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On some qualitative properties of Ćirić's fixed point theorem

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. It is well known that of all the extensions of the Banach-Caccioppoli Contraction Principle, the most general result was established by Ćirić in 1974. In this paper, we will present some results related to Ćirić type operator in complete metric spaces. Existence and uniqueness are re-called and several stability properties (data dependence and Ostrowski stability property) are proved. Using the retraction-displacement condition, we will establish the well-posedness and the Ulam-Hyers stability property of the fixed point equation $x = f(x)$.

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1. Introduction and preliminaries

Let (X, d) be a metric space and $f : X \to X$ be an operator. For $A \subset X$, let $\delta(A) := sup\{d(a, b) : a, b \in A\}$ the diameter of the set A. For each $x \in X$, we denote:

$$O(x, n) = \{x, f(x), ..., f^n(x)\}, n = 1, 2, ...,$$

$$O(x, \infty) = \{x, f(x), ..., f^n(x), ...\}.$$

Definition 1.1. (Ćirić [2]) Let (X, d) be a metric space and $f : X \to X$ be an operator. Then X is said to be f-orbitally complete if every Cauchy sequence which is contained in $O(x, \infty)$, for some $x \in X$ converges in X.

The following classes of operators in a metric space (X, d) are important for our approach.

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Definition 1.2. Let (X, d) be a metric space and $f : X \to X$ be an operator. Then f is said to be an α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X. \tag{1.1}$$

Definition 1.3. (Rus [6]) Let (X, d) be a metric space and $f : X \to X$ be an operator. Then f is said to be a graphic α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X. \tag{1.2}$$

Through this paper we denote $\mathbb{N} := \{0, 1, 2, \cdots\}$ the set of all natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

We recall that, $Fix(f) = \{x \in X | x = f(x)\}$ is the fixed point set of f and we denote by $(f^n(x))_{n\in\mathbb{N}}$ the sequence of Picard iterates for f starting from $x_0 \in X$, where $f^n = f \circ f \circ \cdots \circ f$ for n-times. Notice that the sequence of Picard iterates for f starting from $x_0 \in X$ can be recursively defined by the formula $x_{n+1} = f(x_n)$, for $n \in \mathbb{N}$, where $x_n := f^n(x_0), n \in \mathbb{N}$.

Definition 1.4. (Ćirić [2]) An operator $f : X \to X$ is said to be a generalized contraction if and only if for every $x, y \in X$ there exists nonnegative numbers q, r, s and t, which may depend on both x and y, such that $sup\{q + r + s + 2t : x, y \in X\} < 1$ and

$$\begin{aligned} d(f(x), f(y)) \leq q \cdot d(x, y) + r \cdot d(x, f(x)) + \\ + s \cdot d(y, f(y)) + t \cdot [d(x, f(y)) + d(y, f(x))] . \end{aligned}$$

Definition 1.5. (Ćirić [2]) Let (X, d) be a metric space and $f : X \to X$ be an operator. Then X is said to be a Ćirić type operator (named a quasi-contraction in the original paper [2]) if there exists a number $q \in (0, 1)$, such that

$$d(f(x), f(y)) \leq q \cdot max \{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\} , \tag{1.3}$$

for all $x, y \in X$.

It is well known (see [4]) that of all the extensions of Banach-Caccioppoli Contraction Principle, the most general result was established by Ćirić in 1974 for the above class of operators.

In the following example we present a Ćirić type operator which is not a generalized contraction.

Example 1.6. Let

$$X_1 = \left\{ \frac{m}{n} : m = 0, 1, 2, 4, 6, ...; n = 1, 3, 7, ..., 2k + 1, ... \right\} ,$$

$$X_2 = \left\{ \frac{n}{n} : m = 1, 2, 4, 6, 8, ...; n = 2, 5, 8, ..., 3k + 2, ... \right\} ,$$

where $k \in \mathbb{N}$ and let $X = X_1 \cup X_2$. Let us define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{2}{3}x & , x \in X_1, \\ \\ \frac{1}{5}x & , x \in X_2. \end{cases}$$

