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Relative and mutual monotonicity

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this work we first consider a certain monotonicity relative to some given one-to-one operator and prove the counterparts, adjusted to this new context, of most results obtained before in the joint work with G. Kassay [10]. For two operators with the same status relative to injectivity, such as two local injective operators, we define what we call *mutual h-monotonicity* and prove that every two mutual h-monotone local diffeomorphisms can be obtained from each other via a composition with a h-monotone diffeomorphism.

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1. Introduction

The importance of the Minty-Browder monotonicity stands in its applications to the theory of partial differential equations (see for example [2, 3, 4],[13, 14]) and in its connection with convex analysis, due to the characterization ofconvexity, within the class of semicontinuous functions, through the Minty-Browder monotonicity of the subdifferential operator (see for instance [6]). In the previous joint work with G. Kassay ([10]) we extended the class of Minty-Browder monotone operators to the class of h-monotone operators. While the inverse images of maximal Minty-Browder monotone operators are well-known to be convex sets [16, p. 105], we only proved in [10] that the inverse images of such operators, with finite dimensional source space, are indivisible by closed connected hypersurfaces. In a joint work with G. Kassay and F. Szenkowitz [11] we provided an elementary proof for the convexity of inverse images of Minty-Browder monotone operators. The lack of divisibility of the inverse images of the h-monotone operators through closed connected hypersurfaces allowed us to establish some global injectivity results.

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In this work we first consider a certain monotonicity concept relative to some given one-to-one operator. Note that the Minty-Browder monotonicity as well as the h-monotonicity are particular notions of this relative monotonicity. Indeed, the role of the given operator in the definitions of Minty-Browder monotonicity and in that of h-monotone operators is played by the identity operator. We also prove the counterparts, adjusted to this new context, of most results obtained before in [10]. For two operators with the same status relative to injectivity, such as two local injective operators, we define what we call mutual h-monotonicity and prove that every two mutual h-monotone local diffeomorphisms can be obtained from each other via a composition with a h-monotone diffeomorphism. As a consequence we observe that two mutual h-monotone local diffeomorphisms have the same valence and provide some examples of h-monotone operators relative to the gradient operator of some strictly convex functions.

2. h-monotonicity relative to an injective operator

In this section we first emphasize some geometrical properties of the Minty-Browder monotone operators which suggest an interesting enlargement of this class.

Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere and $d_{S^n} : S^n \times S^n \longrightarrow \mathbb{R}_+$ be the metric associated to the Riemann structure of S^n, i.e., $d_{S^n}(x, y) = \arccos\langle x, y\rangle$, $x, y \in S^n$ is the measure of the angle between the vectors x and y. Note that $0 \leq d_{S^n} \leq \pi$. Denote by pr_{S^n} the radial projection

$$\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n, z \longmapsto \frac{z}{||z||}.$$

The next important geometric characterizations of Minty-Browder monotonicity allow us to enlarge this class.

Let D be a subset of $\mathbb{R}^{n+1}$. The following statements hold:

1. $T : D \longrightarrow \mathbb{R}^{n+1}$ is a Minty-Browder increasing operator if and only if
$$d_{S^n}(pr_{S^n}(x - y), pr_{S^n}(Tx - Ty)) \leq \frac{\pi}{2}$$
for all $x, y \in D,\ Tx \neq Ty$.
2. $T : D \longrightarrow \mathbb{R}^{n+1}$is a Minty-Browder decreasing operator if and only if
$$d_{S^n}(pr_{S^n}(x - y), pr_{S^n}(Tx - Ty)) \geq \frac{\pi}{2}$$
for all $(x, y) \in (D \times D) \setminus \ker T$.

Indeed, the stated facts follow from the following obvious relation

$$\frac{\langle x - y, Tx - Ty\rangle}{||x - y|| \cdot ||Tx - Ty||} = \cos[d_{S^n}(pr_{S^n}(x - y), pr_{S^n}(Tx - Ty))].$$

Taking into account the characterizations above, a natural extension of monotonicity occurs. Recall that $0 \leq d_{S^n} \leq \pi$ for any $x, y \in S^n$.

Definition 2.1. Let $T : D \longrightarrow \mathbb{R}^{n+1}$ be a given operator, where D is a subset of $\mathbb{R}^{n+1}$.

1. T is said to be *h-increasing* if $d_{S^n}(pr_{S^n}(x-y), pr_{S^n}(Tx-Ty)) < \pi$ for all $(x,y) \in (D \times D) \setminus \ker T$.
2. T is said to be *h-decreasing* if $d_{S^n}(pr_{S^n}(x-y), pr_{S^n}(Tx-Ty)) > 0$ for all $(x,y) \in (D \times D) \setminus \ker T$.
3. T is said to be *h-monotone* if T is either h-increasing or T is h-decreasing.

Remark 2.2. Let $T : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be a linear isometry.

1. T is h-increasing if and only if $-1 \notin \mathrm{Spec}(A)$.
2. T is h-decreasing if and only if $1 \notin \mathrm{Spec}(A)$.

Indeed, if T is not h-increasing, then $d_{S^n}(pr_{S^n}(x-y), pr_{S^n}(Tx-Ty)) = \pi$ for some $(x,y) \in (D \times D) \setminus \ker T$, i.e.

$$\frac{Tx-Ty}{\|Tx-Ty\|} = -\frac{x-y}{\|x-y\|} \Leftrightarrow T(x-y) = -(x-y) \Rightarrow -1 \in \mathrm{Spec}(T).$$

Conversely, if $-1 \in \mathrm{Spec}(T)$, then $Tx = -x$ for some $x \neq 0$, i.e.

$$\frac{Tx}{\|x\|} = -\frac{x}{\|x\|} \Leftrightarrow \frac{Tx-To}{\|Tx-T0\|} = -\frac{x-0}{\|x-0\|} \Leftrightarrow d_{S^n}(pr_{S^n}(x-0), pr_{S^n}(Tx-T0)) = \pi$$

which shows that T is not h-increasing. The statement (1) can be similarly proved.

Remark 2.3. The vector-valued function $T : D \longrightarrow \mathbb{R}^{n+1}$ is h-monotone but not Minty-Browder monotone whenever $-1 < i_T < 0$, where i_T stands for

$$\inf\left\{\frac{\langle Tx-Ty, x-y\rangle}{\|Tx-Ty\| \cdot \|x-y\|} \mid (x,y) \in D \times D \setminus \ker T\right\}.$$

Several estimates of some parameters of monotonicity of type i_T are provided in [12].

