k$ for all $k\ge k_0$. For each $z\in C$, we have

$$\begin{aligned}\|x^{k+1}-z\|^2 &= \|P_{C_k}(w^k-\beta_k A w^k)-z\|^2\\ &\le \|(I-\beta_k A)w^k-(I-\beta_k A)z-\beta_k Az\|^2\\ &= (1-\gamma_k)\|w^k-z\|^2-2\beta_k\langle Az, w^k-z-\beta_k Aw^k\rangle\\ &\le (1-\gamma_k)\|w^k-z\|^2-2\beta_k\langle Az, w^k-z\rangle+2\beta_k^2\|Az\|\|Aw^k\|\\ &= (1-\gamma_k)\|w^k-z\|^2+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k-z\rangle\right.\\ &\qquad\left.+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right].\end{aligned}$$

Using (3.5) we arrive at

$$\begin{aligned}\|x^{k+1}-z\|^2 &\le (1-\gamma_k)(\|x^k-z\|+\alpha_k\|x^k-x^{k-1}\|)^2\\ &\quad+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right]\\ &= (1-\gamma_k)(\|x^k-z\|^2+2\alpha_k\|x^k-x^{k-1}\|\|x^k-z\|+\alpha_k^2\|x^k-x^{k-1}\|^2)\\ &\quad+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right].\end{aligned}$$

Therefore, the proof is complete. □

We are now in a position to establish the strong convergence theorem of Algorithm 3.1.

Theorem 3.5. *Assume that the conditions (C1)-(C3) hold. Then any sequence $\{x^k\}$ generated by Algorithm* 3.1 *converges strongly to the unique solution of the variational inequality problem (*1.1*).*

Proof. For each $z \in C$, using the nonexpansive property of projection operators, we have

$$\begin{aligned}\|x^{k+1}-z\|^2 &= \|P_{C_k}(w^k-\beta_k Aw^k)-P_{C_k}w^k+P_{C_k}w^k-P_{C_k}z\|^2\\ &= \|P_{C_k}w^k-P_{C_k}z\|^2+2\beta_k\|w^k-z\|\|Aw^k\|+\beta_k^2\|Aw^k\|^2\\ &\le \|w^k-z\|^2-\|w^k-P_{C_k}w^k\|^2+2\beta_k\|w^k-z\|\|Aw^k\|+\beta_k^2\|Aw^k\|^2\\ &= \|w^k-z\|^2-\|w^k-P_{C_k}w^k\|^2+\beta_k M, \qquad (3.7)\end{aligned}$$

where $M \ge \sup_k\{2\|w^k-z\|\|Aw^k\|+\beta_k\|Aw^k\|^2\}$.
On the other hand, by applying (2.1) we get

$$\begin{aligned}\|w^k-z\|^2 &= \|(1+\alpha_k)(x^k-z)-\alpha_k(x^{k-1}-z)\|^2\\ &= (1+\alpha_k)\|x^k-z\|^2-\alpha_k\|x^{k-1}-z\|^2+\alpha_k(1+\alpha_k)\|x^k-x^{k-1}\|^2\\ &\le (1+\alpha_k)\|x^k-z\|^2-\alpha_k\|x^{k-1}-z\|^2+2\alpha_k\|x^k-x^{k-1}\|^2. \qquad (3.8)\end{aligned}$$

Combining (3.7) and (3.8) we have

$$\|x^{k+1}-z\|^2 \le (1+\alpha_k)\|x^k-z\|^2-\alpha_k\|x^{k-1}-z\|^2+2\alpha_k\|x^k-x^{k-1}\|^2 \\ -\|w^k-P_{C_k}w^k\|^2+\beta_k M.$$

Putting $\Gamma_k := \|x^k-z\|^2$ for all $k\in\mathbb{N}$ we have

$$\|w^k-P_{C_k}w^k\|^2 \le \Gamma_k-\Gamma_{k+1}+\alpha_k(\Gamma_k-\Gamma_{k-1})+2\alpha_k\|x^k-x^{k-1}\|^2+\beta_k M. \quad (3.9)$$

Now, we consider two possible cases:

