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Strong subordination and superordination with sandwich-type theorems using integral operators

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Abstract. The notions of strong differential subordination and superordination have been studied recently by many authors. In the present paper, using these concepts, we obtain some preserving properties of certain nonlinear integral operator defined on the space of normalized analytic functions in $\mathbb{D}\times\overline{\mathbb{D}}$. The sandwich-type theorems and consequences of the main results are also considered.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}^* = \mathcal{H}(\mathbb{D} \times \overline{\mathbb{D}})$ be the class of analytic functions in $\mathbb{D} \times \overline{\mathbb{D}}$. Suppose n is a positive integer and $\mathcal{A}^*_{n\xi}$ is the subclass of $\mathcal{H}^*$ consisting of functions $f(z, \xi)$ of the form

$$f(z, \xi) = z + a_{n+1}(\xi)z^{n+1} + a_{n+2}(\xi)z^{n+2} + \cdots , \ (z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}}),$$

where the coefficients $a_k(\xi)$, $(k \geq n + 1)$ are analytic in $\overline{\mathbb{D}}$. For $n = 1$ we write $\mathcal{A}^*_{\xi} = \mathcal{A}^*_{1\xi}$. Also, if $n = 1$ and $a_k(\xi) = b_k$, then we obtain the usual class of normalized analytic functions $\mathcal{A}$ in $\mathbb{D}$.

For two functions $f, g \in \mathcal{H}$ we say that f is subordinate to g (or g is superordinate to f) and written as $f \prec g$ or $f(z) \prec g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ such that

$$w(0) = 0, |w(z)| < 1 \text{ and } f(z) = g(w(z)).$$

If g is univalent in $\mathbb{D}$, then

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

Let $f(z,\xi)$ and $g(z,\xi)$ be analytic in $\mathbb{D}\times\overline{\mathbb{D}}$. The function $f(z,\xi)$ is said to be strongly subordinate to $g(z,\xi)$ (or $g(z,\xi)$ is strongly superordinate to $f(z,\xi)$) if there exists an analytic function $w(z)$ in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\xi)=g(w(z),\xi)$ for all $\xi\in\overline{\mathbb{D}}$, (see [10]). In such a case we write

$$f(z,\xi)\prec\prec g(z,\xi),\ (z\in\mathbb{D},\xi\in\overline{\mathbb{D}}).$$

If $g(z,\xi)$, as a function of z, is univalent in $\mathbb{D}$ for all $\xi\in\overline{\mathbb{D}}$, then

$$f(z,\xi)\prec\prec g(z,\xi)\Longleftrightarrow f(0,\xi)=g(0,\xi),\xi\in\overline{\mathbb{D}} \text{ and } f(\mathbb{D}\times\overline{\mathbb{D}})\subseteq g(\mathbb{D}\times\overline{\mathbb{D}}).$$

When $f(z,\xi)\equiv f(z)$ and $g(z,\xi)\equiv g(z)$, the strong subordination becomes the usual notion of subordination.

The function $L:\mathbb{D}\times[0,+\infty)\times\overline{\mathbb{D}}\to\mathbb{C}$ is a subordination (or Loewner) chain if $L(z,t;\xi)$, as a function of z, is analytic and univalent in $\mathbb{D}$ for all $t\geq 0,\ \xi\in\overline{\mathbb{D}}$ and is continuously differentiable function of t on $[0,+\infty)$ for all $z\in\mathbb{D},\ \xi\in\overline{\mathbb{D}}$ and $L(z,t_1;\xi)\prec L(z,t_2;\xi)$ when $0\leq t_1\leq t_2$, (see [7]).

