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Baskakov-Kantorovich operators reproducing affine functions

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Abstract. We present a new Kantorovich modification of Baskakov operators which reproduce affine functions. We present an upper estimate for the rate of convergence of the new operators in polynomial weighted spaces and characterize all functions for which there is convergence in the weighted norm.

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1. Introduction

Let $C[0,\infty)$ be the family of all real continuous functions on the semiaxis. Denote $e_k(t) = t^k$, $k \geq 0$. For a function $g : [0,\infty) \to \mathbb{R}$ we set $\|g\| = \sup\{\mid f(x) \mid : x \geq 0\}$. Throughout the paper, we fix $b \geq 0$ and set

$$\varrho(x) = 1/(1+x)^b \quad \text{and} \quad \varphi(x) = \sqrt{x(1+x)}.$$

For $\lambda > 1$ and $f : [0,\infty) \to \mathbb{R}$, the Baskakov operator is defined by

$$V_\lambda(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{\lambda}\right) v_{\lambda,k}(x), \quad v_{\lambda,k}(x) = \binom{\lambda+k-1}{k} \frac{x^k}{(1+x)^{\lambda+k}}, \quad x \geq 0.$$

Here we present Kantorovich-Baskakov operators reproducing linear functions. For the modification we use the notations

$$I_{\lambda,k} = [k/\lambda, (k+1)/\lambda] \qquad \text{and} \qquad a_k = 2k/(2k+1), \quad k \in \mathbb{N}_0.$$

Now, for $f \in C[0,\infty)$ define (whenever the series converges)

$$M_\lambda(f,x) = \lambda \sum_{k=0}^{\infty} Q_{\lambda,k}(f) v_{\lambda,k}(x), \quad Q_{\lambda,k}(f) = \int_{I_{\lambda,k}} f(a_k t)dt.$$

Other authors have introduced Kantarovich-type modifications of some operators defined in subspaces of $C[0,\infty)$ to obtain new ones reproducing affine functions. But these modifications are non-positive or the convergence is only proved in specific proper subintervals of the original domain (see [1] and the references given there).

We will study approximation properties of the operators M_λ in some weighted spaces. The following notations are used

$$C_{\varrho,\beta}[0,\infty)=\{h\in C[0,\infty):\, h(0)=0,\, \|\varrho\varphi^{2\beta}h\|<\infty\},\quad 0\le\beta\le 1.$$

In $C_{\varrho,\beta}[0,\infty)$ we consider the norm $\|f\|_{\varrho,\beta}=\|\varrho\varphi^{2\beta}h\|$.

Section 2 is devoted to study the moments and derivatives of the operators, while some auxiliary results are included in Section 3. In Section 4 we identify all functions in $C_{\varrho,0}[0,\infty)$ for which $\|\varrho(f-M_\lambda(f))\|\to 0$ as $\lambda\to\infty$. The main result is presented in Section 5, where we obtain an estimate of the error $\|\varrho\varphi^{2\beta}(f-M_\lambda(f))\|$ in terms of a K-functional. Analogous results for Kantorovich-Szász-Mirakjan operators will appear in another paper.

In what follows C and C_i will denote absolute constants. They may be different on each occurrence. We remark that our arguments allow to obtain bounds for the constants.

2. Moments and derivatives

For $m\in\mathbb{N}$, the central moment of order m of the operators V_λ and M_λ are defined by $V_{\lambda,m}(x)=V_\lambda((t-x)^m,x)$ and $M_{\lambda,m}(x)=M_\lambda((t-x)^m,x)$ respectively. We also set

$$\theta_\lambda(t)=\frac{1}{\lambda^2}\frac{(2\lambda t)^2}{(1+2\lambda t)^2}. \tag{2.1}$$

Proposition 2.1. (see [2, page 94]) *For each $\lambda>1$ and $x\ge 0$, $V_\lambda(e_0,x)=1$, $V_\lambda(e_1,x)=x$,*

$$V_\lambda(e_2,x)=x^2+\frac{x(1+x)}{\lambda},\quad \textit{and}\quad V_{\lambda,4}(x)=\frac{\varphi^4(x)}{\lambda^2}\left(3+\frac{2}{\lambda}+\frac{(1+2x)^2}{\lambda\varphi^2(x)}\right).$$

Proposition 2.2. *For each $\lambda>1$ and $x\ge 0$, $M_\lambda(e_0,x)=1$, $M_\lambda(e_1,x)=x$,*

$$M_{\lambda,2}(x)=\frac{\varphi^2(x)}{\lambda}+\frac{1}{12}V_\lambda\big(\theta_\lambda,x\big),$$

and

$$M_{\lambda,4}(x)=V_{\lambda,4}(x)+\frac{1}{2}V_\lambda\big(\theta_\lambda(t)(t-x)^2,x\big)+\frac{1}{80}V_\lambda\big(\theta_\lambda^2,x\big).$$

Proof. The first two identities follows from Proposition 2.1, since

$$\lambda\int_{I_{\lambda,k}}dt=1\quad\text{and}\quad\lambda\int_{I_{\lambda,k}}(a_k t)dt=\frac{\lambda k}{2k+1}\Big(\frac{(k+1)^2}{\lambda^2}-\frac{k^2}{\lambda^2}\Big)=\frac{k}{\lambda},$$

For the third equation one has (recall $a_k = (2k)/(2k+1)$)

$$M_\lambda(e_1^2, x) = \frac{\lambda}{3}\sum_{k=0}^{\infty} v_{\lambda,k}(x) a_k^2 \Big(\Big(\frac{k+1}{\lambda}\Big)^3 - \Big(\frac{k}{\lambda}\Big)^3\Big)$$
$$= \sum_{k=0}^{\infty} v_{\lambda,k}(x)\Big(\frac{k^2}{\lambda^2} + a_k^2 \frac{1}{12\lambda^2}\Big)$$
$$= x^2 + \frac{\varphi^2(x)}{n} + \frac{1}{12} V_\lambda\big(\theta_\lambda, x\big),$$

and

$$M_{\lambda,2}(x) = M_\lambda(e_1^2, x) - 2x^2 + x^2 = \frac{\varphi^2(x)}{\lambda} + \frac{1}{12} V_\lambda\big(\theta_\lambda, x\big).$$

For the third moment, first, a straightforward calculus shows that

$$\lambda a_k \int_{I_{\lambda,k}} t(a_k t - x)^2 dt = \frac{k}{\lambda}\Big(\frac{k}{\lambda} - x\Big)^2 + \frac{a_k^2}{4\lambda^2}\Big(\frac{k}{\lambda} - x\Big) + \frac{a_k^2}{12\lambda^2} x.$$

Therefore

$$M_\lambda(e_1(e_1 - x)^2, x) = V_\lambda((e_1(e_1 - x)^2, x) + \frac{1}{4} V_\lambda\Big(\theta_\lambda(t)(t - x), x\Big) + \frac{x}{12} V_\lambda\big(\theta_\lambda, x\big),$$

and

$$M_{\lambda,3}(x) = V_\lambda((e_1(e_1 - x)^2, x) + V_\lambda\big(\theta_\lambda(t)(t - x), x\big)/4 - x^2(1 + x)/\lambda$$
$$= V_{\lambda,3}(x) + x V_{\lambda,2}(x) + \frac{1}{4} V_\lambda\big(\theta_\lambda(t)(t - x), x\big) - x^2(1 + x)/\lambda$$
$$= V_{\lambda,3}(x) + \frac{1}{4} V_\lambda\big(\theta_\lambda(t)(t - x), x\big).$$