The mapping f is a Ćirić type operator with $q = \dfrac{2}{3}$. If both x and y are in X_1 or in X_2, then

$$d(f(x), f(y)) \leq \frac{2}{3} d(x, y)$$

If we take $x \in X_1$ and $y \in X_2$, then we have that

$$x \geq \frac{3}{10} y \text{ implies } d(f(x), f(y)) = \frac{2}{3}\left(x - \frac{3}{10}y\right) \leq \frac{2}{3}\left(x - \frac{1}{5}y\right) = \frac{2}{3} d(x, f(y))$$

$$x < \frac{3}{10} y \text{ implies } d(f(x), f(y)) = \frac{2}{3}\left(\frac{3}{10}y - x\right) \leq \frac{2}{3}(y - x) = \frac{2}{3} d(x, y)$$

Thus, we have that f satisfies the following condition:

$$d(f(x), f(y)) \leq \frac{2}{3} max \left\{ d(x, y), d(x, f(y)), d(y, f(x)) \right\}$$

and, hence, it is Ćirić type operator.

In the following step we show that f is not a generalized contraction on X. Let $x = 1$ and $y = \frac{1}{2}$. Then we have that

$$\begin{aligned} q \cdot d(x, y) + r \cdot d(x, f(x)) + s \cdot d(y, f(y)) + t \cdot [d(x, f(y)) + d(y, f(x))] \\ = \frac{1}{2}q + \frac{1}{3}r + \frac{4}{10}s + \frac{32}{30}t < (q + r + s + 2t)\frac{32}{60} \\ < \frac{32}{60} < \frac{17}{30} = d(f(x), f(y)), \end{aligned}$$

as $q + r + s + 2t < 1$, we can see that f is not a generalized contraction.

In this paper, we will present some results related to Ćirić type operator in complete metric spaces. Existence and uniqueness are re-called and several stability properties (data dependence and Ostrowski stability property) are proved. Using the retraction-displacement condition, we will establish the well-posedness and the Ulam-Hyers stability property of the fixed point equation $x = f(x)$.

Our results generalize and complement some theorems given in [1], [2], [3], [5], [6], [7], [8].

2. Main results

In this section we will consider a metric space (X, d) and $f : X \to X$ a Ćirić type operator. Besides the usual properties which are proved by Ćirić in [2], we will prove some other stability properties. More precisely, we will establish the continuous data dependence property of the fixed point and the Ostrowski stability property for the operator f. Moreover, using the retraction-displacement condition and we also prove that the fixed point equation $x = f(x)$ is well-posed and Ulam-Hyers stable.

Theorem 2.1. *(Ćirić [2]) Let (X, d) be a metric space and $f : X \to X$ be a Ćirić type operator. Suppose that X is f-orbitally complete. Then:*

1. *f has a unique fixed point x^* in X and $\lim_{n\to\infty} f^n(x) = x^*$, i.e., f is a Picard operator;*

2. *$d(f^n(x), x^*) \leq \frac{q^n}{1-q} d(x, f(x))$, for every $x \in X$ and every $n \in \mathbb{N}^*$;*

The idea of the proof is based on the following two relations:

(i) if $n \in \mathbb{N}^*$, then for each $x \in X$ we have that $d(f^i(x), f^j(x)) \leq q\delta(O(x,n))$, for every $i, j \in \mathbb{N}^*$;

(ii) for each $x \in X$ we have that $\delta(O(x, \infty)) \leq \frac{1}{1-q} d(x, f(x))$.

A second result in [2] shows that if there exists $p \in \mathbb{N}$ with $p \geq 2$ such that f^p is a Ćirić type operator, then f is a Picard operator.

Remark 2.2. If $f : X \to X$ satisfies all the assumptions in Theorem 2.1, then we have the following additional conclusion:

3. f satisfies the retraction-displacement condition

$$d(x, x^*) \leq \frac{1}{1-q} d(x, f(x)), \text{ for all } x \in X. \tag{2.1}$$

Remark 2.3. The conclusion 3. follows by 2. in the following way. Take $n = 1$ in 2. Then, we have

$$d(f(x), x^*) \leq \frac{q}{1-q} d(x, f(x)), \text{ for all } x \in X.$$

Hence

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq \frac{1}{1-q} d(x, f(x)), \text{ for all } x \in X.$$

Lemma 2.4 (Cauchy-Toeplitz Lemma). *Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers such that $\sum_{n\geq 0} a_n < \infty$ and $\lim_{n\to\infty} b_n = 0$. Then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k \right) = 0$$

The following notion is essential in our approach.