Definition 2.4. Let $T, A : D \longrightarrow \mathbb{R}^{n+1}$ be given operators with A injective, where D is a subset of $\mathbb{R}^{n+1}$.

1. T is said to be *h-increasing relative to* A or simply A*-increasing* if
$$d_{S^n}(pr_{S^n}(Ax-Ay), pr_{S^n}(Tx-Ty)) < \pi, \forall (x,y) \in (D \times D) \setminus \ker T.$$
2. T is said to be *h-decreasing relative to* A or simply A*-decreasing* if
$$d_{S^n}(pr_{S^n}(Ax-Ay), pr_{S^n}(Tx-Ty)) > 0, \forall (x,y) \in (D \times D) \setminus \ker T.$$
3. T is said to be *h-monotone relative to* A or simply A*-monotone* if T is either A-increasing or T is A-decreasing.

Remark 2.5. Analyzing Definition 2.1, the next (geometric) interpretations become obvious.

1. T is A-increasing if and only if
$$\frac{Tx-Ty}{||Tx-Ty||} \neq \frac{Ay-Ax}{||Ay-Ax||},$$
for all $(x,y) \in (D \times D) \setminus \ker T$.

2. T is A-decreasing if and only if
$$\frac{Tx-Ty}{||Tx-Ty||}\neq\frac{Ax-Ay}{||Ay-Ax||},$$
for all $(x,y)\in(D\times D)\setminus\ker T$. In other words, for A-increasing operators, the action represented by Figure 1(a) is not allowed, while for A-decreasing operators, the action represented by Figure 1(b) is not allowed.

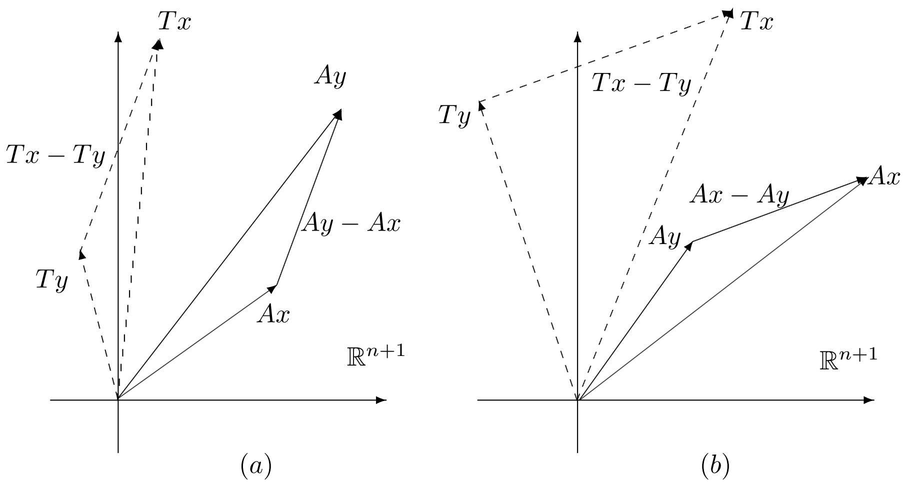

FIGURE 1. Actions not allowed for A-increasing/decreasing operators

3. T is A-increasing if and only if
$$\langle Tx-Ty,Ax-Ay\rangle>-\|Tx-Ty\|\cdot\|Ax-Ay\|,\forall(x,y)\in(D\times D)\setminus\ker T.$$
4. T is A-decreasing if and only if
$$\langle Tx-Ty,Ax-Ay\rangle<\|Tx-Ty\|\cdot\|Ax-Ay\|,\forall(x,y)\in(D\times D)\setminus\ker T.$$
5. T is A-increasing and A-decreasing if and only if
$$|\langle Tx-Ty,Ax-Ay\rangle|<\|Tx-Ty\|\cdot\|Ax-Ay\|,\forall x,y\in(D\times D)\setminus\ker(T).$$
6. If T is h-increasing/decreasing, then $T\circ A$ is A-increasing/decreasing.
7. Let $A:D\longrightarrow\mathbb{R}^{n+1}$ be an injective local homeomorphism/diffeomorphism, i.e. the range of A is open as well as the restriction and the corestriction
$$D\longrightarrow\mathrm{Im}(A),\ x\mapsto Ax$$
is a homeomorphism/diffeomorphism still denoted by A. Then T is A-increasing/decreasing if and only if the composition $T\circ A^{-1}:A(D)\longrightarrow\mathbb{R}^{n+1}$ is h-increasing/decreasing.

8. The h-increasing/decreasing monotonicity coincides with the i_D-increasing/decreasing monotonicity, where $i_D : D \hookrightarrow \mathbb{R}^{n+1}$ stands for the inclusion operator.
9. If $D \subseteq \mathbb{R}^{n+1}$ is a convex open set and $f : D \longrightarrow \mathbb{R}$ is a strictly convex function whose convexity is ensured by the everywhere positive definiteness of its Hessian matrix, then its gradient is an injective local diffeomorphism.

Remark 2.6. In the Definition 2.4 and in Remark 2.5, the role of the injective operator A can be taken over by a (possibly non-injective) local diffeomorphism which we could still denote by A. Thus, we obtain the definition and equivalent forms of monotonicity with respect to (possibly non-injective) local diffeomorphisms.

The increasing A-monotonicity allows the angles between the vectors $Tx-Ty$ and $Ax-Ay$ to exceed $\pi/2$ and approach π arbitrarily close for $(x,y) \in (D \times D) \setminus \ker(T)$, although the upper bound π is never reached by these angles in the case of increasing A-monotone operators. The classes of (A,η)-increasing and (A,η)-decreasing operators $\eta \in (-1,1)$ can still be defined by means of these angles which are not allowed to exceed the upper bound $\arccos \eta$, i.e.

$$\langle Tx - Ty, Ax - Ay \rangle \geq \eta \|Tx - Ty\| \cdot \|Ax - Ay\|, \ \forall x, y \in D$$

for the increasing option and they are not allowed to decrease under the lower bound $\arccos \eta$ for the decreasing option, i.e.

$$\langle Tx - Ty, Ax - Ay \rangle \leq \eta \|Tx - Ty\| \cdot \|Ax - Ay\|, \ \forall x, y \in D.$$

For $\eta = 0$ we call the first type of operators A-Minty-Browder increasing (or shortly $A-M-B$-increasing operators) and the second type A-Minty-Browder decreasing operators (or shortly $A-M-B$-decreasing operators). These angles are therefore allowed to exceed $\pi/2$ when $\eta \in (-1,0)$, for the increasing option, but not to approach π arbitrarily close. This ensure, for the class of (A,η)-increasing operators when $\eta \in (-1,0)$, the status of intermediate class between the class of A-Minty-Browder increasing operators and the class of h-increasing operators. If, on the contrary $\eta \in [0,1)$, then the class of η-monotone operators is contained in the class of A-Minty-Browder operators. A similar discussion can be done for decreasing operators. The η-increasing/decreasing monotonicity corresponds to the (i_D,η)-increasing/decreasing monotonicity, where $i_D : D \hookrightarrow \mathbb{R}^{n+1}$ stands for the inclusion. (see [12]).