The same idea can be used to obtain the other equation. In fact, since

$$\lambda a_k \int_{I_{\lambda,k}} t(a_k t - x)^3 dt = \frac{k}{\lambda}\Big(\frac{k}{\lambda} - x\Big)^3 + \frac{a_k^2}{2\lambda^2}\Big(\Big(\frac{k}{\lambda} - x\Big)^2 + \frac{x}{2}\Big(\frac{k}{\lambda} - x\Big)\Big) + \frac{1}{80\lambda^4} a_k^4,$$

one has

$$M_\lambda(e_1(e_1 - x)^3, x) = V_\lambda((e_1(e_1 - x)^3, x) + \frac{1}{2} V_\lambda\Big(\theta_\lambda(t)(t - x)^2, x\Big)$$
$$+ \frac{x}{4} V_\lambda\Big(\theta_\lambda(t)(t - x), x\Big) + \frac{1}{80} V_\lambda\big(\theta_\lambda^2, x\big),$$
$$= V_{\lambda,4}(x) + x V_{\lambda,3}(x) + \frac{1}{2} V_\lambda\Big(\theta_\lambda(t)(t - x)^2, x\Big)$$
$$+ \frac{x}{4} V_\lambda\Big(\theta_\lambda(t)(t - x), x\Big) + \frac{1}{80} V_\lambda\big(\theta_\lambda^2, x\big),$$

and

$$M_{\lambda,4}(x) = M_\lambda(t(t - x)^3, x) - x M_\lambda((t - x)^3, x)$$
$$= M_\lambda(t(t - x)^3, x) - x V_{\lambda,3}(x) - \frac{x}{4} V_\lambda\big(\theta_\lambda(t)(t - x), x\big)$$
$$= V_{\lambda,4}(x) + \frac{1}{2} V_\lambda\Big(\theta_\lambda(t)(t - x)^2, x\Big) + \frac{1}{80} V_\lambda\big(\theta_\lambda^2, x\big). \qquad \square$$

Corollary 2.3. *For each* $\lambda > 1$ *and* $x \geq 0$*, one has*

$$M_{\lambda,2}(x) \leq \frac{13}{12}\frac{\varphi^2(x)}{\lambda} \quad \text{and} \quad M_\lambda(|\, t-x \,|, x) \leq \sqrt{\frac{13}{12}}\frac{\varphi(x)}{\sqrt{\lambda}}.$$

Moreover, if $x \geq 1/(2(\lambda+1))$*, then* $M_{\lambda,4}(x) \leq 16\varphi^4(x)/\lambda^2$*.*

Proof. Since $4\lambda t \leq (1+2\lambda t)^2$, one has (see (2.1))

$$\theta_\lambda(t) = \frac{1}{\lambda^2}\frac{(2\lambda t)^2}{(1+2\lambda t)^2} = \frac{\lambda t}{\lambda^2}\frac{4\lambda t}{(1+2\lambda t)^2} \leq \frac{\lambda t}{\lambda^2} = \frac{t}{\lambda}.$$

Later we need also the estimate

$$\theta_\lambda(t) = \frac{1}{\lambda^2}\frac{(2\lambda t)^2}{(1+2\lambda t)^2} \leq \frac{1}{\lambda^2}\frac{(1+2\lambda t)^2}{(1+2\lambda t)^2} = \frac{1}{\lambda^2}. \tag{2.2}$$

Therefore (see Proposition 2.2)

$$\begin{aligned} M_{\lambda,2}(x) &= \frac{\varphi^2(x)}{\lambda} + \frac{1}{12}V_\lambda(\theta_\lambda, x) \leq \frac{\varphi^2(x)}{\lambda} + \frac{1}{12\lambda}V_\lambda(t,x) \\ &= \frac{\varphi^2(x)}{\lambda} + \frac{x}{12\lambda} = \frac{\varphi^2(x)}{\lambda} + \frac{\sqrt{x^2}}{12\lambda} \leq \frac{13}{12}\frac{\varphi^2(x)}{\lambda}. \end{aligned}$$

The second inequality follows from the first one by Hölder's inequality. In fact, since M_λ is a positive operator

$$M_\lambda(|\, t-x \,|, x) \leq \sqrt{M_\lambda(1,x)M_\lambda((t-x)^2, x)}.$$

If we set $H(x) = (1+2x)^2/\varphi^2(x)$, then

$$\begin{aligned} H'(x) &= \frac{4(1+2x)(x+x^2) - (1+2x)^2(1+2x)}{(x+x^2)^2} \\ &= (1+2x)\frac{4x+4x^2-1-4x-4x^2}{(x+x^2)^2} < 0. \end{aligned}$$

Thus, if $x \geq 1/(2(\lambda+1))$,

$$H(x) \leq H\Big(\frac{1}{2(1+\lambda)}\Big) = \frac{4(1+\lambda)}{(3+2\lambda)}\Big(\frac{2+\lambda}{1+\lambda}\Big)^2 = \frac{4(2+\lambda)^2}{(3+2\lambda)}$$

On the other hand, since the function

$$\frac{2}{\lambda} + \frac{4(2+\lambda)^2}{\lambda(3+2\lambda)}$$

decreases, for $\lambda > 1$ and $x \geq 1/(2(1+\lambda))$, one has

$$\frac{2}{\lambda} + \frac{(1+2x)^2}{\lambda\varphi^2(x)} \leq \frac{2}{\lambda} + \frac{4(2+\lambda)^2}{\lambda(3+2\lambda)} \leq 2 + \frac{36}{5} < 10.$$

Thus, if $x \geq 1/(2(\lambda+1))$, then (see Proposition 2.1)

$$V_\lambda((t-x)^4, x) = \frac{\varphi^4(x)}{\lambda^2}\left(3 + \frac{2}{\lambda} + \frac{(1+2x)^2}{\lambda\varphi^2(x)}\right) \leq 13\frac{\varphi^4(x)}{\lambda^2}.$$

Also notice that

$$1 \leq \frac{4(1+\lambda)^2}{3+2\lambda}x(1+x) \leq 2(1+\lambda)x(1+x) \leq 4\lambda x(1+x).$$

Therefore, from Proposition 2.1 one has

$$\frac{1}{2}V_\lambda\big(\theta_\lambda(t)(t-x)^2, x\big) \leq \frac{1}{2\lambda^2}V_\lambda\big((t-x)^2, x\big) = \frac{\varphi^2(x)}{2\lambda^3} \leq 2\frac{\varphi^4(x)}{\lambda^2}.$$

Since $2+2\lambda \leq 4\lambda$,

$$\frac{1}{4\lambda} \leq \frac{1}{2(1+\lambda)} \leq x,$$

hence (see (2.2))

$$\frac{1}{80}V_\lambda\big(\theta_\lambda^2, x\big) \leq \frac{1}{80\lambda^4} \leq \frac{1}{5}\frac{x^2}{\lambda^2} \leq \frac{1}{5}\frac{\varphi^4(x)}{\lambda^2}.$$

Therefore

$$M_{\lambda,4}(x) = V_{\lambda,4}(x) + \frac{1}{2}V_\lambda\Big(\theta_\lambda(t)(t-x)^2, x\Big) + \frac{1}{80}V_\lambda\big(\theta_\lambda^2, x\big) \leq 16\frac{\varphi^4(x)}{\lambda^2}. \qquad \square$$

Proposition 2.4. *For each $f \in C_\varrho[0,\infty)$, $\lambda > 1$ and $x \geq 0$, one has*

$$\frac{\varphi^2(x)}{\lambda^2}M'_\lambda(f,x) = \sum_{k=0}^{\infty} Q_{\lambda,k}(f)\Big(\frac{k}{\lambda} - x\Big)v_{\lambda,k}(x),$$

and

$$M'_\lambda(f,x) = \lambda^2\sum_{k=0}^{\infty}\Big(Q_{\lambda,k+1}(f) - Q_{\lambda,k}(f)\Big)v_{\lambda+1,k}(x).$$

If $f(0)=0$, the term corresponding to $k=0$ should be omitted.