Definition 2.5. (Rus [8]) Let (X, d) be a metric space and $f : X \to X$ be an operator such that $Fix(f) \neq \emptyset$. We say that f satisfies the retraction-displacement condition if there exists $c > 0$ and a set retraction $\rho : X \to Fix(f)$ such that

$$d(x, \rho(x)) \leq cd(x, f(x)), \text{ for all } x \in X. \tag{2.2}$$

If $Fix(f) = \{x^*\}$ then we have

$$d(x, x^*) \leq cd(x, f(x)), \text{ for all } x \in X.$$

For example, if $f : X \to X$ is an α-contraction and (X, d) is a complete metric space then f satisfies the following retraction-displacement condition

$$d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X.$$

On the same lines, if $f : X \to X$ is a graphic α-contraction then it satisfies the retraction-displacement condition

$$d(x, \rho(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

where $\rho : X \to Fix(f)$ is defined by

$$\rho(x) = \lim_{n\to\infty} f^n(x).$$

The following theorem is the main result of the paper.

Theorem 2.6. *Let (X, d) be a metric space, $f : X \to X$ be a Ćirić type operator and suppose that X is f-orbitally complete. Denote by $x^* \in X$ the unique fixed point of f. Then the following conclusions hold:*

1. *the fixed point $x = f(x)$ equation has the data dependence property, i.e., for any operator $g : X \to X$ such that $Fix(g) \neq \emptyset$ and $d(f(x), g(x)) \leq \eta$, for all $x \in X$ and some $\eta > 0$, we have*
$$d(x^*, u^*) \leq \frac{1+q}{1-q}\eta,$$
for all $u^* \in Fix(g)$.
2. *the fixed point equation is well-posed, i.e., for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that*
$$d(u_n, f(u_n)) \to 0,$$
as $n \to \infty$, we have that $u_n \to x^*$, as $n \to \infty$;
3. *the fixed point equation is Ulam-Hyers stable, i.e., there exists $c > 0$ such that for any $\varepsilon > 0$ and any $u^* \in X$ an ε-solution of the fixed point equation (in the sense that $d(u^*, f(u^*)) \leq \varepsilon$), we have*
$$d(u^*, x^*) \leq c \cdot \varepsilon.$$
4. *if $q < \frac{1}{2}$, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ with $d(u_{n+1}, f(u_n)) \to 0$ as $n \to \infty$, we have that $u_n \to x^*$;*
5. *if $q < \frac{1}{2}$, then f is a graphic $\frac{q}{1-q}$-contraction;*
6. *if $q < \frac{1}{3}$, then the operator f is a quasi-contraction, in the sense that there exists $\beta := \frac{q}{1-2q} < 1$ such that*
$$d(f(x), x^*) \leq \beta d(x, x^*), \text{ for every } x \in X.$$

Proof.

1. To prove data dependence we will take $u^* \in Fix(g)$ such that $d(f(x), g(x)) \leq \eta$. Then, we will prove that $d(x^*, u^*) \leq \dfrac{1+q}{1-q}\eta$.
$$\begin{aligned} d(x^*, u^*) &= d(f(x^*), g(u^*)) \leq d(f(x^*), f(u^*)) + d(f(u^*), g(u^*)) \\ &\leq q \cdot max\,\{d(x^*, u^*), d(x^*, f(x^*)), d(u^*, f(u^*)), d(x^*, f(u^*)), \\ &\quad\ d(u^*, f(x^*))\} + d(f(u^*), g(u^*)) \\ &\leq q \cdot max\,\{d(x^*, u^*), d(u^*, g(u^*)) + d(g(u^*), f(u^*)), \\ &\quad\ d(x^*, g(u^*)) + d(g(u^*), f(u^*)), d(x^*, u^*)\} + \eta \\ &\leq q \cdot max\,\{d(x^*, u^*), \eta, d(x^*, u^*) + \eta, d(x^*, u^*)\} + \eta \\ &\leq q(d(x^*, u^*) + \eta) + \eta \end{aligned}$$