Remark 2.7. Another direction in which the A-monotonicity can be extended, due to the Remarks 2.5[(1)-(4)], is for operators $T : D \longrightarrow H$, where $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space, $D \subseteq H$ is an open set and $A : D \longrightarrow H$ is injective. This is also the case for (A,η)-monotonicity.

Remark 2.8. Let $T : D \longrightarrow H$ be a given operator. If $A : H \longrightarrow H$ is a linear isomorphism, then the A-Minty-Browder increasing/decreasing monotonicity of T is equivalent with the Minty-Browder increasing/decreasing monotonicity of $A^* \circ T$.

Remark 2.9. 1. Let $A : H \longrightarrow H$ be a linear unitary automorphism, i.e. A is an isometric linear automorphism. Then T is $\left(A\big|_D, \eta\right)$-increasing/decreasing if and only if $A^* \circ T$ is η-increasing/decreasing.

2. If $A : H \longrightarrow H$ is a linear unitary automorphism, then T is $A\big|_D$-increasing/decreasing if and only if $A^* \circ T$ is h-increasing/decreasing.
3. If $A : D \longrightarrow H$ be an injective operator, then T is $\big(A\big|_D, \eta\big)$-increasing/decreasing if and only if $T \circ A^{-1}$ is η-increasing/decreasing.

Proposition 2.10. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $D \subseteq H$ be an open set. Let also $T : D \longrightarrow H$ be a given operator and let $A : H \longrightarrow H$ be a bounded linear isomorphism such that $b_{A^*} := \inf\{\|A^*z\| \ : \ \|z\| = 1\} > 0$. If $-b_{A^*} < \eta a_A \leq 0$ and T is (A, η)-increasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-increasing, where a_A stands for $\|A\|$. If $0 \leq \eta a_A < b_{A^*}$ and T is (A, η)-decreasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-decreasing.*

Proof. Assume that T is (A, η)-increasing for $\eta \in (-1, 0)$, i.e. we have

$$\begin{aligned}
&\langle Tx - Ty, Ax - Ay\rangle \geq \eta\|Tx - Ty\| \cdot \|Ax - Ay\| \Longleftrightarrow \\
&\langle (A^* \circ T)x - (A^* \circ T)y, x - y\rangle \geq \eta\|Tx - Ty\| \cdot \|Ax - Ay\|, \ \forall x, y \in D.
\end{aligned}$$

Therefore, for $x, y \in D$, $x \neq y$, we have

$$\begin{aligned}
&(A^* \circ T)x - (A^* \circ T)y, x - y\rangle \geq \eta\|Tx - Ty\| \cdot \|x - y\| \frac{\|Ax - Ay\|}{\|x - y\|} \\
&= \eta\|A^*(Tx) - A^*(Ty)\| \frac{1}{\left\|A^*\left(\dfrac{Tx - Ty}{\|Tx - Ty\|}\right)\right\|} \cdot \|x - y\| \cdot \left\|A\left(\frac{x - y}{\|x - y\|}\right)\right\| \\
&\qquad \geq \eta \left(\frac{\sup\limits_{\|z\|=1} \|Az\|}{\inf\limits_{\|z\|=1} \|A^*z\|}\right) \|(A^* \circ T)x - (A^* \circ T)y\| \cdot \|x - y\| \\
&\qquad = \frac{\eta a_A}{b_{A^*}} \cdot \|(A^* \circ T)x - (A^* \circ T)y\| \cdot \|x - y\|
\end{aligned}$$

and the proof of the first statement is now complete. The second statement can be similarly proved. $\square$

Corollary 2.11. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a linear isomorphism. If $-b_{A^*} < \eta a_A \leq 0$ and T is (A, η)-increasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-increasing. If $0 \leq \eta a_A < b_{A^*}$ and T is (A, η)-decreasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-decreasing.*

Remark 2.12. The gradients of strictly convex functions whose strict convexity is ensured by the everywhere positive definiteness of the Hessian matrix are good candidates to play the role of the injective operator A. Indeed, the gradient of such a function defined on a convex open subset D of $\mathbb{R}^n$ is injective, as the everywhere positive definiteness of the Hessian matrix is equivalent with the everywhere positive definiteness the Fréchet differentials of the gradient. In fact the Jacobian matrix of the gradient of such a C^2-smooth function is precisely the Hessian matrix of that function.

In fact the h-monotonicity of a certain operator coincides with its ∇f-monotonicity, where $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is the strictly convex function given by

$$f(x) = \frac{1}{2}\|x\|^2.$$

Proposition 2.13. *Let $T, A : D \longrightarrow \mathbb{R}^{n+1}$ be given operators with A injective, where D is a subset of $\mathbb{R}^{n+1}$. The operator $T + A$ is A-increasing and A-decreasing if and only if T is A-increasing and A-decreasing.*

Proof. Indeed,

$$\begin{aligned}\langle Tx + Ax - Ty - Ay, Ax - Ay\rangle^2 &= \big(\|Ax - Ay\|^2 + \langle Tx - Ty, Ax - Ay\rangle\big)^2\\ = \|Ax - Ay\|^4 + 2\|Ax - Ay\|^2&\langle Tx - Ty, Ax - Ay\rangle + \langle Tx - Ty, Ax - Ay\rangle^2.\end{aligned}$$

and

$$\begin{aligned}\|Tx + Ax - Ty - Ay\|^2 \cdot \|Ax - Ay\|^2 &= \|Tx - Ty + Ax - Ay\|^2 \cdot \|Ax - Ay\|^2\\ = \big(\|Ax - Ay\|^2 + 2\langle Tx - Ty, Ax - Ay\rangle &+ \|Tx - Ty\|^2\big) \cdot \|Ax - Ay\|^2\\ = \|Ax - Ay\|^4 + 2\|Ax - Ay\|^2\langle Tx - Ty, Ax &- Ay\rangle + \|Tx - Ty\|^2\|Ax - Ay\|^2.\end{aligned}$$