Proof. Since, $v_{\lambda,k}(x)$ satisfies (we set $v_{\lambda,-1}=0$)

$$v'_{\lambda,k}(x) = \frac{k-\lambda x}{x(1+x)}v_{\lambda,k}(x) = \lambda\big(v_{\lambda+1,k-1}(x) - v_{\lambda+1,k}(x))\big),$$

we have

$$\begin{aligned} M'_\lambda(f,x) &= \lambda\sum_{k=0}^{\infty} Q_{\lambda,k}(f)\frac{k-\lambda x}{x(1+x)}v_{\lambda,k}(x) \\ &= \frac{\lambda^2}{x(1+x)}\sum_{k=0}^{\infty} Q_{\lambda,k}(f)\Big(\frac{k}{\lambda} - x\Big)v_{\lambda,k}(x) \\ &= \lambda^2\sum_{k=0}^{\infty} Q_{\lambda,k+1}(f)v_{\lambda+1,k}(x) - \lambda^2\sum_{k=0}^{\infty} Q_{\lambda,k}(f)v_{\lambda+1,k}(x) \\ &= \lambda^2\sum_{k=0}^{\infty}\Big(Q_{\lambda,k+1}(f) - Q_{\lambda,k}(f)\Big)v_{\lambda+1,k}(x). \qquad \square \end{aligned}$$

Proposition 2.5. *For $\lambda > 1$ and $x \geq 0$,*

$$M_\lambda\Big(\frac{1}{1+t}, x\Big) \leq \frac{2\lambda}{(\lambda-1)(1+x)}.$$

Proof. For $k \geq 1$, $4k \geq 1+2k$. Therefore, if $k \geq 0$,

$$1+\frac{k}{\lambda} \leq 1+\frac{k}{\lambda}\frac{4k}{(1+2k)} \qquad \text{and} \qquad \frac{1}{1+a_k\, k/\lambda} \leq \frac{2}{1+k/\lambda}.$$

Hence

$$M_\lambda\Big(\frac{1}{1+t},x\Big) \leq \sum_{k=0}^{\infty}\frac{v_{\lambda,k}(x)}{1+a_k(k/\lambda)} \leq 2\sum_{k=0}^{\infty}\frac{\lambda}{\lambda+k}v_{\lambda,k}(x)$$

$$\leq \frac{2}{(1+x)^\lambda}\Big(1+\sum_{k=0}^{\infty}\frac{(\lambda-1)\lambda\cdots(\lambda+(k-2))}{k!(1+x)^{\lambda+k-1}}x^k\Big) \leq 2\frac{\lambda}{(\lambda-1)(1+x)}. \qquad \square$$

3. Auxiliary results

Proposition 3.1. *Assume $\gamma \in [0,2)$ and set*

$$I_\gamma(x,t)=\left|\int_x^t \frac{t-u}{\varphi^{2\gamma}(u)\varrho(u)}du\right|.$$

(i) *If $t > x > 0$ and $1/2 \leq \gamma \leq b$, then*

$$I_\gamma(x,t) \leq 2\mid t-x\mid^{3/2}(1+t)^{b-\gamma}x^{1/2-\gamma}.$$

(ii) *If $t > x > 0$, $0 \leq \gamma < 1/2$ and $b \geq \gamma$, then*

$$I_\gamma(x,t) \leq \frac{2(t-x)^{2-\gamma}}{(1+x)(1+t)^{\gamma-b-1}}.$$

(iii) *If $t > 0$, $x > 0$, $0 \leq \gamma < 2$ and $\gamma \leq b$, then*

$$I_\gamma(x,t) \leq \frac{(t-x)^2}{2-\gamma}\Big(\frac{1}{\varrho(x)\varphi^{2\gamma}(x)}+\frac{(1+t)^{b-\gamma}}{x^\gamma}\Big).$$

Proof. (i) If $t > x$ and $1/2 \leq \gamma \leq b$, then

$$\left|\int_x^t \frac{(t-u)}{u^\gamma(1+u)^{\gamma-b}}du\right| \leq \frac{(t-x)(1+t)^{b-\gamma}}{x^{\gamma-1/2}}\int_x^t \frac{1}{u^{1/2}}du$$

$$=2\frac{(1+t)^{b-\gamma}(t-x)(\sqrt{t}-\sqrt{x})}{x^{\gamma-1/2}} \leq \frac{2(1+t)^{b-\gamma}\mid t-x\mid^{3/2}}{x^{\gamma-1/2}}.$$

(ii) If $t > x$ and $0 \leq \gamma < 1/2$, since the function $(t-u)/(1+u)$ decreases in the interval $[x,t]$, then

$$\int_x^t \frac{(t-u)}{u^\gamma(1+u)^{\gamma-b}}du=\int_x^t \frac{t-u}{(1+u)}\frac{1}{u^\gamma(1+u)^{\gamma-b-1}}du$$

$$\leq \frac{(t-x)}{(1+x)(1+t)^{\gamma-b-1}}\int_x^t \frac{du}{u^\gamma} \leq \frac{2(t-x)^{2-\gamma}}{(1+x)(1+t)^{\gamma-b-1}}.$$

(iii) First note that

$$I_\gamma(x,t) = \left| \int_x^t \frac{(t-u)}{u^\gamma}(1+u)^{b-\gamma} du \right|$$
$$\leq \left((1+x)^{b-\gamma} + (1+t)^{b-\gamma}\right) \left| \int_x^t \frac{|t-u|}{u^\gamma} du \right|.$$

Now we estimate the last integral in the expression above. If $t < x$, then, putting $u - t = \tau(x-t)$, we have

$$\left| \int_x^t \frac{|t-u|}{u^\gamma} du \right| = \int_t^x \frac{(u-t)}{u^\gamma} du = (x-t)^2 \int_0^1 \frac{\tau}{\left((1-\tau)t + \tau x\right)^\gamma} d\tau$$
$$\leq \frac{(x-t)^2}{x^\gamma} \int_0^1 \tau^{1-\gamma} d\tau = \frac{(x-t)^2}{(2-\gamma)x^\gamma}.$$

If $x < t$, then

$$\left| \int_x^t \frac{|t-u|}{u^\gamma} du \right| = \int_x^t \frac{(t-u)}{u^\gamma} du \leq \frac{1}{x^\gamma} \int_x^t (t-u) du = \frac{(t-x)^2}{2x^\gamma}.$$

Hence, the inequality in (iii) holds. □

Proposition 3.2. *For each $b \geq 0$, $x \geq 0$, and $\lambda \geq 2(1+b)$, one has*

$$V_\lambda((1+t)^b, x) \leq 2^b(1+x)^b \qquad \text{and} \qquad M_\lambda((1+t)^b, x) \leq 2^{2b}(1+x)^b.$$

Moreover, for each $b \geq 0$, $x \geq 0$, and $\lambda \geq 1$, one has

$$V_\lambda((1+t)^b, x) \leq (2+b)^b(1+x)^b.$$

Proof. The case $b = 0$ is trivial, because $V_\lambda(1,x) = M_\lambda(1,x) = 1$. Thus we will assume that $b > 0$.

(i) First we prove that, for $m \in \mathbb{N}$

$$(1+k/\lambda)^m \, v_{\lambda,k}(x) \leq 2^{m-1}(1+x)^m v_{\lambda+m,k}(x). \tag{3.1}$$

If $1 \leq k < m$, from the definition of $v_{\lambda,k}(x)$, we have

$$\frac{v_{\lambda+m,k}(x)}{v_{\lambda,k}(x)} = \frac{1}{(1+x)^m} \frac{(\lambda+m)(\lambda+m+1)\cdots(\lambda+k+m-1)}{\lambda(\lambda+1)\cdots(\lambda+k-1)}$$
$$= \frac{1}{(1+x)^m} \prod_{j=0}^{k-1} \left(1 + \frac{m}{\lambda+j}\right)$$

But, for $1 \leq k < m$, $1 + k/\lambda \leq 1 + m/\lambda \leq 2$ and

$$\frac{1}{2^{m-1}}\left(1 + \frac{k}{\lambda}\right)^m \leq 1 + \frac{k}{\lambda} \leq 1 + \frac{m}{\lambda} \leq \prod_{j=0}^{k-1}\left(1 + \frac{m}{\lambda+j}\right). \tag{3.2}$$

Here the condition $\lambda \geq m$ is needed.
Hence

$$\frac{1}{2^{m-1}}\left(1 + \frac{k}{\lambda}\right)^m v_{\lambda,k}(x) \leq \prod_{j=0}^{k-1}\left(1 + \frac{m}{\lambda+j}\right) v_{\lambda,k}(x) = (1+x)^m v_{\lambda+m,k}(x).$$

For $k \geq m$, one has the equality

$$\frac{v_{\lambda+m,k}(x)}{v_{\lambda,k}(x)} = \frac{1}{(1+x)^m}\frac{(\lambda+k)(\lambda+k+1)\cdots(\lambda+k+m-1)}{\lambda(\lambda+1)\cdots(\lambda+m-1)}$$
$$= \frac{1}{(1+x)^m}\prod_{j=0}^{m-1}\Big(1+\frac{k}{\lambda+j}\Big).$$

But for $j = 1, \cdots, m-1$,

$$\frac{1}{2}\Big(1+\frac{k}{\lambda}\Big) \leq \frac{\lambda}{\lambda+j}\Big(1+\frac{k}{\lambda}\Big) \leq \Big(1+\frac{k}{\lambda+j}\Big).$$

Therefore Hence

$$\Big(1+\frac{k}{\lambda}\Big)^m v_{\lambda,k}(x) \leq 2^{m-1}\prod_{j=0}^{m-1}\Big(1+\frac{k}{\lambda+j}\Big)v_{\lambda,k}(x) = 2^{m-1}(1+x)^m v_{\lambda+m,k}(x).$$

We have proved (3.1) and it is sufficient to verify the first inequality in Proposition 3.2 in the case of an integer b and $\lambda \geq b$.