Suppose that $f(z,\xi),F(z,\xi)\in\mathcal{A}^*_{n\xi},\ f(z,\xi)\neq 0$ and $F(z,\xi)F'(z,\xi)\neq 0$ for all $z\in\mathbb{D}\setminus\{0\}$ and $\xi\in\overline{\mathbb{D}}$ with $F'(z,\xi)=\frac{\partial F(z,\xi)}{\partial z}$. We introduce the integral operator $I^*_{F,\beta}:\mathcal{A}^*_{n\xi}\to\mathcal{A}^*_{n\xi}$ as follows:

$$I^*_{F,\beta}(f)(z,\xi)=\left(\beta\int_0^z f^{\beta}(t,\xi)\frac{F'(t,\xi)}{F(t,\xi)}dt\right)^{1/\beta},(z\in\mathbb{D},\xi\in\overline{\mathbb{D}},Re\beta>0). \tag{1.1}$$

Note that all powers in (1.1) are principal ones.
When $f(z,\xi)\equiv f(z)$ and $F(z,\xi)\equiv F(z)$ the integral operator (1.1) becomes

$$I_{F,\beta}(f)(z)=\left(\beta\int_0^z f^{\beta}(t)\frac{F'(t)}{F(t)}dt\right)^{1/\beta}$$

which has been studied by Bulboaca [2].

The notions of strong subordination and superordination have been used by many authors (see, for example [1, 6, 8, 10]). Motivated by the recent works in the literature (see [2, 3, 4, 11]), in the present investigation we obtain some strong subordination and superordination preserving properties for the integral operator $I^*_{F,\beta}$ defined by (1.1) with the sandwich-type theorems. Applications of the main results are also mentioned.

To prove our main results we shall need the following lemmas.

Lemma 1.1. ([8]) *Let $p(z,\xi)$ be analytic in $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$ and, as a function of z, univalent in $\overline{\mathbb{D}}$ for all $\xi\in\overline{\mathbb{D}}$ with $p(0,\xi)=a$, and let*

$$q(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\cdots\in\mathcal{H}^*$$

with $n\geq 1$ and $q(z,\xi)\not\equiv a$. If $q(z,\xi)$ is not strongly subordinate to $p(z,\xi)$ then there exist points $z_0=r_0e^{i\theta_0}\in\mathbb{D},\ \xi_0\in\partial\mathbb{D}$ and an $m\geq n\geq 1$ such that

$$q(z_0,\xi)=p(\xi_0,\xi),z_0q'(z_0,\xi)=m\xi_0p'(\xi_0,\xi),\ \xi\in\overline{\mathbb{D}}$$

and $q(\mathbb{D}_{r_0}\times\overline{\mathbb{D}_{r_0}})\subseteq p(\mathbb{D}\times\overline{\mathbb{D}})$ where $\mathbb{D}_{r_0}=\{z\in\mathbb{C}:|z|<r_0\}$.

Lemma 1.2. ([8]) *Let* $h(z,\xi)$ *be analytic in* $\mathbb{D}\times\overline{\mathbb{D}}$,

$$q(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\cdots\in\mathcal{H}^*,\ (z\in\mathbb{D},\xi\in\overline{\mathbb{D}},n\in\mathbb{N})$$

and $\psi:\mathbb{C}^2\times\overline{\mathbb{D}}\times\overline{\mathbb{D}}\longrightarrow\mathbb{C}$. *Suppose that*

$$\psi(q(z,\xi),tzq'(z,\xi);\zeta,\xi)\in h(\mathbb{D}\times\overline{\mathbb{D}})$$

for $z\in\mathbb{D},\ \zeta\in\partial\mathbb{D},\ \xi\in\overline{\mathbb{D}}$ *and* $0<t\le\frac{1}{n}\le 1$. *If* $p(z,\xi)$ *is analytic in* $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$ *and univalent in* $\overline{\mathbb{D}}$ *for all* $\xi\in\overline{\mathbb{D}}$, $p(0,\xi)=a$ *and* $\psi(p(z,\xi),zp'(z,\xi);z,\xi)$ *is analytic in* $\mathbb{D}\times\overline{\mathbb{D}}$ *and univalent in* $\mathbb{D}$ *for all* $\xi\in\overline{\mathbb{D}}$, *then*

$$h(z,\xi)\prec\prec\psi(p(z,\xi),zp'(z,\xi);z,\xi)\Longrightarrow q(z,\xi)\prec\prec p(z,\xi).$$