The statement follows easily by using Remark 2.5(5). □

Corollary 2.14. *Let $D \subseteq \mathbb{R}^{n+1}$ be a convex open set and $f : D \longrightarrow \mathbb{R}$ be a C^2-smooth strictly convex function whose convexity is ensured by the everywhere positive definiteness of its Hessian matrix. The operator $T + \nabla f$ is ∇f-increasing and ∇f-decreasing if and only if $T : D \longrightarrow \mathbb{R}^{n+1}$ is ∇f-increasing and ∇f-decreasing.*

3. On the degree of some spherical projections

Since in our study on h-monotone operators the *degree* of differentiable maps plays an important role, in this section we discuss some of its properties. In this respect we first recall the notions of *critical/regular points* and *critical/regular values.*

Let M, N be differential manifolds and $f : M \to N$ be a differentiable mapping. We first define the *rank* of f at a point $p \in M$ as $\text{rank}_p f := \text{rank}(df)_p = \dim[\text{Im}(df)_p$, where $(df)_p : T_p(M) \longrightarrow T_{f(p)}(N)$ is the differential (or tangent map) of f at p, and observe that $\text{rank}_p f \leq \min\{m, n\}$, where $m = \dim(M)$ and $n = \dim(N)$. The point $p \in M$ is called a *regular* point of f if $\text{rank}_p f = \min\{m, n\}$ and it is called *critical point* of f if $\text{rank}_p f < \min\{m, n\}$. One can immediately observe that the set $\mathcal{R}(f)$ of all regular points of f is open while the set $\mathcal{C}(f) := M \backslash \mathcal{R}(f)$ of all critical points of f is closed. A value $y \in f(\mathcal{C}(f)) =: \mathcal{B}(f)$ is called *critical value* of f, and a point $q \in N \backslash \mathcal{B}(f)$ is called *regular value* of f.

If $m = n$, then a point $x \in M$ is a regular point of $f : M \longrightarrow N$ if and only if f is a local diffeomorphism at x. Consequently the preimage $f^{-1}(y)$ of a regular value y of f is discrete. If f is additionally proper (i.e. the inverse images of compact sets are compact), then the preimage $f^{-1}(y)$ of such a regular value is finite.

If $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a hypersurface, i.e. an n-dimensional submanifold, and $p \in \mathcal{H}$, then denote by $\mathcal{T}_p(\mathcal{H})$ the collection of all tangent vectors $\gamma'(0)$, where

$\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{H}$ is a parameterized differentiable curve such that $\gamma(0) = p$ and recall that $\mathcal{T}_p(\mathcal{H})$ is an n-dimensional vector subspace of $\mathbb{R}^{n+1}$. Denote by $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathbb{R}^{n+1}$ the inclusion mapping and recall that two hyperplanes of $\mathbb{R}^{n+1}$ are orthogonal if their normal vectors are orthogonal. A compact hypersurface of $\mathbb{R}^{n+1}$ without boundary (in the sense of manifold theory) will be called *closed hypersurface.*
If M, N are compact connected oriented n-dimensional manifolds, $f : M \longrightarrow N$ is a differentiable map and $y \in N$ is a regular value of f, then

$$\deg_y(f) := \begin{cases} \displaystyle\sum_{x \in f^{-1}(y)} \varepsilon_x & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

where

$$\varepsilon_x := \begin{cases} 1 & \text{if } (df)_x \text{ preserves the orientation} \\ -1 & \text{if } (df)_x \text{ reverses the orientation.} \end{cases}$$

In fact $\deg_y(f)$ does not depend on y and is called the *degree* of f, being simply denoted by $\deg(f)$ (see [1], pp. 253]). If f is not onto, observe that $\deg(f) = 0$, since every $y \in N \setminus \mathrm{Im}(f)$ is a regular value of f. On the other hand, one can show that deg is invariant on differential homotopy classes of maps from M to N. Since every continuous homotopy class of maps from M to N contains a differentiable map, the notion of *degree* can be extended to the class of all continuous maps and its invariance on continuous homotopy classes is part of the extension procedure. For more details the reader could consult [8], pp. 165, 166, 21-221]. A different approach of degree theory for continuous maps appears in [7], pp. 62-65, 266-271].

Proposition 3.1. *If X is a topological space and $f, g : X \longrightarrow S^n, n \geq 1$ are continuous maps such that $d_{S^n}(f(x), g(x)) < \pi$ for all $x \in X$, then $f \simeq g$, i.e. f and g are homotopic.*

Proof. Indeed, the following homotopy

$$H : X \times [0,1] \longrightarrow S^n, \; H(x,t) := \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is well defined and $H(\cdot, 0) = f, H(\cdot, 1) = g$. □

Remark 3.2. If X is a topological space and $f, g : X \longrightarrow S^n, n \geq 1$ are continuous maps such that $d_{S^n}(f(x), g(x)) > 0$ for all $x \in X$, then $f \simeq -g$, i.e. f and $-g$ are homotopic.

For a given function $f : X \longrightarrow Y$ we define its kernel as the equivalence relation on X whose graph is $\ker(f) := \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$. The next statements reveal some important homotopy properties of the A-monotone operators.

Corollary 3.3. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be an injective local diffeomorphism. If $T : D \to \mathbb{R}^{n+1}$ is an A-monotone operator, then the map*

$$D \times D \setminus \ker(T) \to S^n, (x, y) \longmapsto pr_{S^n}(Tx - Ty)$$

is homotopic to one of the maps

$$D \times D \setminus \ker(T) \to S^n, (x, y) \longmapsto pr_{S^n}(Ax - Ay)$$

or

$$D \times D \setminus \ker(T) \to S^n, (x, y) \longmapsto pr_{S^n}(Ay - Ax).$$

Corollary 3.4. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be an injective local diffeomorphism. If $T : D \to \mathbb{R}^{n+1}$ is a differentiable A-monotone operator and $\mathcal{H} \subset D$ is a closed connected hypersurface, then the degree of $\mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az)$ is invariant over every connected component of $D \setminus T^{-1}(T(\mathcal{H}))$.*

Proof. We assume that T is A-increasing, as the decreasing option can be similarly treated. Let us consider a continuous path $\gamma : [0, 1] \longrightarrow D \setminus T^{-1}(T(\mathcal{H}))$. Therefore $\gamma(0) = z_0$ and $\gamma(1) = z_1$ belong to the same connected component of $T^{-1}(T(\mathcal{H}))$. By using Corollary 3.3 one can deduce that

$$\mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0) \simeq \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0)$$

and

$$\mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1) \simeq \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1)$$

along with

$$\deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0) = \deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0) \tag{3.1}$$

and

$$\deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1) = \deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1). \tag{3.2}$$