If $b > 0$ is not an integer, let $\lceil b \rceil$ be the ceiling of b, that is the least integer greater than b. Note that $\lambda > 2(1+b) \geq 2\lceil b \rceil$. Applying Hölder's inequality we obtain

$$V_\lambda\big((1+t)^b, x\big) \leq \Big(V_\lambda\big((1+t)^{\lceil b \rceil}, x\big)\Big)^{b/\lceil b \rceil} \leq 2^{(\lceil b \rceil-1)b/\lceil b \rceil}(1+x)^b \leq 2^b(1+x)^b.$$

(ii) Since $0 \leq a_k \leq 1$, for $k/\lambda \leq t \leq (k+1)/\lambda$, one has

$$1 + a_k t \leq 1 + (k+1)/\lambda \leq 2 + k/\lambda \leq 2(1+k/\lambda).$$

Therefore $M_\lambda((1+t)^b, x) \leq 2^b V_\lambda((1+t)^b, x)$.

(iii) The condition $\lambda \geq 2(1+b)$ was first used to prove equation (3.2). This restriction can be omitted if, for $1 \leq k < m$, we consider the inequalities

$$\Big(1+\frac{k}{\lambda}\Big)^m \leq \Big(1+\frac{m}{\lambda}\Big)^{m-1}\Big(1+\frac{m}{\lambda}\Big) \leq (1+m)^{m-1}\prod_{j=0}^{k-1}\Big(1+\frac{m}{\lambda+j}\Big).$$

Thus for an integer m equation (3.1) is replaced by

$$(1+k/\lambda)^m v_{\lambda,k}(x) \leq (1+m)^{m-1}(1+x)^m v_{\lambda+m,k}(x)$$

Moreover, if $b > 0$ is not an integer,

$$V_\lambda\big((1+t)^b, x\big) \leq \Big(V_\lambda\big((1+t)^{\lceil b \rceil}, x\big)\Big)^{b/\lceil b \rceil}$$
$$\leq \Big((1+\lceil b \rceil)^{\lceil b \rceil-1}\Big)^{b/\lceil b \rceil}(1+x)^b \leq (2+b)^b(1+x)^b. \qquad \square$$

Proposition 3.3. *For $\beta \in (0,2]$, $x > 0$ and any real λ satisfying $\lambda \geq 4$, one has*

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^\beta v_{\lambda,k}(x) \leq \frac{4^\beta}{x^\beta}. \qquad (3.3)$$

Proof. (i) First note that, for $x > 0$,

$$x\Big(\frac{\lambda}{k}\Big)\frac{v_{\lambda,k}(x)}{v_{\lambda-1,k+1}(x)} = \Big(\frac{\lambda}{k}\Big)\frac{(k+1)!}{k!}\frac{\lambda(\lambda+1)\cdots(\lambda+k-1)}{(\lambda-1)(\lambda)\cdots(\lambda-1+k))}$$

$$= \Big(\frac{\lambda}{k}\Big)\frac{(k+1)!}{k!}\frac{1}{(\lambda-1)} = \frac{k+1}{k}\frac{\lambda}{\lambda-1} \le 4.$$

With similar arguments, we prove that

$$x^2\Big(\frac{\lambda}{k}\Big)^2\frac{v_{\lambda,k}(x)}{v_{\lambda-2,k+2}(x)} = \Big(\frac{\lambda}{k}\Big)^2\frac{(k+2)!}{k!}\frac{1}{(\lambda-1)(\lambda-2)} = \frac{k+1}{k}\frac{\lambda}{\lambda-1} \le 4^2.$$

Therefore

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)v_{\lambda,k}(x) \le \frac{4}{x}\sum_{k=1}^{\infty}v_{\lambda-1,k+1}(x) \le \frac{4}{x} \tag{3.4}$$

and

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^2 v_{\lambda,k}(x) \le \frac{4^2}{x^2}\sum_{k=1}^{\infty}v_{\lambda-2,k+2}(x) \le \frac{4^2}{x^2}$$

(ii) Finally if $0 < \beta < 1$, using Hölder's inequality, we have

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^{\beta} v_{\lambda,k}(x) \le \Big(\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)v_{\lambda,k}(x)\Big)^{\beta} \le \frac{4^{\beta}}{x^{\beta}},$$

and, if $1 < \beta < 2$, then

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^{\beta} v_{\lambda,k}(x) \le \Big(\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^2 v_{\lambda,k}(x)\Big)^{\beta/2} \le \frac{4^{\beta}}{x^{\beta}},$$

and this proves the result. □

Proposition 3.4. *If $\gamma \in [0,2)$, $b \ge \gamma$, $\lambda > 1$ and $k > 0$, then*

$$\int_{k/\lambda}^{(k+1)/\lambda}\frac{dt}{\varrho(a_k t)\varphi^{2\gamma}(a_k t)} \le \frac{2^b}{\lambda}\frac{1}{\varrho(k/\lambda)\varphi^{2\gamma}(k/\lambda)}.$$

When $\gamma = 0$ the inequality also holds for $k = 0$.

Proof. If $k > 0$, since $1/2 \le a_k \le 1$, then

$$\int_{k/\lambda}^{(k+1)/\lambda}\frac{(1+a_k u)^{b-\gamma}}{a_k^{\gamma}u^{\gamma}}du \le \frac{C_2}{\lambda}\frac{(1+k/n)^b}{\varphi^{\gamma}(k/n)}.$$

□

Lemma 3.5. *Assume $\gamma \in [0,2)$, $b \ge 2$ and $\lambda > 2(1+b)$. There exists a constant C such that*

$$A_{\lambda,\gamma}(x) := \sum_{k=1}^{\infty}v_{\lambda,k}(x)\int_x^{k/\lambda}\frac{du}{\varphi^{2\gamma}(u)\varrho(u)} \le C\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)}.$$

and

$$B_{\lambda,\gamma}(x) := \sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varphi^{2\gamma}(k/\lambda)\varrho(k/\lambda)} \le C\lambda\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)},$$

Proof. Notice that $1+k/\lambda \le 4(1+(k-1)/(\lambda+1))$ for $k\ge 1$.

(i) If $\gamma=0$, the result follows from Proposition 3.2 taking into account that

$$A_{\lambda,0}(x) \le \sum_{k=1}^{\infty} \frac{k}{\lambda}(1+k/\lambda)^b v_{\lambda,k}(x)$$

$$\le 4^b x \sum_{k=1}^{\infty} \frac{(\lambda+1)\cdots(\lambda+k-1)}{\varrho((k-1)/(\lambda+1))} \frac{1}{(k-1)!} \frac{x^{k-1}}{(1+x)^{\lambda+1+k-1}}$$

$$= 4^b x V_{\lambda+1}((1+t)^b, x) \le \frac{C_1 \varphi^2(x)}{\varrho(x)}.$$

(ii) Assume $0<\gamma<1$. Notice that

$$\int_x^{k/\lambda} \frac{du}{\varphi^{2\gamma}(u)\varrho(u)} \le (1+k/\lambda)^{b-\gamma} \int_0^{k/\lambda} \frac{u}{u^{\gamma}} = \frac{(1+k/\lambda)^{b-\gamma}}{1-\gamma}(k/\lambda)^{1-\gamma}.$$

Hence, by Hölder inequality we obtain

$$A_{\lambda,\gamma}(x) \le C_2 \sum_{k=1}^{\infty} \Big(\frac{k}{\lambda}\Big)^{1-\gamma} (1+k/\lambda)^{b-\gamma} v_{\lambda,k}(x)$$

$$\le C_2 (V_\lambda(t,x))^{1-\gamma} \big(V_\lambda((1+t)^{(b-\gamma)/\gamma}, x\big)^{\gamma} = C_2 x^{1-\gamma} \big(V_\lambda((1+t)^{(b-\gamma)/\gamma}, x\big)^{\gamma},$$

where we use Proposition 2.1. It follows from the last inequality in Proposition 3.2 that

$$\big(V_\lambda((1+t)^{(b-\gamma)/\gamma}, x\big)^{\gamma} \le C(b)(1+x)^{b-\gamma} \le C(b)(1+x)^{b+1-\gamma}.$$