Lemma 1.3. ([7], p. 4) *Let*

$$L(z,t;\xi)=a_1(t,\xi)z+a_2(t,\xi)z^2+\cdots,\ (z\in\mathbb{D},t\ge 0,\xi\in\overline{\mathbb{D}})$$

with $a_1(t,\xi)\ne 0,\ \lim_{t\to+\infty}|a_1(t,\xi)|=+\infty$ *for all* $t\ge 0,\ \xi\in\overline{\mathbb{D}}$. *Suppose that* $L(z,t;\xi)$, *as a function of* z, *is analytic in* $\mathbb{D}$ *and continuously differentiable function of* t *on* $[0,+\infty)$ *for all* $z\in\mathbb{D}$ *and* $\xi\in\overline{\mathbb{D}}$. *If* $L(z,t;\xi)$ *satisfies*

$$Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right)>0,\ (z\in\mathbb{D},\ t\ge 0),$$

and

$$|L(z,t;\xi)|\le k_0|a_1(t,\xi)|,\ (|z|<r_0<1,\ t\ge 0),$$

for some positive constants k_0 *and* r_0, *then* $L(z,t;\xi)$ *is a subordination chain.*

Lemma 1.4. ([7], pp. 30-35, [9]) *Let* $Re a>0$ *and the function*

$$p(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\cdots,$$

is analytic in $\mathbb{D}\times\overline{\mathbb{D}}$. *Suppose that the function* $J:\mathbb{C}^2\times\mathbb{D}\times\overline{\mathbb{D}}\to\mathbb{C}$ *satisfies the condition*

$$Re\{J(is,t;z,\xi)\}\le 0,\ \left(s\in\mathbb{R},\ t\le\frac{-n(a^2+s^2)}{2\ Re a}\right).$$

If

$$Re\{J(p(z,\xi),zp'(z,\xi);z,\xi)\}>0,\quad (z\in\mathbb{D},\ \xi\in\overline{\mathbb{D}}),$$

then $Re\{p(z,\xi)\}>0$ *in* $\mathbb{D}\times\overline{\mathbb{D}}$.

From here and throughout the paper we will assume that $f,g,F,G\in\mathcal{A}^*_{n\xi}$, $f(z,\xi)\ne 0$, $g(z,\xi)\ne 0$, $F(z,\xi)F'(z,\xi)\ne 0, G(z,\xi)G'(z,\xi)\ne 0$ for all $z\in\mathbb{D}\backslash\{0\}$ and $\xi\in\overline{\mathbb{D}}$.

2. Main results

We begin with the following theorem which gives the sufficient conditions so that the integral operator $I^*_{F,\beta}$ are preserved under the strong subordination.

Theorem 2.1. *Let* $\left(z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D}$ *and* $\xi \in \overline{\mathbb{D}}$. *Suppose, also that*

$$Re\left\{1 + \frac{z\varphi''(z,\xi)}{\varphi'(z,\xi)}\right\} > -\delta \tag{2.1}$$

for all $z \in \mathbb{D}$ *and* $\xi \in \overline{\mathbb{D}}$, *with*

$$\varphi(z,\xi) = z\left(\frac{g(z,\xi)}{z}\right)^{\beta}\frac{zG'(z,\xi)}{G(z,\xi)},\ \delta = \frac{n(1+|\beta-1|^2 - |1-(\beta-1)^2|)}{4Re(\beta-1)},\ Re\beta > 1. \tag{2.2}$$

If $I^*_{F,\beta}, I^*_{G,\beta}$ *are the integral operators defined by* (1.1), then

$$z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)} \prec\prec z\left(\frac{g(z,\xi)}{z}\right)^{\beta}\frac{zG'(z,\xi)}{G(z,\xi)} \tag{2.3}$$

implies that

$$z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}.$$

Proof. We define the functions H_1 and H_2 by

$$H_1(z,\xi) = z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta},\ H_2(z,\xi) = z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}. \tag{2.4}$$