Case 1. Assume that there exists $k_0\ge 0$ such that for each $k\ge k_0$, $\Gamma_{k+1}\le\Gamma_k$.
In this case, $\lim\limits_{k\to\infty}\Gamma_k$ exists and $\lim\limits_{k\to\infty}(\Gamma_k-\Gamma_{k+1})=0$.
Since $\lim\limits_{k\to\infty}\beta_k=0$ and $\lim\limits_{k\to\infty}\alpha_k\|x^k-x^{k-1}\|^2=0$, it follows from (3.9) that

$$\lim_{k\to\infty}\|w^k-P_{C_k}w^k\|^2=0. \quad (3.10)$$

We now show that $\omega_w(x^k)\subset C$. Let $\bar{x}\in\omega_w(x^k)$ be an arbitrary element. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ that converges weakly to $\bar{x}\in C_k$. Note that

$$\lim_{k\to\infty}\|w^k-x^k\|=\lim_{k\to\infty}\alpha_k\|x^k-x^{k-1}\|=0. \quad (3.11)$$

It follows from (3.11) that $\{w^{k_l}\}$ also converges weakly to $\bar{x}$. Next we verify that $\bar{x}\in C$.
Due to $P_{C_{k_l}}w^{k_l}\in C_{k_l}$, it follows from the definition of C_{k_l} that

$$c(w^{k_l})+\langle\xi^{k_l},P_{C_{k_l}}w^{k_l}-w^{k_l}\rangle\le 0,$$

where $\xi^{k_l}\in\partial c(w^{k_l})$. The use of the Cauchy-Schwart inquality implies that

$$c(w^{k_l})\le\|\xi^{k_l}\|\big\|P_{C_{k_l}}w^{k_l}-w^{k_l}\big\|. \quad (3.12)$$

From the boundedness assumption of ξ^{k_l} and (3.10), (3.12), we have

$$c(w^{k_l})\le\|\xi^{k_l}\|\big\|P_{C_{k_l}}w^{k_l}-w^{k_l}\big\|\to 0. \quad (3.13)$$

From the weak lower-semicontinuity of the convex function $c(x)$ and since $w^{k_l}\rightharpoonup\bar{x}$, it follows from (3.13) that

$$c(\bar{x})\le\liminf_{l\to\infty}c(w^{k_l})\le 0,$$

which means that $\bar{x}\in C$.
Using Lemma 3.4 we have

$$\|x^{k+1}-z\|^2\le(1-\gamma_k)(\|x^k-z\|^2+2\alpha_k\|x^k-x^{k-1}\|\|x^k-z\|+\alpha_k^2\|x^k-x^{k-1}\|^2) \\ +\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az,w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right]. \quad (3.14)$$

Besides, we obtain

$$\begin{aligned}\|x^k - z\|^2 + 2\alpha_k \|x^k - z\| \big\|x^k - x^{k-1}\big\| + \alpha_k^2 \|x^k - x^{k-1}\|^2 \\ \leq \|x^k - z\|^2 + 2\alpha_k \|x^k - z\| \big\|x^k - x^{k-1}\big\| + \alpha_k \|x^k - x^{k-1}\|^2 \\ \leq \|x^k - z\|^2 + 3M_1 \alpha_k \|x^k - x^{k-1}\|, \end{aligned} \tag{3.15}$$

where $M_1 = \sup_{k\in\mathbb{N}}\{\|x^k - z\|, \|x^k - x^{k-1}\|\}$.

Combining (3.14) and (3.15) we get

$$\begin{aligned}\|x^{k+1} - z\|^2 &\leq (1-\gamma_k)\|x^k - z\|^2 + 3M_1(1-\gamma_k)\alpha_k \|x^k - x^{k-1}\| \\ &\quad + \gamma_k \left[\frac{-4}{2\eta - \beta_k L^2}\langle Az, w^k - z\rangle + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\|\right] \\ &\leq (1-\gamma_k)\|x^k - z\|^2 + \gamma_k \left[3M_1(1-\gamma_k)\frac{\alpha_k}{\gamma_k}\|x^k - x^{k-1}\|\right. \\ &\quad \left. + \frac{-4}{2\eta - \beta_k L^2}\langle Az, w^k - z\rangle + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\|\right]. \end{aligned} \tag{3.16}$$