(iii) If $\gamma=1$, then

$$A_{\lambda,1}(x) \le \frac{1}{\sqrt{x}} \sum_{k=1}^{\infty} \Big(\int_0^{k/\lambda} \frac{du}{\sqrt{u}}\Big) \frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{-b}} \le \frac{C_4}{\sqrt{x}} \sum_{k=1}^{\infty} \sqrt{\frac{k}{\lambda}} \frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{-b}}$$

$$\le \frac{C_4}{\sqrt{x}} \sqrt{V_\lambda(t,x) V_\lambda((1+t)^{2b}, x)} \le \frac{C_5}{\varrho(x)}.$$

(iv) If $1<\gamma<2$, then

$$V_{\lambda,\gamma}(x) \le \frac{1}{x} \sum_{k=1}^{\infty} \frac{v_{\lambda,k}(x)}{\varrho(k/\lambda)} \int_0^{k/\lambda} \frac{du}{u^{\gamma-1}} = \frac{C_6}{x} \sum_{k=1}^{\infty} \Big(\frac{k}{\lambda}\Big)^{2-\gamma} \frac{v_{\lambda,k}(x)}{\varrho(k/\lambda)}$$

$$\le \frac{C_6}{x} \Big(\sum_{k=1}^{\infty} \Big(\frac{k}{\lambda}\Big) v_{\lambda,k}(x)\Big)^{2-\gamma} \Big(\sum_{k=1}^{\infty} \frac{v_{\lambda,k}(x)}{\varrho^{\gamma-1}(k/\lambda)}\Big)^{\gamma-1} \le \frac{C_7 x^{1-\gamma}}{\varrho(x)}.$$

For the second inequality we also consider several cases.

(a) If $\gamma=0$, the results follows from Proposition 3.2 taking into account that

$$B_{\lambda,0}(x) = \lambda x \sum_{k=1}^{\infty} \frac{(\lambda+1)\cdots(\lambda+k-1)}{k!(1+k/\lambda)^{-b}(1+x)^{k-1+\lambda+1}} x^{k-1} \le \lambda x V_{\lambda+1}((1+t)^b, x).$$

(b) If $0<\gamma<1$, we use Hölder inequality, Propositions 2.1 and 3.2 to obtain

$$B_{\lambda,\gamma}(x)=\lambda\sum_{k=1}^{\infty}\frac{1}{\lambda}\left(\frac{k}{\lambda}\right)^{-\gamma}\frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{\gamma-b}(x)}\leq\lambda\sum_{k=1}^{\infty}\left(\frac{k}{n}\right)^{1-\gamma}\frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{\gamma-b}}$$
$$\leq\lambda\big(V_\lambda(t,x)\big)^{1-\gamma}\big(V_\lambda((1+t)^{(b-\gamma)/\gamma},x)\big)^{\gamma}\leq C_2\lambda\, x^{1-\gamma}(1+x)^{b-\gamma}.$$

(c) If $\gamma=1$, the proof is simpler, because $B_{\lambda,1}(x)\leq\lambda V_\lambda((1+t)^{b-1},x)$.
(d) If $1<\gamma<2$, then

$$B_{\lambda,\gamma}(x)=\sum_{k=1}^{\infty}\frac{\lambda}{k}\Big(\frac{\lambda}{k}\Big)^{-1+\gamma}\frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{\gamma-b}(x)}$$
$$\leq 2\lambda\sum_{k=1}^{\infty}\Big(\frac{k}{\lambda}\Big)^{1-\gamma}\frac{\lambda(\lambda+1)\cdots(\lambda+k-1)}{(1+k/\lambda)^{\gamma-b}}\frac{1}{(k+1)k!}\frac{x^k}{(1+x)^{\lambda+k}}$$
$$\leq 2\lambda\frac{1+x}{x}\sum_{k=1}^{\infty}\frac{1}{\lambda+k}\Big(\frac{k}{\lambda}\Big)^{1-\gamma}\frac{v_{\lambda,k+1}(x)}{(1+k/\lambda)^{\gamma-b}}$$
$$\leq C_1\lambda\frac{1+x}{x}\sum_{k=1}^{\infty}\Big(\frac{k+1}{\lambda}\Big)^{2-\gamma}\frac{v_{\lambda,k+1}(x)}{(1+(k+1)/\lambda)^{\gamma-b}}$$
$$\leq C_2\lambda\frac{1+x}{x}x^{2-\gamma}(1+x)^{b-\gamma}=C_2\lambda\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)}.\qquad\square$$

Lemma 3.6. *Assume $\gamma\in[0,2)$, $b\geq\gamma$ and $\lambda>2(1+b)$. There exists a constant C such that, if $0<x\leq 1/(2\lambda)$, then*

$$C_{\lambda,\gamma}(x):=\sum_{k=1}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\frac{((k+1)/\lambda-u)}{\varphi^{2\gamma}(u)\varrho(u)}du\leq\frac{C}{\lambda}\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)},$$

and, if $\alpha,\beta\in[0,1]$, $\alpha+\beta\leq b$, then

$$D_{\lambda,\alpha+\beta}(x):=\sum_{k=1}^{\infty}v_{n,k}(x)\int_{x}^{k/\lambda}\frac{(k/\lambda-s)ds}{\varrho(s)\varphi^{2(\alpha+\beta)}(s)}\leq\frac{C\varphi^{2(1-\alpha)}(x)}{\lambda\, x^{\beta}\varrho(x)}.$$

Proof. Since the functions $\varphi^{2\gamma}(x)$ and $1/\varrho(x)$ increase,

$$C_{\lambda,\gamma}(x)\leq C_1\sum_{k=1}^{\infty}\frac{1}{\varphi^{2\gamma}(k/\lambda)\varrho(k/\lambda)}\int_{k/\lambda}^{(k+1)/\lambda}((k+1)/\lambda-u)du\,v_{\lambda,k}(x)$$
$$\leq\frac{C_2}{\lambda^2}\sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varphi^{2\gamma}(k/\lambda)\varrho(k/\lambda)}=\frac{C_2}{\lambda^2}B_{\lambda,\gamma}(x)\leq\frac{C_3}{\lambda}\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)}.$$

If $0<x\leq 1/(2\lambda)$ and $1/2\leq\alpha+\beta\leq b$, then

$$D_{\lambda,\alpha+\beta}(x)\leq\frac{1}{x^{\alpha+\beta-1/2}}\sum_{k=1}^{\infty}v_{n,k}(x)\frac{|\,k/\lambda-x\,|}{(1+k/n)^{\alpha+\beta-b}}\int_{x}^{k/\lambda}\frac{ds}{\sqrt{s}}$$
$$\leq\frac{C_1}{x^{\alpha+\beta-1/2}}\sum_{k=1}^{\infty}v_{n,k}(x)\frac{|\,k/\lambda-x\,|^{3/2}}{(1+k/n)^{\alpha+\beta-b}}$$

$$\leq \frac{C_1}{x^{\alpha+\beta-1/2}(1+x)^{\alpha+\beta-b}}\big(V_\lambda(t-x)^2,x)\big)^{3/4} \leq C_2\frac{x^{3/4}}{\lambda^{3/4}x^{\alpha-1/2}x^\beta\varrho(x)}$$
$$= C_2\frac{x}{\lambda^{3/4}x^\alpha x^\beta\varrho(x)}x^{1/4} \leq \frac{C_2\varphi^{2(1-\alpha)}(x)}{\lambda\, x^\beta\varrho(x)}.$$

If $0 < x \leq 1/(2\lambda)$ and $\alpha+\beta < 1/2 \leq b$, from (ii) in Proposition 3.1 and Hölder inequality we obtain

$$D_{\lambda,\alpha+\beta}(x) \leq \frac{2}{(1+x)}\sum_{k=1}^{\infty} v_{n,k}(x)\frac{(k/\lambda-x)^{2-\alpha-\beta}}{(1+k/n)^{\alpha+\beta-b-1}}$$
$$\leq \frac{C_3}{(1+x)}\Big(\frac{x}{\lambda}\Big)^{(2-\alpha-\beta)/2}\frac{1}{(1+x)^{\alpha+\beta-b-1}} \leq C_3\frac{x^{\beta/2}\lambda^{\beta/2}}{\lambda^{1-\alpha/2}}\frac{x^{1-\alpha}x^{\alpha/2}}{(1+x)^\alpha x^\beta\varrho(x)}. \qquad \square$$