Note that H_1 and H_2 are analytic in $\mathbb{D}\times\overline{\mathbb{D}}$. First, we show that if the function $q(z,\xi)$ is defined by

$$q(z,\xi) = 1 + \frac{zH_2''(z,\xi)}{H_2'(z,\xi)},\ (z \in \mathbb{D}, \xi \in \overline{\mathbb{D}}) \tag{2.5}$$

then $Re\{q(z,\xi)\} > 0$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. By a simple calculation, using (2.4) and (2.5), we obtain the following relation

$$1 + \frac{z\varphi''(z,\xi)}{\varphi'(z,\xi)} = q(z,\xi) + \frac{zq'(z,\xi)}{\beta - 1 + q(z,\xi)} \equiv Q(z,\xi). \tag{2.6}$$

From the definition of $q(z,\xi)$ and assumption of the theorem it is clear that $q(z,\xi)$ is analytic in $\mathbb{D}\times\overline{\mathbb{D}}$ and $q(0,\xi) = Q(0,\xi) = 1$. Now we define the function $J : \mathbb{C}^2 \to \mathbb{C}$ by

$$J(u,v) = u + \frac{v}{u+\beta-1} + \delta.$$

From the above relations we obtain $Re\{J(q(z,\xi), zq'(z,\xi))\} > 0$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. Next, we show that

$$Re\{J(is,t)\} \leq 0,\ \left(s \in \mathbb{R},\ t \leq \frac{-n(1+s^2)}{2},\ n \in \mathbb{N}\right).$$

We have

$$Re\{J(is,t)\} = Re\left\{is + \frac{t}{is+\beta-1} + \delta\right\} = t\frac{Re(\beta-1)}{|\beta-1+is|^2} + \delta$$
$$\leq -\frac{I_\delta(s)}{2|\beta-1+is|^2}$$

where

$$I_\delta(s) = (nRe(\beta-1)-2\delta)s^2 - 4\delta(Im(\beta-1))s$$
$$+ (nRe(\beta-1)-2\delta|\beta-1|^2).$$

The definition of δ shows that $nRe(\beta-1) \geq 2\delta$ and

$$(nRe(\beta-1)-2\delta|\beta-1|^2)(nRe(\beta-1)-2\delta) - 4\delta^2(Im(\beta-1))^2 = 0.$$

Therefore

$$I_\delta(s) = (nRe(\beta-1)-2\delta)\left(s - \frac{2\delta(Im(\beta-1))}{nRe(\beta-1)-2\delta}\right)^2 \geq 0,$$

and we obtain $Re\{J(is,t)\} \leq 0$. By using Lemma 1.4, with $a=1$, we conclude that

$$Re\{q(z,\xi)\} = Re\left\{1 + \frac{zH_2''(z,\xi)}{H_2'(z,\xi)}\right\} > 0$$

and $H_2(z,\xi)$, as a function of z, is convex (univalent) function in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$. Next, we prove that $H_1(z,\xi) \prec\prec H_2(z,\xi)$. Without loss of generality we can assume that $H_2(z,\xi)$ is analytic in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ and univalent in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$. The function $L : \mathbb{D} \times [0,+\infty) \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ given by

$$L(z,t;\xi) = \frac{\beta-1}{\beta}H_2(z,\xi) + \frac{1+t}{\beta}zH_2'(z,\xi)$$

is analytic in $\mathbb{D}$ for all $t \geq 0$ and $\xi \in \overline{\mathbb{D}}$, and is continuously differentiable function of t on $[0,+\infty)$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. Since H_2 is convex in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$ and $Re\beta > 1$, we have

$$a_1(t,\xi) = \left.\frac{\partial L}{\partial z}\right|_{z=0} = 1 + \frac{t}{\beta} \neq 0.$$

Also, $\lim_{t\to+\infty} |a_1(t,\xi)| = +\infty$ for all $\xi \in \overline{\mathbb{D}}$. A simple calculation shows that

$$Re\left\{\frac{z\partial L/\partial z}{\partial L/\partial t}\right\} = Re(\beta-1) + (1+t)Re\left\{1 + \frac{zH_2''(z,\xi)}{H_2'(z,\xi)}\right\} > 0$$

for all $\xi \in \overline{\mathbb{D}}$.