It is easy to see that

$$\lim_{k\to\infty}\left[(1-\gamma_k)\frac{\alpha_k}{\gamma_k}\|x^k - x^{k-1}\| + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\|\right] = 0. \tag{3.17}$$

To apply Lemma 2.3, it remains to show that $\limsup_{k\to\infty}\langle Az, w^k - z\rangle \geq 0$. Indeed, since $z \in \mathrm{Sol}(C, A)$, we get that

$$\limsup_{k\to\infty}\langle Az, w^k - z\rangle = \max_{\hat{z}\in\omega_w(\{w^k\})}\langle Az, \hat{z} - z\rangle \geq 0.$$

By applying Lemma 2.3 to (3.16) with the data

$$\begin{aligned} a_k &:= \|x^k - z\|^2, \quad \delta_k := \gamma_k, \quad c_k := 0, \\ b_k &:= 3M_1(1-\gamma_k)\frac{\alpha_k}{\gamma_k}\|x^k - x^{k-1}\| + \frac{-4}{2\eta - \beta_k L^2}\langle Az, w^k - z\rangle \\ &\quad + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\| \end{aligned}$$

we immediately deduce that the sequence $\{x^k\}$ converges strongly to $z \in \mathrm{Sol}(C, A)$.

Case 2. Assume that there exists a subsequence $\{\Gamma_{k_m}\} \subset \{\Gamma_k\}$ such that $\Gamma_{k_m} \leq \Gamma_{k_m+1}$ for all $m \in \mathbb{N}$. In this case, we can define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(k) = \max\{n \leq k : \Gamma_n < \Gamma_{n+1}\}.$$

Then we have from Lemma 2.4 that $\tau(k) \to \infty$ as $k \to \infty$ and $\Gamma_{\tau(k)} < \Gamma_{\tau(k)+1}$. So, we have from (3.9) that

$$\begin{aligned}\|w^{\tau(k)} - P_{C_{\tau(k)}} w^{\tau(k)}\|^2 &\le \Gamma_{\tau(k)} - \Gamma_{\tau(k)+1} + \alpha_{\tau(k)}(\Gamma_{\tau(k)} - \Gamma_{\tau(k)-1}) \\ &\qquad + 2\alpha_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|^2 + \beta_{\tau(k)}M \\ &\le \alpha_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|\left(\sqrt{\Gamma_{\tau(k)}} + \sqrt{\Gamma_{\tau(k)-1}}\right) \\ &\qquad + 2\alpha_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|^2 + \beta_{\tau(k)}M \\ &\to 0. \end{aligned} \tag{3.18}$$

Following the same lines as in the proof of Case 1, we get from (3.18) that

$$\begin{aligned} &\lim_{k\to\infty} \|w^{\tau(k)} - P_{C_{\tau(k)}} w^{\tau(k)}\|^2 = 0, \\ &\limsup_{k\to\infty} \langle Az, w^{\tau(k)} - z\rangle = \max_{\hat{z}\in\omega_w(\{w^{\tau(k)}\})} \langle Az, \hat{z} - z\rangle \ge 0 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|x^{\tau(k)+1} - z\|^2 &\le (1-\gamma_{\tau(k)})\|x^{\tau(k)} - z\|^2 \\ &\quad + \gamma_{\tau(k)}\Bigg[3M_1(1-\gamma_{\tau(k)})\frac{\alpha_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| \\ &\quad + \frac{-4}{2\eta - \beta_{\tau(k)}L^2}\langle Az, w^{\tau(k)} - z\rangle + \frac{4\beta_{\tau(k)}}{2\eta - \beta_{\tau(k)}L^2}\|Az\|\|Aw^{\tau(k)}\|\Bigg]. \end{aligned} \tag{3.20}$$

Since $\Gamma_{\tau(k)} < \Gamma_{\tau(k)+1}$, we have from (3.20) that

$$\begin{aligned}\|x^{\tau(k)} - z\|^2 &\le 3M_1(1-\gamma_{\tau(k)})\frac{\alpha_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| \\ &\quad + \frac{-4}{2\eta - \beta_{\tau(k)}L^2}\langle Az, w^{\tau(k)} - z\rangle + \frac{4\beta_{\tau(k)}}{2\eta - \beta_{\tau(k)}L^2}\|Az\|\|Aw^{\tau(k)}\|. \end{aligned} \tag{3.21}$$