On the other hand

$$H : \mathcal{H} \times [0, 1] \longrightarrow T^{-1}(T(\mathcal{H})),\ H(x, t) = \frac{Tx - T(\gamma(t))}{\|Tx - T(\gamma(t))\|}$$

realizes a homotopy between $\mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0)$ and $\mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1)$. Therefore

$$\deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0) = \deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1)$$

which combined with (3.1) and (3.2) leads us to the equality

$$\deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0) = \deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1).$$

□

Remark 3.5. Let X be a compact differential n-dimensional manifold and

$$f, g : X \longrightarrow S^n,\ n \geq 1$$

be continuous maps such that

1. If $d_{S^n}(f(x), g(x)) < \pi$ for all $x \in X$, then $\deg(g) = \deg(f)$. Indeed, f and g are, according to Proposition 3.1, homotopic to each other.

2. If $d_{S^n}(f(x), g(x)) > 0$ for all $x \in X$, then $\deg(g) = (-1)^{n+1}\deg(f)$. Indeed,
$$d_{S^n}(f(x), g(x)) > 0,\ \forall\, x \in X \Leftrightarrow d_{S^n}(-f(x), g(x)) < \pi,\ \forall\, x \in X,$$
which shows, according to Proposition 3.1, that
$$\deg(g) = \deg(-f) = \deg(A \circ f) = \deg(A)\deg(f) = (-1)^{n+1}\deg(f),$$
where $A : S^n \longrightarrow S^n, Ax = -x$ is the antipodal map.
Consequently, if $\deg(g) \neq (-1)^{n+1}\deg(f)$, then the *coincidence set*
$$C(f, g) := \{x \in X : f(x) = g(x)\}$$
is not empty.

If $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a closed connected hypersurface, then, according to [9], Theorem 4.6] and the related results therein, $\mathcal{H}$ separates $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1} \setminus \mathcal{H}$ has precisely two connected components, one of which is bounded and denoted by $int(\mathcal{H})$ and another one which is unbounded and denoted by $ext(\mathcal{H})$.
On the other hand $\partial\big[int(\mathcal{H})\big] = \mathcal{H} = \partial\big[ext(\mathcal{H})\big]$, where ∂S stands for the topological frontier of S.

Proposition 3.6. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism. If $\mathcal{H} \subset D \subseteq \mathbb{R}^{n+1}$ is a closed connected hypersurface, then*
$$\deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = 0,\ \ \forall z \in A^{-1}(ext(A(\mathcal{H}))),$$
and either
$$\deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = 1, \forall z \in A^{-1}(int(A(\mathcal{H})))$$
or
$$\deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = -1, \forall z \in A^{-1}(int(A(\mathcal{H})).$$

Proof. Since $A : D \longrightarrow \mathbb{R}^{n+1}$ is an injective local diffeomorphism, the image $A(\mathcal{H})$ of $\mathcal{H}$ through A is a closed connected hypersurface and according to [10, Proposition 3.7] we conclude that
$$\deg[pr_{S^n} \circ (i_{A(\mathcal{H})} - Az)] = 0 \text{ for all } z \in A^{-1}(ext(\mathcal{A}(\mathcal{H}))),$$
as well as either
$$\deg[pr_{S^n} \circ (i_{\mathcal{H}} - Az)] = 1 \text{ for all } z \in A^{-1}(int(A(\mathcal{H})))$$
or
$$\deg[pr_{S^n} \circ (i_{A(\mathcal{H})} - Az)] = -1 \text{ for all } z \in A^{-1}(int(A(\mathcal{H}))).$$
Note that
$$pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = [pr_{S^n} \circ (i_{A(\mathcal{H})} - Az)] \circ r,$$
where r stands for the restriction and corestriction $\mathcal{H} \longrightarrow A(\mathcal{H}),\ x \mapsto Ax$, which is a diffeomorphism. The multiplicative property of the degree combined with the obvious fact that either $\deg r \equiv 1$ or $\deg r \equiv -1$, concludes the proof. $\square$

4. Properties of the inverse images of A-monotone operators

In this section we provide some examples of closed subsets of the Euclidean space $\mathbb{R}^{n+1}$ which can be separated by closed connected hypersurfaces. We close this section by proving that the inverse images of continuous A-monotone operators cannot be separated by closed smooth hypersurfaces.

Definition 4.1. A subset X of $\mathbb{R}^{n+1}$ is separated by a closed connected hypersurface $\mathcal{H}$ of $\mathbb{R}^{n+1}$ if $\mathcal{H} \subseteq \mathbb{R}^{n+1} \setminus X$ and each $int(\mathcal{H})$, $ext(\mathcal{H})$ contains a connected component of X at least. We say that X is *divisible by closed connected hypersurfaces* if X is separated by one closed connected hypersurface, at least. Otherwise we say that X is *indivisible by closed connected hypersurfaces.*

Theorem 4.2. ([10]) *If the closed set $C \subset \mathbb{R}^{n+1}$ has a compact connected component K such that $C \setminus K$ is nonempty and closed, then C is divisible by closed connected hypersurfaces.*

Theorem 4.3. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. If $y \in \operatorname{Im}(T)$, then $T^{-1}(y)$ is indivisible by closed connected hypersurfaces.*

Proof. Assume that $T^{-1}(y)$ is divisible by closed connected hypersurfaces, for some $y \in \operatorname{Im}(T)$ and consider a closed connected hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ with the property that one component of $T^{-1}(y)$, say C, is contained in $int(\mathcal{H})$ and another component of $T^{-1}(y)$, say K, is contained in $ext(\mathcal{H})$.
If $z_0 \in C \subseteq A^{-1}(int(A(\mathcal{H}))$ and $z_1 \in K \subseteq A^{-1}(ext(A(\mathcal{H}))$, then, according to Proposition 3.6 and Corollary 3.4, one gets

$$\pm 1 = \deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0)] = \pm \deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1)] = 0,$$

which is absurd. □

Corollary 4.4. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. If $y \in \operatorname{Im}(T)$, then either $T^{-1}(y)$ is connected or the set $T^{-1}(y) \setminus K$ is not closed for every compact connected component K of $T^{-1}(y)$.*

Proof. Assume that $T^{-1}(y)$ is not connected and $T^{-1}(y) \setminus K$ is closed for some compact connected component K of $T^{-1}(y)$. Then $T^{-1}(y) \setminus K$ is nonempty and the statement follows by using Theorems 4.2, 4.3. □