From the definition of $L(z,t;\xi)$, for all $t \geq 0$ and arbitrary (fixed) point $\xi_0 \in \overline{\mathbb{D}}$, we have

$$\frac{|L(z,t;\xi_0)|}{|a_1(t,\xi_0)|} = \frac{|(\beta-1)H_2(z,\xi_0) + (1+t)zH_2'(z,\xi_0)|}{|\beta+t|}$$
$$\leq |\beta-1||H_2(z,\xi_0)| + |H_2'(z,\xi_0)|. \tag{2.7}$$

We know that $|H_2(z,\xi_0)|$ and $|H_2'(z,\xi_0)|$ are both continuous real-valued functions in each subdisk $|z|\le r_0<1$. So, there exist positive numbers k_1 and k_2 such that

$$\frac{|L(z,t;\xi_0)|}{|a_1(t,\xi_0)|}\le|\beta-1|k_1+k_2=k_0,\quad (|z|\le r_0<1,\ t\ge 0).$$

Therefore, by Lemma 1.3, $L(z,t;\xi)$ is a subordination chain and we have

$$\varphi(z,\xi)=L(z,0;\xi)\prec L(z,t;\xi)$$

for $t\ge 0$ and $\xi\in\overline{\mathbb{D}}$. From the last relation we see that

$$L(\zeta,t;\xi)\notin L(\mathbb{D}\times\{0\}\times\{\xi\})=\varphi(\mathbb{D}\times\{\xi\}) \tag{2.8}$$

where $\zeta\in\partial\mathbb{D}, t\ge 0$ and $\xi\in\overline{\mathbb{D}}$. Now, suppose that $H_1(z,\xi)$ is not strongly subordinate to $H_2(z,\xi)$. Then by Lemma 1.1 there exist points $z_0\in\mathbb{D},\xi_0\in\partial\mathbb{D}$ and $t\ge 0$ such that

$$H_1(z_0,\xi)=H_2(\xi_0,\xi), z_0H_1'(z_0,\xi)=(1+t)\xi_0H_2'(\xi_0,\xi)$$

for all $\xi\in\overline{\mathbb{D}}$. So we obtain

$$\begin{aligned}L(\xi_0,t;\xi)&=\frac{\beta-1}{\beta}H_2(\xi_0,\xi)+\frac{1+t}{\beta}\xi_0H_2'(\xi_0,\xi)\\&=\frac{\beta-1}{\beta}H_1(z_0,\xi)+\frac{1}{\beta}z_0H_1'(z_0,\xi)\\&=\left(\frac{f(z_0,\xi)}{z_0}\right)^{\beta}\frac{{z_0}^2F'(z_0,\xi)}{F(z_0,\xi)}.\end{aligned}$$

Condition (2.3) then shows that $L(\xi_0,t;\xi)\in\varphi(\mathbb{D}\times\{\xi\})$ for all $\xi\in\overline{\mathbb{D}}$. But this contradicts (2.8) and we conclude that $H_1(z,\xi)\prec\prec H_2(z,\xi)$. ☐

Next, we investigate the dual problem of Theorem 2.1. In this case the subordinations are replaced by superordinations.

Theorem 2.2. *Let* $\left(z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi)\neq 0$ *for all* $z\in\mathbb{D}$ *and* $\xi\in\overline{\mathbb{D}}$. *Suppose, also that*

$$Re\left\{1+\frac{z\varphi''(z,\xi)}{\varphi'(z,\xi)}\right\}>-\delta,\quad (z\in\mathbb{D},\ \xi\in\overline{\mathbb{D}}),$$

where δ *and* $\varphi(z,\xi)$ *are given by* (2.2) and $Re\beta>1$. In addition, assume that

$$\psi(z,\xi)=z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$$

as a function of z, is univalent in $\mathbb{D}$ for all $\xi\in\overline{\mathbb{D}}$ and that $z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$, as a function of z, is univalent in $\overline{\mathbb{D}}$ for all $\xi\in\overline{\mathbb{D}}$. If $I^*_{F,\beta}$ and $I^*_{G,\beta}$ are the integral operators defined by (1.1), then the superordination condition

$$z\left(\frac{g(z,\xi)}{z}\right)^{\beta}\frac{zG'(z,\xi)}{G(z,\xi)}\prec\prec z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$$

implies that

$$z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^\beta \prec\prec z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^\beta.$$