Combining (3.17), (3.19) and (3.21) yields

$$\limsup_{k\to\infty} \|x^{\tau(k)} - z\|^2 \le 0,$$

and hence

$$\lim_{k\to\infty} \|x^{\tau(k)} - z\|^2 = 0.$$

From (3.20), we have

$$\limsup_{k\to\infty} \|x^{\tau(k)+1} - z\|^2 \le \limsup_{k\to\infty} \|x^{\tau(k)} - z\|^2.$$

Thus

$$\lim_{k\to\infty} \|x^{\tau(k)+1} - z\|^2 = 0.$$

Therefore, by Lemma 2.4, we obtain

$$0 \le \|x^k - z\| \le \max\{\|x^{\tau(k)} - z\|, \|x^k - z\|\} \le \|x^{\tau(k)+1} - z\| \to 0.$$

Consequently, $\{x^k\}$ converges strongly to $z \in \mathrm{Sol}(C, A)$. □

4. Numerical results

Example 4.1. Image restoration problems can be formulated as an inverse problem as follows:

$$y = Ax + v, \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}$ represents a known blurring operator (which is called the point spread function: PSF), $y \in \mathbb{R}^{m\times 1}$ represents the blurred image, and $v \in \mathbb{R}^{m\times 1}$ stands for the additive noises or perturbation signals, $x \in \mathbb{R}^{n\times 1}$ is the unknown original image whose size is assumed to be the same as that of y (that is, $m = n$). In most cases, this problem is ill-posed, hence directly inverting A would lead to bad and possibly multiple solutions. To overcome this difficulty, a popular strategy is to use a regularization based method, which provides the prior knowledge of images that one wants to reconstruct. In this paper, the problem (4.1) is approximately solved by the following optimization model:

$$\min_{x\in\mathbb{R}^{n^2}} f(x) := \frac{1}{2}\|Ax - y\|^2 + \frac{1}{2}\alpha\|x\|^2, \quad \text{s.t. } \|x\|_1 \le t, \tag{4.2}$$

where α is a positive parameter, and $\|.\|_1$ is the ℓ_1-norm, which is to make small component of x to become zero. The objective function of the problem (4.2) is strongly convex. Note that, the objective f is strongly convex and differentiable with the gradient given by

$$\nabla f(x) = A^*(Ax - y) + \alpha x,$$

where A^* is the adjoint of A.
We observe that the gradient ∇f is $(\|A\|^2 + \alpha)$-Lipschitz continuous and α-strongly monotone. We already know that x^* solves (4.2) if and only if x^* solves the variational inequality problem of finding $x \in C$ such that

$$\langle \nabla f(x), y - x\rangle \ge 0 \quad \forall y \in C,$$

where $C := \{x \in \mathbb{R}^{n^2} : \|x\|_1 \le t\}$.

The quality of the restoration is measured by the peak signal-to-noise ratio (PSNR) in decibel (dB):

$$\mathrm{PSNR}(x) = 20\log_{10}\frac{x_{\max}}{\sqrt{\mathrm{Var}(x,\bar{x})}},$$

where

$$\mathrm{Var}(x,\bar{x}) = \frac{\sum_{j=1}^{n^2}[\bar{x}(j) - x(j)]^2}{n^2},$$

and $\bar{x}$ is the true image and $x_{\max}$ is the maximum possible pixel value of the image.

FIGURE 1. Cameraman original and blurred and noisy images on top; Lena original and blurred and noisy images below.