Proposition 3.7. *Assume $\alpha,\beta\in[0,1]$, $\alpha+\beta<2\leq b$. There exists a constant C such that, for $\lambda>2(1+b)$ and $x>0$, one has*

$$x^\beta\varrho(x)M_\lambda(|\ I_{\alpha+\beta}(x,t)\ |,x) \leq C\frac{\varphi^{2(1-\alpha)}(x)}{\lambda}.$$

Proof. (a) First we consider the case $0\leq x\leq 1/(2\lambda)$. When $k=0$, since $a_k=0$,

$$x^\beta\varrho(x)\left|\lambda v_{\lambda,0}(x)\int_0^{1/\lambda}\int_0^x \frac{s}{\varphi^{2(\alpha+\beta)}(s)\varrho(u)}dsdt\right|$$
$$= x^\beta\int_0^x \frac{s^{1-\alpha-\beta}(1+s)^{b-\alpha-\beta}}{(1+x)^{\lambda+b}}\,ds$$
$$\leq \frac{x^\beta}{(1+x)^{\alpha+\beta}}\int_0^x s^{1-\alpha-\beta}ds \leq C_1\frac{x^{2-\alpha}}{(1+x)^\alpha} \leq C_2\frac{1}{\lambda}\varphi^{2(1-\alpha)}(x).$$

If $k>0$ and $x\leq 1/(2\lambda)$, then $x<a_kt\leq t$, for $t\in I_{\lambda,k}$. Hence

$$\lambda\int_{I_{\lambda,k}}\Big|\int_x^{a_kt}\frac{a_kt-u}{\varphi^{2\gamma}(u)\varrho(u)}du\Big|dt \leq \int_x^{(k+1)/\lambda}\frac{((k+1)/\lambda-u)}{\varphi^{2\gamma}(u)\varrho(u)}du$$
$$= \int_{k/\lambda}^{(k+1)/\lambda}\frac{(k+1)/\lambda-u)du}{\varphi^{2\gamma}(u)\varrho(u)} + \frac{1}{\lambda}\int_x^{k/\lambda}\frac{du}{\varphi^{2\gamma}(u)\varrho(u)} + \int_x^{k/\lambda}\frac{(k/\lambda-u)du}{\varphi^{2\gamma}(u)\varrho(u)}.$$

If $R^*_{\lambda,\alpha,\beta}(x) = x^\beta\varrho(x)M^*_\lambda(|\ I_\lambda(x,t)\ |,x)$ (M^*_λ means that we omit the term corresponding to $k=0$), then

$$R^*_{n,\alpha,\beta}(x) \leq x^\beta\varrho(x)\Big(\frac{1}{\lambda}A_{\lambda,\alpha+\beta}(x)+C_{\lambda,\alpha+\beta}(x)+D_{\lambda,\alpha+\beta}(x)\Big),$$

and the result follows from the estimates given in Lemmas 3.5 and 3.6, with $\gamma=\alpha+\beta$.

(b) Now assume $x>1/(2\lambda)$. By (iii) in Proposition 3.1 and Corollary 2.3, one has

$$x^\beta\varrho(x)M_\lambda\Big(\Big|\int_x^t\frac{t-u}{\varphi^{2(\alpha+\beta)}(u)\varrho(u)}du\Big|,x\Big)$$
$$\leq C_1x^\beta\frac{M_\lambda((t-x)^2,x)}{\varphi^{2(\alpha+\beta)}(x)} + C_2\frac{\varrho(x)}{x^\alpha}M_\lambda\Big((t-x)^2(1+t)^{b-\alpha-\beta},x\Big)$$

$$\leq C_3 \frac{\varphi^{2(1-\alpha)}(x)}{\lambda} + C_2 \frac{\varrho(x)}{x^\alpha} \sqrt{M_\lambda((t-x)^4, x) M_\lambda((1+t)^{2(b-\alpha-\beta)} x)}$$

$$\leq C_4 \Big(\frac{\varphi^{2(1-\alpha)}(x)}{\lambda} + \frac{\varrho(x)}{x^\alpha} \sqrt{\frac{\varphi^4(x)}{\lambda^2}} (1+x)^{b-\alpha} \Big) \leq C_5 \frac{\varphi^{2(1-\alpha)}(x)}{\lambda}. \quad \square$$

4. Which functions can be approximated?

For $\phi(x) = \ln(1+x)$ set

$$q_\lambda = \sup_{x \geq 0} M_\lambda(|\phi(t) - \phi(x)|, x) \text{and} \quad r_\lambda = \sup_{x \geq 0} \varrho(x) M_\lambda(| \, 1/\varrho(t) - 1/\varrho(x) \, |, x).$$

Proposition 4.1. *For $b \geq 1$ there exists a constant C such that, for $\lambda > 2$,*

$$q_\lambda \leq \frac{2}{\sqrt{\lambda - 1}} \quad \text{and} \quad r_\lambda \leq \frac{C}{\sqrt{\lambda}}.$$

Proof. (i) For any $x, t \in [0, \infty)$, using the inequality $| \ln c - \ln d | \leq | c - d | / \sqrt{cd}$ for $c, d > 0$ (see [6, page 40]), we obtain

$$| \ln(1+x) - \ln(1+t) | \leq | x - t | / (\sqrt{(1+t)(1+x)}).$$

Hence (see Corollary 2.3 and Proposition 2.5)

$$M_\lambda(| \, \phi(t) - \phi(x)|, x) \leq \frac{1}{\sqrt{1+x}} \sqrt{M_\lambda((t-x)^2, x) M_\lambda\big((1+t)^{-1}, x\big)}$$

$$\leq \frac{1}{\sqrt{1+x}} \sqrt{2 \frac{\varphi^2(x)}{\lambda} \frac{2\lambda}{(\lambda-1)(1+x)}} = \frac{2}{\sqrt{1+x}} \sqrt{\frac{x}{\lambda - 1}}.$$

This provides the estimate for q_λ.
Note that, for $t \in I_{\lambda,k}$ and $x \geq 0$,

$$| \, t - x \, | \leq \max\{| \, k/\lambda - x \, |, | \, (k+1)/\lambda - x \, |\} \leq 1/\lambda + | \, x - k/\lambda \, |$$

and

$$\frac{(1-a_k)t}{1+a_k t} = \frac{1}{2k+1} \frac{t}{1+a_k t} \leq \frac{1}{(2k+1)} \frac{k+1}{\lambda} \leq \frac{1}{\lambda}.$$

Hence taking into account Proposition 2.1, one has

$$M_\lambda\Big(\Big|\frac{1}{1+t} - \frac{1}{1+x}\Big|, x\Big) = \lambda \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{I_{\lambda,k}} \Big|\frac{1}{1+a_k t} - \frac{1}{1+x}\Big| dt$$

$$= \frac{\lambda}{1+x} \sum_{k=0}^{\infty} v_{\lambda,k}(x) \Big(\int_{I_{\lambda,k}} \Big| \frac{(a_k - 1)t}{1+a_k t} + \frac{t-x}{1+a_k t} \Big| dt \Big)$$

$$\leq \frac{1}{1+x} \sum_{k=0}^{\infty} \Big(\frac{2}{\lambda} + \Big| \frac{k}{\lambda} - x \Big| \Big) v_{\lambda,k}(x) \leq \frac{2}{\lambda(1+x)} + \frac{\sqrt{x(1+x)}}{\sqrt{\lambda}(1+x)} \leq \frac{3}{\sqrt{\lambda}}.$$

Finally, if $x \geq 0$ and $t \in I_{\lambda,k}$, using the mean value theorem, we know that there exists a point between x and $a_k t$ such that

$$| \, (1+a_k t)^b - (1+x)^b \, | \leq b(1+\theta)^{b-1} \, | \, a_k t - x \, |$$

$$\leq b\big((1+(k+1)/\lambda)^{b-1}+(1+x)^{b-1}\big)\big((1-a_k)t+\mid t-x\mid\big)$$
$$\leq C_1\big((1+k/\lambda)^{b-1}+(1+x)^{b-1}\big)\big(2/\lambda+\mid x-k/\lambda\mid\big).$$