Proof. The first part of the proof is similar to that of Theorem 2.1. As before we define the functions $H_1(z,\xi)$ and $H_2(z,\xi)$ by (2.4). From the definitions of H_1 and H_2 we obtain

$$\psi(z,\xi) = \frac{\beta-1}{\beta}H_1(z,\xi) + \frac{1}{\beta}zH_1'(z,\xi)$$

and

$$\varphi(z,\xi) = \frac{\beta-1}{\beta}H_2(z,\xi) + \frac{1}{\beta}zH_2'(z,\xi),$$

respectively. Let $q(z,\xi)$ be as in (2.5). Using the same techniques as in the proof of Theorem 2.1 we can prove that $Re\{q(z,\xi)\} > 0$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. This means that $H_2(z,\xi)$, as a function of z, is convex (univalent) function in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$. Now, we define the function $L : \mathbb{D} \times [0,+\infty) \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by

$$L(z,t;\xi) = \frac{\beta-1}{\beta}H_2(z,\xi) + \frac{t}{\beta}zH_2'(z,\xi).$$

As in the proof of Theorem 2.1, we see that $L(z,t;\xi)$ is a subordination chain. Therefore its definition shows that

$$L(z,t;\xi) \prec L(z,1;\xi),\ (z \in \mathbb{D},\ 0 < t \le 1,\ \xi \in \overline{\mathbb{D}}).$$

From the last relation we obtain

$$\frac{\beta-1}{\beta}H_2(z,\xi) + \frac{t}{\beta}zH_2'(z,\xi) \in \varphi(\mathbb{D}\times\overline{\mathbb{D}}),\ (z \in \mathbb{D},\ 0 < t \le 1,\ \xi \in \overline{\mathbb{D}}).$$

If we define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by $\psi(r,s) = \frac{\beta-1}{\beta}r + \frac{1}{\beta}s$, then we have

$$\psi(H_2(z,\xi), tzH_2'(z,\xi)) \in \varphi(\mathbb{D}\times\overline{\mathbb{D}}),\ (z \in \mathbb{D},\ 0 < t \le 1,\ \xi \in \overline{\mathbb{D}}).$$

Since all conditions of Lemma 1.2 are satisfied with

$$h(z,\xi) = \varphi(z,\xi),\ p(z,\xi) = H_1(z,\xi) \text{ and } q(z,\xi) = H_2(z,\xi)$$

we conclude that $H_2(z,\xi) \prec\prec H_1(z,\xi)$, and the proof is complete. □

Combining Theorems 2.1 and 2.2 we obtain the following sandwich-type result.

Corollary 2.3. *Let* $g_i, G_i \in \mathcal{A}^*_{n\xi}$, $g_i(z,\xi) \neq 0$, *and* $G_i(z,\xi)G_i'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$, $\xi \in \overline{\mathbb{D}}$ *and* $i = 1,2$. *Also, let* $\left(z\left(\frac{I^*_{G_i,\beta}(g_i)(z,\xi)}{z}\right)^\beta\right)'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D}$, $\xi \in \overline{\mathbb{D}}$ *and* $i = 1,2$. *Suppose, also that*

$$Re\left\{1 + \frac{z\varphi_i''(z,\xi)}{\varphi_i'(z,\xi)}\right\} > -\delta,\quad (z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}},\ i = 1,2) \tag{2.9}$$

where δ *is given by* (2.2) *and*

$$\varphi_i(z,\xi) = z\left(\frac{g_i(z,\xi)}{z}\right)^\beta \frac{zG_i'(z,\xi)}{G_i(z,\xi)},\ (i = 1,2,\ Re\beta > 1). \tag{2.10}$$