All the codes were written in Matlab (R2016a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz. In the numerical results reported in the following tables, 'Iter.' and 'Sec.' stand for the number of iterations and the CPU time in seconds, respectively. We now apply our proposed algorithm - Algorithm 3.1 (IGPM) and the strongly convergent algorithms in the literature including Algorithm 1 of Hieu and Thong [10] (VPRGM), Algorithm 3.1 of Khanh and Vuong [12] (GPM), and the golden ratio algorithm of Malitsky [21] (GRA) with diminishing step sizes to recover the blurred Lena and Cameraman images. The size of the image is $m = n = 256$. The original and the blurred images are shown in Figure 1. For all tested algorithms, we use the same starting points $x^0 = x^1 = \mathbf{0}$ ($\mathbf{0}$ is a vector in $\mathbb{R}^{n^2}$ in which all components are zero) and limit the number of iterations by 2500 for all algorithms as well. Moreover, we set $A = RW$, where R is the blur matrix and W denotes the inverse wavelet transform. The blur kernel is taken to be $h_{ij} = \frac{1}{1+i^2+j^2}$, for $i, j = -4, ..., 4$. An additive zero-mean white Gaussian noise with standard deviation 10^{-3} was added to the images.

Moreover, for Algorithm 3.1 (IGPM), we take $\epsilon_k = \frac{1}{k^{1.1}}$, $\theta = 0.6$; α_k is computed by (3.2).

We take the same stepsizes $\lambda_k = \frac{1}{k^{0.3}}$, the regularization parameter $\alpha = 2e{-}5$ for all algorithms. Besides, we choose $\theta_k = 1$ for VPRGM of [10]. The comparison of four algorithms with Cameraman and Lena images are reported in Table 1 and Table 2, respectively. The reconstructed images are presented in Figures 2, 4. The convergence behaviour of algorithms is given in Figures 3, 5. In these figures, the value of PSNR for all algorithms is represented by the y-axis, the running time is represented by the x-axis.

	Sec.	Iter	PSNR
GRA	112.9531	2500	28.4390
VPRGM	113.5313	2500	31.8681
GPM	113.6406	2500	31.8692
Our algorithm (IGPM)	**83.6**	2500	**37.0024**

Table 1. Comparison of four algorithms for reconstructing the blurred Cameraman image.

	Sec.	Iter	PSNR
GRA	112.9531	2500	31.9244
VPRGM	117.6094	2500	35.2395
GPM	117.0469	2500	35.7691
Our algorithm (IGPM)	**94.2031**	2500	**44.5633**

Table 2. Comparison of four algorithms for reconstructing the blurred Lena image.