Hence, from Propositions 2.1 and 3.2,

$$M_\lambda(\mid 1/\varrho(t)-1/\varrho(x)\mid ,x)\leq \frac{C_2}{\lambda}V_\lambda((1+t)^{b-1},x)+\frac{2}{\lambda}(1+x)^{b-1}$$
$$+C_1V_\lambda((1+t)^{b-1}\mid t-x\mid ,x)+C_1(1+x)^{b-1}V_\lambda(\mid t-x\mid ,x)$$
$$\leq C_3\Big(\frac{1}{\lambda}+\sqrt{V_\lambda((t-x)^2,x)}\Big)(1+x)^{b-1}$$
$$\leq C_4\Big(\frac{1}{\sqrt{\lambda}}+\sqrt{x(1+x)}\Big)\frac{1}{\sqrt{\lambda}}(1+x)^{b-1}\leq \frac{C_5(1+x)^b}{\sqrt{\lambda}}. \qquad \square$$

For a continuous bounded function $f:[0,\infty)\to\mathbb{R}$ and $\phi(x)=\ln(1+x)$ define

$$\Omega_\phi(f,\delta)=\sup_{t,x\in[0,\infty),|\phi(t)-\phi(x)|\leq\delta}\mid f(t)-f(x)\mid .$$

Proposition 4.2. (see [5]) *Let $f\in C[0,\infty)$ be a bounded function, $\phi(x)=\ln(1+x)$ and ϕ^{-1} be the inverse function.*

(i) *For any $\delta>0$, $\Omega_\phi(f,\delta)=\omega(f\circ\phi^{-1},\delta)$, where ω is the usual first modulus of continuity.*

(ii) *The function $f\circ\phi^{-1}$ is uniformly continuous on $[0,\infty)$ if and only if for any sequence $\delta_n\to 0$ of positive numbers one has $\Omega_\phi(f,\delta_n)\to 0$.*

(iii) *For any $\delta>0$ and $x,t\in[0,\infty)$,*

$$|f(t)-f(x)|\leq\Big(1+\frac{\mid\phi(t)-\phi(x)\mid}{\delta}\Big)\Omega_\phi(f,\delta).$$

Theorem 4.3. *Assume $b\geq 1$, $\phi(x)=\ln(1+x)$, and ϕ^{-1} is the inverse function. For a function $f\in C_{\varrho,0}[0,\infty)$ one has $\lim_{\lambda\to\infty}\|\varrho(M_\lambda(f)-f)\|=0$ if and only if the function $(\varrho f)\circ\phi^{-1}$ is uniformly continuous on $[0,\infty)$.*

Proof. Let q_n and r_n be given as above. Assume $(\varrho f)\circ\phi^{-1}$ is uniformly continuous. From (iii) in Proposition 4.2 we know that, for any $\delta>0$,

$$|f(t)-f(x)|\leq\mid(\varrho f)(t)\mid\Big|\frac{1}{\varrho(t)}-\frac{1}{\varrho(x)}\Big|+\frac{1}{\varrho(x)}\cdot\mid(\varrho f)(t)-(\varrho f)(x)\mid\,|$$
$$\leq\|\varrho f\|\cdot\Big|\frac{1}{\varrho(t)}-\frac{1}{\varrho(x)}\Big|+\frac{1}{\varrho(x)}\Big(1+\frac{\mid\phi(x)-\phi(t)\mid}{\delta}\Big)\Omega_\phi(\varrho f,\delta).$$

Therefore

$$\varrho(x)\mid M_\lambda(f,x)-f(x)\mid\leq\|\varrho f\|\,\varrho(x)\,M_\lambda\Big(\Big|\frac{1}{\varrho(t)}-\frac{1}{\varrho(x)}\Big|,x\Big)$$
$$+\Big(1+\frac{1}{q_\lambda}M_\lambda(\mid\phi(x)-\phi(t)\mid ,x)\Big)\Omega_\phi(\varrho f,q_\lambda)\leq r_\lambda\|\varrho f\|+2\Omega_\phi(\varrho f,q_\lambda).$$

Since $r_\lambda,q_\lambda\to 0$ as $\lambda\to\infty$ (Proposition 4.1), if we assume that $(\varrho f)\circ\phi^{-1}$ is uniformly continuous on $[0,\infty)$, it follows from Proposition 4.2 that $\Omega_\phi(\varrho f,q_\lambda)\to 0$ as $\lambda\to\infty$. We have proved that $\|\varrho(M_\lambda(f)-f)\|\to 0$.

Now assume $\lim_{\lambda\to\infty} \|\varrho(M_\lambda(f)-f)\| = 0$. Taking into account Proposition 4.2, it is sufficient to prove that $\Omega_\phi(\varrho f, 1/\lambda^2) \to 0$, as $\lambda \to \infty$. By using the properties of the first modulus of continuity one has

$$\Omega_\phi\big(\varrho f, 1/\lambda^2\big) \le 2\|\varrho(f - M_\lambda(f))\| + \Omega_\phi\big(\varrho M_\lambda(f), 1/\lambda^2\big).$$

It remains to prove that the second term goes to zero. By definition we should estimate the difference $\mid \varrho(x)M_\lambda(f,x) - \varrho(y)M_\lambda(f,y) \mid$ for all point x, y satisfying

$$\mid \phi(x) - \phi(y) \mid \le 1/\lambda^2.$$

Case 1. If $0 \le y < x \le 1/\lambda$ and $\mid \phi(x)-\phi(y) \mid\le 1/\lambda^2$, there exists a point θ between x and y such that

$$\mid x - y \mid /2 \le\mid x-y \mid /(1+\theta) \le\mid \ln(1+x) - \ln(1+y) \mid\le 1/\lambda^2.$$

Therefore

$$\begin{aligned}
\mid \varrho(x)M_\lambda(f,x) - \varrho(y)M_\lambda(f,y) \mid &\le 2\|\varrho(f - M_\lambda(f)\| + \mid (\varrho f)(x) - (\varrho f)(y) \mid \\
&\le 2\|\varrho(f - M_\lambda(f)\| + \|\varrho\|_{[0,1]} \mid f(x) - f(y) \mid + \|f\|_{[0,1]} \mid \varrho(x) - \varrho(y) \mid \\
&\le 2\|\varrho(f - M_\lambda(f)\| + \|\varrho\|_{[0,1]}\omega(f, x-y)_{[0,1]} + \|f\|_{[0,1]}\omega(\varrho, x-y)_{[0,1]} \\
&\le 2\|\varrho(f - M_\lambda(f)\| + \|\varrho\|_{[0,1]}\omega\Big(f, \frac{2}{\lambda^2}\Big)_{[0,1]} + \|f\|_{[0,1]}\omega\Big(\varrho, \frac{2}{\lambda^2}\Big)_{[0,1]},
\end{aligned} \tag{4.1}$$

where the usual modulus of continuity is computed in the interval $[0,1]$.

Case 2. If $0 \le y < 1/\lambda < x$, we consider the inequality

$$\mid (\varrho f)(x) - (\varrho f)(y) \mid\le\mid (\varrho f)(y) - (\varrho f)(1/\lambda) \mid + \mid (\varrho f)(1/\lambda) - (\varrho f)(x) \mid. \tag{4.2}$$

The first term was estimated in Case 1, the second one will be considered in Case 3.