In addition, assume that

$$z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$$

as a function of z*, is univalent in* $\mathbb{D}$ *for all* $\xi\in\overline{\mathbb{D}}$ *and that* $z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$*, as a function of* z*, is univalent in* $\overline{\mathbb{D}}$ *for all* $\xi\in\overline{\mathbb{D}}$*. If* $I^*_{F,\beta}$ *and* $I^*_{G,\beta}$ *are the integral operators defined by* (1.1)*, then the condition*

$$z\left(\frac{g_1(z,\xi)}{z}\right)^{\beta}\frac{zG_1'(z,\xi)}{G_1(z,\xi)}\prec\prec z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}\prec\prec z\left(\frac{g_2(z,\xi)}{z}\right)^{\beta}\frac{zG_2'(z,\xi)}{G_2(z,\xi)}$$

implies that

$$z\left(\frac{I^*_{G_1,\beta}(g_1)(z,\xi)}{z}\right)^{\beta}\prec\prec z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}\prec\prec z\left(\frac{I^*_{G_2,\beta}(g_2)(z,\xi)}{z}\right)^{\beta}.$$

In Corollary 2.3 we assumed that $z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$ is univalent function of z in $\mathbb{D}$ for all $\xi\in\overline{\mathbb{D}}$ and that $z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$, as a function of z, is univalent in $\overline{\mathbb{D}}$ for all $\xi\in\overline{\mathbb{D}}$. In the following result we replace these assumptions by another condition.

Corollary 2.4. *Let* $g_i, G_i\in\mathcal{A}^*_{n\xi}$*,* $g_i(z,\xi)\neq 0$*, and* $G_i(z,\xi)G_i'(z,\xi)\neq 0$ *for all* $z\in\mathbb{D}\setminus\{0\}$*,* $\xi\in\overline{\mathbb{D}}$ *and* $i=1,2$*. Also, let* $\left(z\left(\frac{I^*_{G_i,\beta}(g_i)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi)\neq 0$ *and* $\left(z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi)\neq 0$ *for all* $z\in\mathbb{D}$*,* $\xi\in\overline{\mathbb{D}}$ *and* $i=1,2$*. Suppose, also that the conditions* (2.9) *and* (2.10) *are satisfied and that*

$$Re\left\{1+\frac{z\psi''(z,\xi)}{\psi'(z,\xi)}\right\}>\frac{-\delta}{n},\quad(z\in\mathbb{D},\ \xi\in\overline{\mathbb{D}}),\tag{2.11}$$

where δ *is given by* (2.2) *and* $\psi(z,\xi)=z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$*. If* $I^*_{F,\beta}$ *and* $I^*_{G_i,\beta}$ *are the integral operators given by* (1.1)*, then the condition*

$$z\left(\frac{g_1(z,\xi)}{z}\right)^{\beta}\frac{zG_1'(z,\xi)}{G_1(z,\xi)}\prec\prec z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}\prec\prec z\left(\frac{g_2(z,\xi)}{z}\right)^{\beta}\frac{zG_2'(z,\xi)}{G_2(z,\xi)}$$

implies that

$$z\left(\frac{I^*_{G_1,\beta}(g_1)(z,\xi)}{z}\right)^{\beta}\prec\prec z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}\prec\prec z\left(\frac{I^*_{G_2,\beta}(g_2)(z,\xi)}{z}\right)^{\beta}.$$

Proof. It is sufficient to show that the condition (2.11) implies the univalence of $\psi(z,\xi)$, as a function of z, in $\mathbb{D}$ and the univalence of $H_1(z,\xi)=z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$, as a function of z, in $\overline{\mathbb{D}}$ for all $\xi\in\overline{\mathbb{D}}$. Since $0\leq\delta\leq\frac{n}{2}$, the condition (2.11) implies that $\psi(z,\xi)$, as a function of z, is close-to-convex (univalent) in $\mathbb{D}$ for all $\xi\in\overline{\mathbb{D}}$, (see Kaplan's Theorem [5]). In addition, by using the same techniques as in the proof of Theorem 2.1 we conclude that $H_1(z,\xi)$ is convex (univalent) function in $\mathbb{D}$ for all

$\xi \in \overline{\mathbb{D}}$ (In fact, without loss of generality, we can assume that $H_1(z, \xi)$ is univalent in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$). Therefore all conditions of Corollary 2.3 are satisfied and we obtain the result. □

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