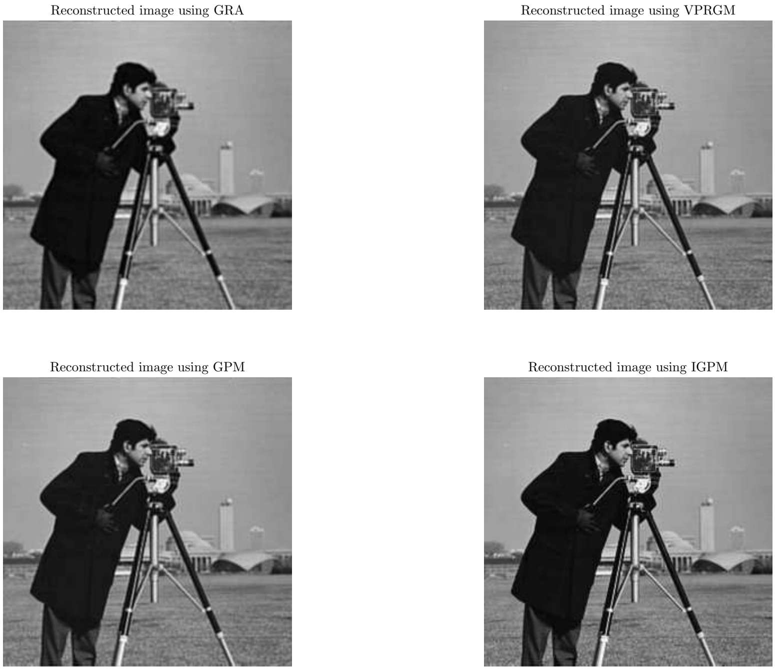

Figure 2. The reconstructed images with the Cameraman image

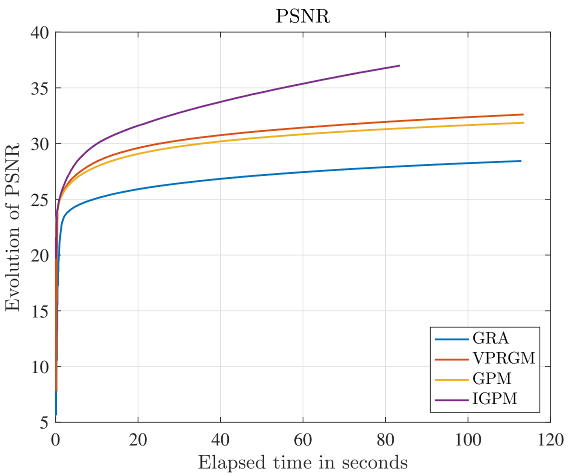

Figure 3. Evolution of PSNR with the Cameraman image

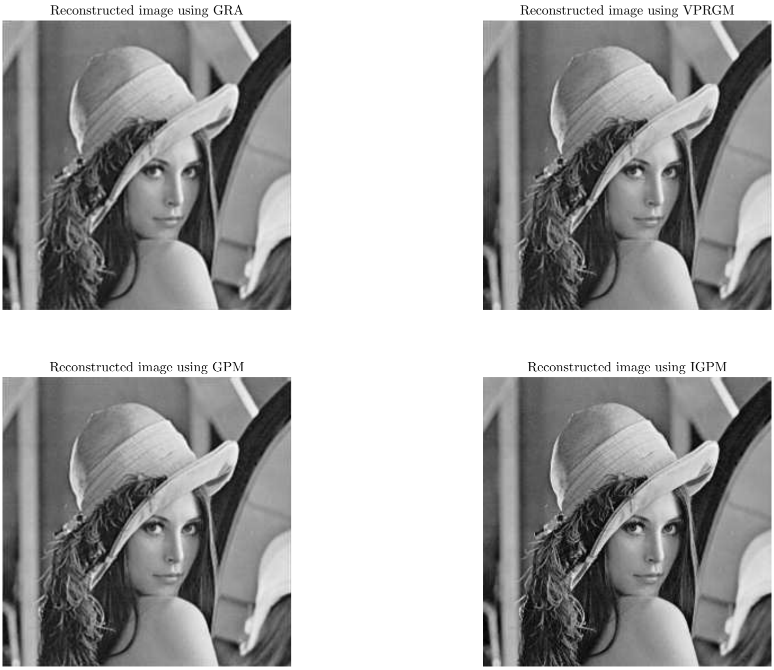

FIGURE 4. The reconstructed images with the Lena image

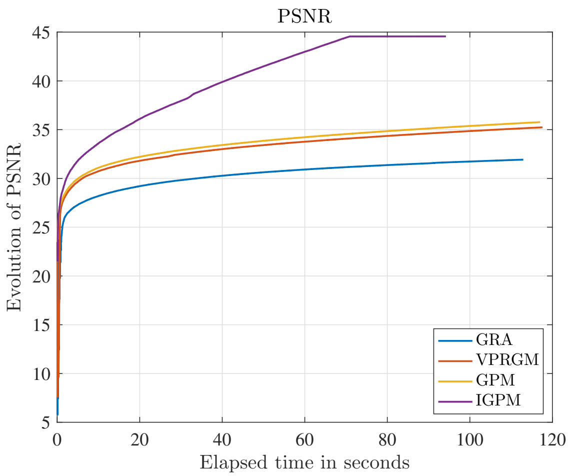

FIGURE 5. Evolution of PSNR with the Lena image

Figures 3, 5 clearly demonstrate that IGPM gives lower running time compared to others. Clearly, our method provides clearer images and improved PSNR values. We emphasize here that these numerical results are very preliminary.

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References

[1] Ansari, Q.H., Lalitha, C.S., Mehta, M., *Generalized Convexity, Nonsmooth Variational Inequalities, and Nonsmooth Optimization*, CRC Press, Boca Raton, FL, 2014.

[2] Cao, Y., Guo, K., *On the convergence of inertial two-subgradient extragradient method for variational inequality problems*, Optimization, **69**(2019), 1-17.