Theorem 4.5. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. If $q \in \operatorname{Im}(T)$, then either $T^{-1}(q)$ is a singleton or $\dim\big(T^{-1}(q)\big) \geq 1$.*

Proof. Recall that, according to Remark 2.5(7), the operator $T : D \longrightarrow \mathbb{R}^{n+1}$ is A-monotone if and only if $T \circ A^{-1} : A(D) \longrightarrow \mathbb{R}^{n+1}$ is h-monotone. By using [10, Theorem 4.8] one gets that either $(T \circ A^{-1})^{-1}(q) = A(T^{-1}(q))$ is a singleton or

$$\dim(T \circ A^{-1})^{-1}(q) = \dim(A(T^{-1}(q)) \geq 1,$$

i.e., $T^{-1}(q)$ is a singleton or $\dim\big(T^{-1}(q)\big) \geq 1$. □

The next properties of the inverse images of A-monotone operators are immediate consequences of Theorem 4.5.

Corollary 4.6. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. Then either T is injective or $\dim\left(T^{-1}(y)\right) \geq 1$ for some $y \in \mathrm{Im}(T)$.*

Proof. If T is not injective, then $\mathrm{card}\left(T^{-1}(y)\right) \geq 2$ for some $y \in \mathrm{Im}(T)$. According to Theorem 4.5, $\dim\left(T^{-1}(y)\right) \geq 1$. □

Definition 4.7. ([5]) A continuous map $f : X \to Y$ is said to be *light* if

$$\dim\left(f^{-1}(y)\right) \leq 0 \text{ for every } y \in Y.$$

Observe that locally injective operators are light, as their inverse images are discrete sets.

Corollary 4.8. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ a continuous A-monotone light operator, then T is injective.*

Proof. We need to prove that $\mathrm{card}\left[A^{-1}(q)\right] = 1$ for each $q \in \mathrm{Im}(A)$. Indeed, if $\mathrm{card}\left[A^{-1}(q)\right]$ were at least 2 for some $q \in \mathrm{Im}(A)$, then, according to Theorem 4.5, we would get $\dim\left(A^{-1}(q)\right) \geq 1$. □

Corollary 4.9. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ a continuous A-monotone local homeomorphism, then T is injective.*

Corollary 4.10. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ is a C^1-smooth A-monotone operator, then T is locally injective. Indeed, according to Corollary 4.9, the restriction $T\big|_U$, where $U \subseteq D$ is an open set, is injective whenever the restriction $A\big|_U$ is injective. The later type of restrictions are one-to-one for suitable choices of the open set $U \subseteq D$, as A is a local diffeomorphism. If T is additionally open, then one can conclude that T is a local homeomorphism.*

Remark 4.11. Observe that Corollary 4.9 can be also obtained from Theorem 4.3. Indeed the inverse images of local diffeomorphisms, as discrete sets, are divisible by closed connected hypersurfaces provided their cardinality is at least two.

5. Pairs of mutual monotone local homeomorphisms

In this section we deal with pairs operators having *a priori* the same status relative to injectivity, i.e. they are local homeomorphisms.

Definition 5.1. Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local homeomorphisms. The two local homeomorphisms are said to be

1. *mutual* h-increasing if

$$d_{S^n}\left(pr_{S^n}(Tx - Ty), pr_{S^n}(Qx - Qy)\right) < \pi, \ \forall (x, y) \in (D \times D) \setminus (\ker T \cup \ker Q).$$

2. *mutual* h-decreasing if

$$d_{S^n}(pr_{S^n}(Tx - Ty), pr_{S^n}(Qx - Qy)) > 0, \forall (x, y) \in (D \times D) \setminus (\ker S \cup \ker Q).$$

3. *mutual* h-monotone if they are either *mutual* h-increasing or *mutual* h-decreasing.

Remark 5.2. Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ is an open C^1-smooth A-monotone operator, then, according to Corollary 4.10, the operators T and A have *a posteriori* some rather close status with respect to injectivity, i.e. A is a local diffeomorphism and T is a local homeomorphism and the two operartors T and A are mutual h-monotone local homeomorphisms.

Remark 5.3. Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local homeomorphisms.

1. The two local homeomorphisms are *mutual* h-increasing if and only if

$$\langle Tx - Ty, Qx - Qy\rangle > -\|Tx - Ty\| \cdot \|Qx - Qy\|, \forall (x, y) \in (D \times D) \setminus (\ker T \cup \ker Q);$$

2. The two local homeomorphisms are *mutual* h-decreasing if and only if

$$\langle Tx - Ty, Qx - Qy\rangle < \|Tx - Ty\| \cdot \|Qx - Qy\|, \forall (x, y) \in (D \times D) \setminus (\ker S \cup \ker Q).$$

3. The relation of being *mutual* h-increasing/decreasing is symmetric.

Theorem 5.4. *If $D \subseteq \mathbb{R}^n$ is an open set and $T, Q : D \longrightarrow \mathbb{R}^n$ are two mutual h-monotone local diffeomorphisms, then* $\ker S = \ker T$.

Proof. By using Corollary 4.9 it follows that the two local diffeomorphisms are simultaneously injective or non-injective. Their injectivity is equivalent with

$$\ker T = \ker Q = \Delta_D := \{(x, x) \ : \ x \in D\}.$$

We now assume that none of them is injective as well as $\ker T \setminus \ker Q \neq \emptyset$ and consider $(u, v) \in \ker T \setminus \ker Q$, i.e. $Tu = Tv$ and $Qu \neq Qv$. Let $r, \varepsilon > 0$ be such that $T(B(u, r + \varepsilon))$, $Q(B(u, r + \varepsilon))$ are open and the restrictions

$$\bar{B}(u, r) \longrightarrow T(\bar{B}(u, r)), \ x \mapsto Tx$$
$$\bar{B}(u, r) \longrightarrow Q(\bar{B}(u, r)), \ x \mapsto Qx$$

are diffeomorphisms and $Qv \notin Q(\bar{B}(u, r))$. In particular the sphere $S(p, r)$ is mapped by T onto a closed hypersurface $T(S(p, r))$. Since the local diffeomorphisms T, Q are mutual h-monotone, it follows that either

$$d_{S^n}(pr_{S^n} \circ (T\big|_{S(u,r)} - Tv), pr_{S^n} \circ (T\big|_{S(u,r)} - Tv)) < \pi$$

or

$$d_{S^n}(pr_{S^n} \circ (T\big|_{S(u,r)} - Tv), pr_{S^n} \circ (Q\big|_{S(u,r)} - Qv)) > 0.$$