Case 3. Assume that $1/\lambda \le y < x$. From the Cauchy mean value theorem, for any point $x, y \in (0,\infty)$, there is z between x and y, such that

$$\frac{\varrho(y)M_\lambda(f,y) - \varrho(x)M_\lambda(f,x)}{\phi(y) - \phi(x)} = (1+z)\Big(\varrho'(z)M_\lambda(f,z) + \varrho(z)M'_\lambda(f,z)\Big).$$

It is easy to see that (see Proposition 3.4)

$$(1+z) \mid \varrho'(z)M_\lambda(f,z) \mid\le C_3\varrho(z)\|\varrho f\| \sum_{k=0}^{\infty} \frac{v_{\lambda+1,k}(z)}{(1+k/(\lambda+1))^{-b}} \le C_4\|\varrho f\|.$$

On the other hand, from Propositions 2.4 and 3.4 we obtain

$$\begin{aligned}
z(1+z) \mid M'_\lambda(f,z) \mid &= \lambda^2 \Big| \sum_{k=0}^{\infty} \Big(\frac{k}{\lambda} - z\Big) v_{\lambda,k}(z) \int_{I_{\lambda,k}} f(a_k t)dt \Big| \\
&\le C_5\lambda\|\varrho f\|_\infty \sum_{k=0}^{\infty} \Big|\frac{k}{\lambda} - z\Big| \frac{1}{\varrho(k/n)} v_{\lambda,k}(z) \\
&\le C_6\lambda\|\varrho f\|_\infty \frac{\varphi(z)}{\sqrt{\lambda}} \frac{1}{\varrho(z)}.
\end{aligned}$$

Thus if $z \ge 1$, then

$$(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid\le C_6\|\varrho f\| \sqrt{(1+z)/z}\sqrt{\lambda} \le C_7\sqrt{\lambda}\|\varrho f\|.$$

On the other hand, from the other equation in Proposition 2.4.

$$(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid\leq C\lambda\|\varrho f\|\varrho(z)\sum_{k=0}^{\infty}\frac{v_{\lambda+1,k}(z)}{\varrho(k/(\lambda+1))}.$$

Therefore, for $1/\lambda \leq z \leq 1$, $(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid\leq C_4\,\lambda\|\varrho f\|$.
We have proved that, for $\lambda > 1$ and $1/\lambda \leq z$,

$$(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid\leq C_8\lambda\|\varrho f\|.$$

Thus if $\mid \phi(y)-\phi(x) \mid\leq 1/\lambda^2$ and $1/\lambda \leq y < x$, then

$$\mid \varrho(y)M_\lambda(f,y)-\varrho(x)M_\lambda(f,x) \mid\leq \frac{C_9(1+\lambda)}{\lambda^2}\|\varrho f\| \leq \frac{C_{10}}{\lambda}\|\varrho f\|. \tag{4.3}$$

From (4.1)-(4.3) we know that if $x, y \geq 0$ and $\mid \phi(x)-\phi(y) \mid\leq 1/\lambda^2$, then

$$\mid \varrho(y)M_\lambda(f,y)-\varrho(x)M_\lambda(f,x) \mid$$

$$\leq C(f,\varrho)\Big(\|\varrho(f-M_\lambda(f)\| + \omega\Big(f,\frac{2}{\lambda^2}\Big)_{[0,1]} + \omega\Big(\varrho,\frac{2}{\lambda^2}\Big)_{[0,1]} + \frac{2}{\lambda}\|\varrho f\|\Big). \qquad \square$$

5. Main results

In Theorem 5.1 we estimate the norm of the operator M_λ.

Theorem 5.1. *If $\beta \in [0,1]$ and $b \geq \beta$, there exists a constant C such that, for all $\lambda > 2(1+b)$ and every $f \in C_{\varrho,\beta}[0,\infty)$, one has $\|\varrho\varphi^{2\beta}M_\lambda(f)\| \leq C\|\varrho\varphi^{2\beta}f\|$.*

Proof. First we consider the case $0 < \beta \leq 1$. If $f \in C_{\varrho,\beta}[0,\infty)$, and $x > 0$, we use Proposition 3.4 to obtain

$$(\varrho\varphi^{2\beta})(x) \mid M_\lambda(f,x) \mid\leq (\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\lambda\sum_{k=1}^{\infty}\int_{I_{\lambda,k}}\frac{dt}{\varrho(a_kt)\varphi^{2\beta}(a_kt)}\,v_{\lambda,k}(x)$$

$$\leq C_1(\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varrho(k/\lambda)\varphi^{2\beta}(k/\lambda)}$$

$$\leq C_1(\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\Big(\sum_{k=1}^{\infty}(\lambda/k)^{2\beta}v_{\lambda,k}(x)\Big)^{1/2}\Big(\sum_{k=1}^{\infty}(1+k/\lambda)^{2(b-\beta)}v_{\lambda,k}(x)\Big)^{1/2}$$

$$\leq C_2(\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\frac{(1+x)^{b-\beta}}{x^\beta} \leq C_2\|\varrho\varphi^{2\beta}f\|,$$

where we use Propositions 3.3 and 3.2.
The case $\beta = 0$ follows analogously (we do not need to use Proposition 3.3. $\square$
In the main result we use the following notations. For $\alpha, \beta \in [0,1]$ set

$$K_{\alpha,\beta}(f,t)_\varrho = \inf\big\{\|\varrho\varphi^{2\beta}(f-g)\| + t\|\varrho\varphi^{2(\alpha+\beta)}g''\| : g \in D(\alpha,\beta,\varrho)\big\},$$

where $D(\alpha,\beta,\varrho) = \{g \in C_{\varrho,\beta} : g' \in AC_{loc} : \|\varrho\varphi^{2(\alpha+\beta)}g''\| < \infty\}$.

Theorem 5.2. *If $\alpha, \beta \in [0,1]$, $\alpha+\beta < 2 \le b$, then there exists a constant C such that, for all $\lambda > 2(1+b)$, every $f \in C_{\varrho,\beta}[0,\infty)$ and $x > 0$,*

$$\varrho(x)\varphi^{2\beta}(x) \mid M_\lambda(f,x) - f(x) \mid \le CK_{\alpha,\beta}\Big(f, \frac{\varphi^{2(1-\alpha)}(x)}{n}\Big)_\varrho .$$

Proof. We know that the operators $M_\lambda : C_{\varrho,\beta}[0,\infty) \to C_{\varrho,\beta}[0,\infty)$ are uniformly bounded. If $x > 0$ and $g \in C^2_\varrho[0,\infty)$, use the representation

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)du.$$

Therefore, by setting $W(g) = \|\varrho\varphi^{2(\alpha+\beta)}g''\|$, it follows from Proposition 3.7 that

$$\begin{aligned}
&(\varrho\varphi^{2\beta})(x) \mid M_\lambda(g,x) - g(x) \mid \\
&\quad \le (\varrho\varphi^{2\beta})(x)\lambda \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{I_{\lambda,k}} \left| \int_x^{a_k t} g''(u)(a_k t - u)du \right| dt \\
&\quad \le (\varrho\varphi^{2\beta})(x)W(g) \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{I_{\lambda,k}} \left| \int_x^{a_k t} \frac{a_k t - u}{\varrho(u)\varphi^{2(\alpha+\beta)}(u)} du \right| dt \\
&\quad = (\varrho\varphi^{2\beta})(x)W(g)\, M_\lambda \left(\left| \int_x^t \frac{\mid t-u \mid du}{\varphi^{2(\alpha+\beta)}(u)\varrho(u)} \right|, x \right) \le CW(g)\frac{\varphi^{2(1-\alpha)}(x)}{\lambda}.
\end{aligned}$$

By the definition of the K-functional we obtain

$$\varrho(x)\varphi^{2\beta}(x) \mid M_\lambda(f,x) - f(x) \mid \le CK_{\alpha,\beta}\big(f, \frac{\varphi^{2(1-\alpha)}(x)}{\lambda}\big)_\varrho . \qquad \square$$

Remark 5.3. Theorem 5.2 combine pointwise estimates ($\alpha \in [0,1)$) with norm estimates ($\alpha = 1$). When $\beta = 0$, we pass to usual approximation in polynomial-type weighted spaces, in such a case, taking into account Theorem 6.1.1 of [3], the result can be written as: There exists a constant C such that, for all $\lambda > 2(1+b)$, every $f \in C_\varrho[0,\infty)$, and $x \ge 0$,

$$\varrho(x) \mid f(x) - M_\lambda(f,x) \mid \le C\omega_{\varphi^\alpha}\big(f, \varphi^{(1-\alpha)}(x)/\sqrt{\lambda}\big)_\varrho ,$$

where $\omega_{\varphi^\alpha}(f,t)_\varrho = \sup\{ \mid \varrho(x)\Delta^2_{h\varphi^\alpha(x)} f(x) \mid ;\ 0 < h \le t,\ x \ge h\varphi^\alpha(x)\}$.

In particular, if we chose $\alpha = 1$, then $\|\varrho(M_\lambda(f) - f)\| \to 0$, as $n \to \infty$, if $\omega_{\varphi^\alpha}(f,t)_\varrho \to 0$, as $t \to 0$.

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