Hence, we get that

$$d(x^*, u^*) \leq \frac{1+q}{1-q}\eta$$

2. We will prove that the fixed point equation is well-posed. Let us estimate the distance between u_n and x^*, where $(u_n)_n \in \mathbb{N}$ is a sequence in X such that $d(u_n, f(u_n)) \to 0$ as $n \to \infty$.
   In order to prove this, we will use the retraction-displacement condition (2.1). We have:

$$d(u_n, x^*) \leq \frac{1}{1-q} d(u_n, f(u_n)) \to 0 \text{ as } n \to \infty$$

3. Let $\varepsilon > 0$ and $u^* \in X$ be an ε-solution of the fixed point equation $x = f(x)$, i.e., $d(u^*, f(u^*)) \leq \varepsilon$. Using the retraction-displacement condition (2.1) we will estimate the distance between x^* and u^*:

$$d(x^*, u^*) = d(u^*, x^*) \leq \frac{1}{1-q} d(u^*, f(u^*)) \leq \frac{1}{1-q}\varepsilon$$

   There exists $c > 0$ such that $c := \frac{1}{1-q}$. Then it follows that

$$d(x^*, u^*) \leq c \cdot \varepsilon$$

   which proves that the fixed point equation $x = f(x)$ is Ulam-Hyers stable.
4. We will show that the operator $f : X \to X$ has the Ostrowski property. We observe that:

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, f(u_n)) + d(f(u_n), x^*) \tag{2.3}$$

   We take separately $d(f(u_n), x^*)$ from the above inequality and we have:

$$\begin{aligned} d(f(u_n), x^*) &= d(f(u_n), f(x^*)) \\ &\leq q \cdot max\,\{d(u_n, x^*), d(u_n, f(u_n)), d(x^*, f(x^*), d(u_n, f(x^*)), \\ &\quad d(x^*, f(u_n))\} \\ &\leq q \cdot max\,\{d(u_n, x^*), d(u_n, x^*) + d(x^*, f(u_n)), d(u_n, x^*), \\ &\quad d(x^*, f(u_n))\} \\ &\leq q(d(u_n, x^*) + d(x^*, f(u_n))). \end{aligned}$$

   Thus

$$d(f(u_n), x^*) \leq \frac{q}{1-q} d(u_n, x^*) \tag{2.4}$$

   We replace in (2.3) the relation obtained in inequality (2.4):

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, f(u_n)) + \frac{q}{1-q} d(u_n, x^*) \tag{2.5}$$

We denote: $\alpha := \dfrac{q}{1-q} < 1$.

We will use Cauchy-Toeplitz Lemma and we obtain:

$$\begin{aligned} d(u_{n+1}, x^*) &\leq d(u_{n+1}, f(y_n)) + \alpha d(u_n, x^*) \\ &\leq d(u_{n+1}, f(u_n)) + \alpha[d(u_n, f(u_{n-1})) + \alpha d(u_{n-1}, x^*)] \\ &\leq d(u_{n+1}, f(u_n)) + \alpha d(u_n, f(u_{n-1})) + \alpha^2 d(u_{n-1}, x^*) \leq \dots \leq \\ &\leq d(u_{n+1}, f(u_n)) + \alpha d(u_n, f(u_{n-1})) + \alpha^2 d(u_{n-1}, f(u_{n-2})) \\ &\quad + \dots + \alpha^n d(u_1, f(u_0)) + \alpha^{n+1} d(u_0, x^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

5. If we put $y := f(x)$ in the Ćirić type operator condition, we get
$$\begin{aligned} d(f(x), f^2(x)) &\leq q \max\{d(x, f(x)), d(f(x), f^2(x)), d(x, f^2(x))\} \\ &\leq q\left(d(x, f(x)) + d(f(x), f^2(x))\right). \end{aligned}$$
Thus, we get that $d(f(x), f^2(x)) \leq \frac{q}{1-q} d(x, f(x))$, for every $x \in X$.
6. We will show now that f is a quasi-contraction, in the sense that
$$d(f(x), x^*) \leq \beta d(x, x^*), \text{ for every } x \in X,$$
where $\beta := \frac{q}{1-2q} < 1$. Indeed, by the second conclusion of Theorem 2.1 for $n = 1$, we have $d(f(x), x^*) \leq \frac{q}{1-q} d(x, f(x))$, for every $x \in X$. Then, we can write successively:
$$d(f(x), x^*) \leq \frac{q}{1-q} d(x, f(x)) \leq \frac{q}{1-q}\left(d(x, x^*) + d(f(x), x^*)\right).$$
As a consequence,
$$d(f(x), x^*) \leq \frac{q}{1-2q} d(x, x^*), \text{ for each } x \in X. \qquad \square$$

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