In both cases we get, via Remark 3.5, that

$$\deg pr_{S^n} \circ (Q\big|_{S(u,r)} - Qv) = \pm \deg pr_{S^n} \circ (T\big|_{S(u,r)} - Tv) = \pm \deg pr_{S^n} \circ (T\big|_{S(u,r)} - Tu).$$

On the other hand, by using Proposition 3.6

$$\deg pr_{S^n} \circ (Q\big|_{S(u,r)} - Qv) = 0,$$

as $Qv \in ext\ (S(u,r))$ and

$$\deg pr_{S^n} \circ (T\big|_{S(u,r)} - Tu) = \pm 1,$$

as $Tu \in int\ T(S(u,r))$, which is absurd.
Therefore $\ker T \setminus \ker Q = \emptyset \iff \ker T \subseteq \ker Q$. The opposite inclusion can be similarly done by interchanging the roles of T and Q. □

Theorem 5.5. *Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local diffeomorphisms. Then T, Q are mutual h-monotone if and only if there exists a h-monotone diffeomorphism*

$$\Phi : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$$

such that $T = \Phi \circ Q$.

Proof. If Φ is h-increasing and $T = \Phi \circ Q$, then

$$\begin{aligned}\langle Tx - Ty, Qx - Qy\rangle &= \langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle \\ &> -\|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\| \\ &= -\|Tx - Ty\| \cdot \|Qx - Qy\|,\end{aligned}$$

i.e. $T = \Phi \circ Q$ is h-increasing. If Φ is h-decreasing and $T = \Phi \circ Q$, then

$$\begin{aligned}\langle Tx - Ty, Qx - Qy\rangle &= \langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle < \|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\| \\ &= \|Tx - Ty\| \cdot \|Qx - Qy\|\end{aligned}$$

i.e. $T = \Phi \circ Q$ is h-decreasing. Conversely, if T, Q are mutual h-monotone, then $\ker T = \ker Q$, due to Theorem 5.4. The functions

$$\alpha : D/\ker T \longrightarrow \mathrm{Im}\ T,\ \ \alpha(d + \ker T) = T(d)$$
$$\beta : D/\ker Q \longrightarrow \mathrm{Im}\ Q,\ \ \beta(d + \ker Q) = Q(d)$$

are well-defined, bijective and $T = \alpha \circ \pi_{\ker T}$ and $Q = \beta \circ \pi_{\ker Q}$, where

$$\pi_{\ker T} : D \longrightarrow D/\ker T \text{ and } \pi_{\ker Q} : D \longrightarrow D/\ker Q$$

are the canonical projections. The bijections α and β are also unique with their corresponding properties. Since $\ker T = \ker Q$, it follows that $D/\ker T = D/\ker Q$ and

$$\mathrm{Im}\ Q \xleftarrow{\beta} D/\ker Q = D/\ker T \xrightarrow{\alpha} \mathrm{Im}\ T$$

are bijections. Therefore $\Phi := \alpha \circ \beta^{-1} : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$ is a bijection and

$$\Phi \circ Q = \alpha \circ \beta^{-1} \circ Q = \alpha \circ \pi_{\ker Q} = \alpha \circ \pi_{\ker T} = T.$$

Since T and Q are local diffeomorphisms it follows that Φ is differentiable and a diffeomorphism therefore. Finally

$$\begin{aligned}\langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle &= \langle Tx - Ty, Qx - Qy\rangle > -\|Tx - Ty\| \cdot \|Qx - Qy\| \\ &= -\|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\|\end{aligned}$$

if T, Q are mutual h-increasing and

$$\begin{aligned}\langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle &= \langle Tx - Ty, Qx - Qy\rangle < \|Tx - Ty\| \cdot \|Qx - Qy\| \\ &= \|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\|\end{aligned}$$

if T, Q are mutual h-decreasing. In other words, Φ is h-increasing/decreasing if T, Q are mutual h-increasing/decreasing. $\square$

Corollary 5.6. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ is an open C^1-smooth A-monotone operator, then there exists a h-monotone homeomorphism*

$$\Phi : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$$

such that $T = \Phi \circ Q$.

Proof. According to Remark 5.2, T is a local homeomorphism and T, A are obviously mutual h-monotone local homeomorphisms. From now on the proof works along the same lines with the proof of Theorem 5.5. $\square$

Corollary 5.7. *Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local diffeomorphisms. If T, Q are mutual h-monotone, then*

$$\mathrm{Val}(T) = \mathrm{Val}(Q),$$

where

$$\mathrm{Val}(F) := \sup\{\mathrm{card} F^{-1}(y) : y \in \mathbb{R}^n\}$$

stands for the valence of $F : D \longrightarrow \mathbb{R}^n$, as defined in [15].

Proof. Indeed, according to Theorem 5.5 there exists an h-monotone diffeomorphism

$$\Phi : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$$

such that $T = \Phi \circ Q$. Thus $T^{-1}(y) = (\Phi \circ Q)^{-1}(y) = Q^{-1}(\Phi^{-1}(y))$ for every $y \in \mathrm{Im}(T)$, which implies that

$$\mathrm{card}\ T^{-1}(y) = \mathrm{card}\ Q^{-1}(\Phi^{-1}(y)), \forall y \in \mathrm{Im}(T)$$

and shows that

$$\begin{aligned}\mathrm{Val}(T) &= \sup\{\mathrm{card}\ T^{-1}(y) : y \in \mathrm{Im}(T)\}\\ &= \sup\{\mathrm{card}\ Q^{-1}(\Phi^{-1}(y)) : y \in \mathrm{Im}(T)\}\\ &= \sup\{\mathrm{card}\ Q^{-1}(z) : z \in \mathrm{Im}(Q)\} = \mathrm{Val}(Q).\end{aligned}$$

$\square$

Remark 5.8. In the proof of Corollary 5.7, we only used the quality of Φ to be globally injective, not its quality to be differentiable with differentiable inverse.