[3] Cegielski, A., Zalas, R., *Methods for variational inequality problem over the intersection of fixed point sets of quasi-nonexpansive operators*, Numer. Funct. Anal. Optim., **34**(2013), 255-283.

[4] Censor, Y., Gibali, A., Reich, S., *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., **148**(2011), 318-335.

[5] Goldstein, A.A., *Convex programming in Hilbert space*, Bull. Amer. Math. Soc. (N.S.), **70**(1964), 709-710.

[6] He, B., *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim., **35**(1997), 69-76.

[7] He, S., Dong, Q.L., Tian, H., *Relaxed projection and contraction methods for solving Lipschitz continuous monotone variational inequalities*, RACSAM, **113**(2019), 2773-2791.

[8] He, S., Tian, H., *Selective projection methods for solving a class of variational inequalities*, Numer. Algorithms, **80**(2019), 617-634.

[9] He, S.N., Wu, T., Gibali, A., *Totally relaxed, self-adaptive algorithm for solving variational inequalities over the intersection of sub-level sets*, Optimization, **67**(2018), 1487-1504.

[10] Hieu, D.V., Thong, D.V., *A new projection method for a class of variational inequalities*, Appl. Anal., **98**(2019), 2423-2439.

[11] Iusem, A.N., Svaiter, B.F., *A variant of Korpelevich's method for variational inequalities with a new search strategy*, Optimization, **42**(1997), 309-321.

[12] Khanh, P.D., Vuong, P.T., *Modified projection method for strongly pseudomonotone variational inequalities*, J. Global Optim., **58**(2014), 341-350.

[13] Khobotov, E.N., *A modification of the extragradient method for solving variational inequalities and some optimization problems*, Zh. Vychisl. Mat. Mat. Fiz., **27**(1987), 1462-1473.

[14] Konnov, I.V., *Equilibrium Models and Variational Inequalities*, Elsevier, Amsterdam, Boston, Heidelberg, London, New York, 2007.

[15] Korpelevich, G.M., *An extragradient method for finding saddle points and for other problems*, Ekon. Mat. Metody, **12**(1976), 747-756.

[16] Maingé, P.E., *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **325**(2007), 469-479.

[17] Maingé, P.E., *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal., **16**(2008), 899-912.

[18] Maingé, P.E., *Numerical approach to monotone variational inequalities by a one-step projected reflected gradient method with line-search procedure*, Comput. Math. Appl., **72**(2016), 720-728.

[19] Maingé, P., Gobinddass, M. L., *Convergence of one-step projected gradient methods for variational inequalities*, J. Optim. Theory Appl., **171**(2016), 146-168.

[20] Malitsky, Y., *Projected reflected gradient methods for monotone variational inequalities*, SIAM J. Optim., **25**(2015), 502-520.

[21] Malitsky, Y., *Golden ratio algorithms for variational inequalities*, Math. Program., **184**(2020), 383-410.

[22] Malitsky, Y., Semenov, V.V., *An extragradient algorithm for monotone variational inequalities*, Cybernet. Systems Anal., **50**(2014), 271-277.

[23] Popov, L., *A modification of the Arrow-Hurwicz method for search for saddle points*, Math. Notes, **28**(1980), 845-848.

[24] Solodov, M.V., Svaiter, B.F., *A new projection method for variational inequality problems*, SIAM J. Control Optim., **37**(1999), 765-776.

[25] Tinti, F., *Numerical solution for pseudomonotone variational inequality problems by extragradient methods*, (English Summary), Variational Analysis and Applications, Nonconvex Optim. Appl., 79, Springer, New York, (2005), 1101-1128.

[26] Tseng, P., *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431-446.

[27] Xu, H.K., *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc., **66**(2002), no. 2, 240-256.

Nguyen The Vinh
Department of Mathematical Analysis,
University of Transport and Communications,
3, Cau Giay Street, Hanoi, Vietnam
e-mail: thevinhbn@utc.edu.vn
ORCID: 0000-0003-3373-1881

Ngo Thi Thuong
Department of Mathematics - Mechanics - Informatics,
University of Science, Vietnam National University,
334, Nguyen Trai Street, Thanh Xuan, Hanoi, Vietnam
e-mail: ngothithuong.k56@hus.edu.vn