6. Final comments and remarks

Throughout the section we make use of the notation described below (see [18]). Let D be a nonempty open convex subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a C^2-smooth convex function. The Hessian matrix of f at an arbitrary point $x \in D$ will be denoted by $H_x(f)$. Recall that $H_x(f)$ is a symmetric matrix and it defines a symmetric bilinear functional

$$\mathcal{H}_x(f) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},\ \mathcal{H}_x(f)(u, v) := u \cdot H_x(f) \cdot v^T.$$

The following region

$$\mathrm{Hess}^+(f) := \{x \in D | H_x(f) \text{ is positive definite}\}$$

associated to some C^2-smooth regular function $f : D \longrightarrow \mathbb{R}$ was described in [17] for the particular polynomial function

$$f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2).$$

Denote by $h_x(f) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the linear transformation defined by the following equality $\mathcal{H}_x(f)(u, v) := \langle h_x(f)u, v\rangle$, $\forall u, v \in \mathbb{R}^n$. and set

$$\sigma_f := \sup_{z \in D} \|h_f(z)\|.$$

Let further $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, and let $T : D \to \mathbb{R}^n$ be the vector-valued function defined by $Tx := \nabla f(x) + Ax$. We shall denote by $[A]$ the matrix representation of A with respect to the standard basis of $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere (i.e., centered at the origin) in $\mathbb{R}^n$, and let

$$W(A) := \{\langle Ax, x\rangle \mid x \in S^{n-1}\}$$

be the *numerical range* of A. It is well known that $W(A) = [\lambda_A, \mu_A]$, where λ_A and μ_A denote the smallest and the greatest eigenvalue, respectively, of the symmetric operator $(A + A^*)/2$. Let also λ^*_A and μ^*_A denote the smallest and the greatest eigenvalue, respectively, of the symmetric positive semidefinite operator A^*A. It is well known that

$$\|A\| := \max_{x \in S^{n-1}} \|Ax\| = \sqrt{\mu^*_A} \qquad \text{and} \qquad \min_{x \in S^{n-1}} \|Ax\| = b_A. \tag{6.1}$$

Sometimes we set, for brevity, $a_A := \|A\| = \sqrt{\mu^*_A}$ and $b_A := 1/\|A^{-1}\|$ if A is invertible. Since $\|A^{-1}\|$ equals the square root of the greatest eigenvalue of

$$(A^{-1})^* A^{-1} = (AA^*)^{-1},$$

it follows that $b_A = \sqrt{\lambda^*_A}$.

Theorem 6.1. ([18]) *Let $D \subseteq \mathbb{R}^n$ be a convex open set, let $f : D \longrightarrow \mathbb{R}$ be a C^2-smooth convex function and let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear automorphism. If the following inequalities are satisfied*

$$\sigma_f < b_A + \lambda_A \textit{ and } \inf_{z \in D} \|h_f(z)\| < -\mu_A,$$

then $-1 < i_{\nabla f + A} < 0$, namely $\nabla f + A$ is h-monotone but not monotone.

Theorem 6.2. ([18]) *Let $D \subseteq \mathbb{R}^n$ be a convex open set, let $f : D \longrightarrow \mathbb{R}$ be a C^2-smooth convex function and let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear automorphism. If the following inequality is satisfied*

$$\sigma_f < \min\{b_A + \lambda_A,\ -\mu_A\}, \tag{6.2}$$

then $T := \nabla f + A$ is injective.

Remark 6.3. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^2-smooth functions such that $\mathrm{Hess}^+(f) \neq \emptyset$ and $\mathrm{Hess}^+(g) = \mathbb{R}^n$. Then $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ and $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are $\nabla g\big|_D$-increasing for every convex open subset D of $\mathbb{R}^n$ such that the range of $\nabla g\big|_D$ is contained in $\mathrm{Hess}^+(f)$, where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear automorphism related with f through the inequality (6.2). Thus $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ and $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are one-to-one as ∇g is a Minty-Browder monotone global diffeomorphism.

Example 6.4. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^2-smooth function such that $\mathrm{Hess}^+(f) \neq \emptyset$ and the smooth function

$$g : \mathbb{R}^n \longrightarrow \mathbb{R},\ g(x) = \frac{1}{2} e^{\|x\|^2}.$$

Then its gradient $(\nabla g)_x = e^{\|x\|^2} \cdot x$ is a Minty-Browder monotone global diffeomorphism, as the Hessian matrix $H_x(g) = e^{\|x\|^2}(I_n + 2x \cdot x^T)$ of g, which is actually the Jacobian matrix of ∇g, is positive definite. Indeed, the diagonal determinants $\Delta_k = 1 + 2(x_1^2 + \cdots + x_k^2)$ of $I_n + 2x \cdot x^T$ are all positive and the positive definiteness of $I_n + 2x \cdot x^T$ follows via the Sylvester criterion. Therefore $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ along with $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are $\nabla g\big|_D$-increasing for every convex open subset D of $\mathbb{R}^n$ such that the range of $\nabla g\big|_D$ is contained in $\mathrm{Hess}^+(f)$, where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear automorphism related with f through the inequality (6.2). Thus $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ and $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are one-to-one as ∇g is a Minty-Browder monotone global diffeomorphism.

Remark 6.5. For the global injectivity of ∇g alone, in Example 6.4, we need neither the Minty-Browder monotonicity of ∇g nor the positive definiteness of $H(g)$, as the injectivities of its restrictions to the spheres centered at the origin and to the half lines starting from the origin are rather obvious. For example the injectivity of the restriction of ∇g to the half line $\{\lambda x |\ \lambda > 0\}$ generated by $x \neq 0$ reduces to the injectivity of the function

$$\varphi : (0, \infty) \longrightarrow \mathbb{R},\ \varphi(\lambda) = \frac{\|(\nabla g)_{\lambda x}\|}{\|x\|^2} = \lambda e^{\lambda^2 \|x\|^2}.$$

Note however that the outcome of the Minty-Browder monotonicity of ∇g along with the positive definiteness of $H(g)$, in Example 6.4, does not reduce to the global injectivity of ∇g alone, but also ensure the differentiability of its inverse.

Remark 6.6. Let $D \subseteq \mathbb{R}^n$ be a convex open set and $f, g : D \longrightarrow \mathbb{R}$ be C^2-smooth functions such that $\mathrm{Hess}^+(f) \neq \emptyset$. Then $(\nabla f + A) \circ \nabla f\big|_{\mathrm{Hess}^+(f)}$ is $\nabla f\big|_{\mathrm{Hess}^+(f)}$-increasing, where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear automorphism related with f through the inequality (6.2). Indeed, $\nabla f + A$ is, according to Theorems 6.1 and 6.2, an h-monotone global injective operator. Therefore, according to Corollary 5.7 and the Remark 5.8 afterwards, we obtain:

$$\mathrm{Val}\,(\nabla f + A) \circ \nabla f\big|_{\mathrm{Hess}^+(f)} = \mathrm{Val}\left(\nabla f\big|_{\mathrm{Hess}^+(f)}\right).$$

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