

On a property of the generalized Brauer pairs

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Abstract. In this paper we give a generalization of a result of Puig and Zhou to the context of group graded algebras. We use this generalization for an alternative approach of the proof of a result involving group graded basic Morita equivalences.

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1. Introduction

The extended Brauer quotient for G -interior algebras was first introduced in [5]. The construction is useful for extending to the blocks of the normalizers of the local pointed groups the local equivalences induced by a basic Morita equivalence. The construction was then generalized in [4] to the case of H -interior G -algebras, where H is normal in G . Further remarks and properties are given in [2] and [3] for the case of G/H -graded algebras, serving for the generalization of the main result of [5], which was achieved in [3].

In the current paper we generalize [6, Lemma 4.18] to the case of graded algebras. We obtain this generalization by employing a generalized Brauer pair and we first state and prove, without referring to graded algebras, Theorem 4.2. Secondly we give Corollary 4.4. This approach is necessary due to some technical difficulties involving the blocks of the generalized Brauer pairs. On the other hand the mentioned generalization can be used for giving an alternative proof for the main result of [2]. We discuss these situations in section 4.

In Sections 2 and 3 we introduce the standard notations and the basic results needed for the main result.

2. Assumptions and general settings

Throughout this paper \mathcal{O} denotes a discrete complete valuation ring with $k := \mathcal{O}/J(\mathcal{O})$ its residue class field, of characteristic p . G is always a finite group such that p divides its order. All modules and algebras are finitely generated or of finite dimension. A G -algebra over \mathcal{O} is an algebra A such that there exists a group homomorphism $G \mapsto \text{Aut}_{\mathcal{O}}(A)$ compatible with the algebra structure of A . For any $a \in A$ and any $g \in G$ we denote by a^g the result of g acting on a .

A G -interior algebra over \mathcal{O} is an algebra A endowed with a group homomorphism $G \rightarrow A^*$ such that $g \mapsto g \cdot 1_A = 1_A \cdot g$, where A^* denotes the units of A . Any G -interior algebra becomes a G -algebra via conjugation with the elements of G . Explicitly for any $a \in A$ and $g \in G$ we have $a^g = g^{-1} \cdot a \cdot g$.

For all subgroups K in G , on a G -algebra A , we denote by A^K the set of all K -fixed elements of A . If L is also a subgroup of G with $L \leq K$ then the \mathcal{O} -module homomorphism

$$\text{Tr}_L^K : A^L \rightarrow A^K \text{ with } \text{Tr}_L^K(a) = \sum_{g \in K/L} a^g,$$

for all $a \in A^L$, is called the relative trace map. In this sum g runs through a set of representatives of K/L . For a p -subgroup P in G we denote by $A(P)$ the quotient

$$A^P / \sum_{R < P} A_R^P$$

and by Br_P the $N_G(P)$ -algebra epimorphism

$$A^P \rightarrow A(P).$$

Any G -interior algebra is naturally endowed with a $G \times G$ -module structure given by

$$(g_1, g_2) \cdot a = g_1 \cdot a \cdot g_2^{-1}$$

for any $g_1, g_2 \in G$ and $a \in A$. If A' is another G -algebra a homomorphism of G -algebras is any homomorphism of \mathcal{O} -algebras $\psi : A \rightarrow A'$ that verifies $\psi(a^g) = \psi(a)^g$, for any $a \in A$ and $g \in G$.

If H is a normal subgroup of G , an H -interior G -algebra A is an algebra simultaneously endowed with an H -interior and G -algebra structure with G acting on the natural algebra homomorphism

$$\mathcal{O}H \rightarrow A.$$

We refer the reader to [7] for the language of pointed groups and defect pointed groups on G -algebras. Throughout the proof of Theorem 4.2 we freely use the fact that, for any finite group G , the epimorphism of G -interior algebras

$$\mathcal{O}G \rightarrow kG \tag{2.1}$$

induces a defect group preserving bijection between the blocks of the two group algebras. Furthermore, if G is normal in some finite group \tilde{G} the epimorphism of \tilde{G} -algebras 2.1 also induces a defect group preserving bijection between the \tilde{G} -invariant blocks of $\mathcal{O}G$ and those of kG . We also frequently use the results of [1].

3. The extended Brauer quotient

Let A be a G -interior algebra and consider P , a p subgroup of G . For any $\varphi \in \text{Aut}(P)$ we set $\Delta_\varphi(P) : P \rightarrow P \times P$ the φ -twisted diagonal subgroup of $P \times P$ defined as

$$\Delta_\varphi(P) = \{(u, \varphi(u)) \mid u \in P\}.$$

Since A is a $G \times G$ -module we use the definition of $\Delta_\varphi(P)$ to introduce the \mathcal{O} -module

$$A^{\Delta_\varphi(P)} = \{a \in A \mid (u, \varphi(u)) \cdot a = a \text{ for all } u \in P\}. \tag{3.1}$$

We briefly present here the construction of the extended Brauer quotient. Let R be a subgroup of P , we also introduce the \mathcal{O} -module

$$A_{\Delta_\varphi(R)}^{\Delta_\varphi(P)} = \{\text{Tr}_{\Delta_\varphi(R)}^{\Delta_\varphi(P)}(a) \mid a \in A^{\Delta_\varphi(R)}\}. \tag{3.2}$$

It is easily checked that $A^{\Delta_\varphi(P)} \cdot A^{\Delta_\psi(P)} \subseteq A^{\Delta_{\varphi\psi}(P)}$ for all $\varphi, \psi \in \text{Aut}(P)$. Denote by $\varphi_g \in \text{Aut}(P)$ the group homomorphism $\varphi_g(u) = g^{-1}ug$ for all $g \in N_G(P)$ and $u \in P$. Then the external direct sum

$$N_A^{\text{Aut}(P)}(P) = \bigoplus_{\varphi \in \text{Aut}(P)} A^{\Delta_\varphi(P)} \tag{3.3}$$

is an $N_G(P)$ -interior algebra via $N_G(P) \ni g \mapsto g \cdot 1_A \in A^{\Delta_{\varphi_g}(P)}$, while the sum

$$I := \bigoplus_{\varphi \in \text{Aut}(P)} \left(\sum_{R < P} A_{\Delta_\varphi(R)}^{\Delta_\varphi(P)} \right) \tag{3.4}$$

is a two-sided ideal of $N_A^{\text{Aut}(P)}(P)$.

Definition 3.1. The extended Brauer quotient associated with the interior G -algebra A and the p -group P is the $N_G(P)$ -interior algebra

$$\begin{aligned} \bar{N}_A^{\text{Aut}(P)}(P) &:= N_A^{\text{Aut}(P)}(P)/I \\ &= \bigoplus_{\varphi \in \text{Aut}(P)} A^{\Delta_\varphi(P)} / \left(\sum_{R < P} A_{\Delta_\varphi(R)}^{\Delta_\varphi(P)} \right) \\ &=: \bigoplus_{\varphi \in \text{Aut}(P)} A(\Delta_\varphi(P)) \end{aligned}$$

Remark 3.2. Definition 3.1 can be given for any subgroup of $\text{Aut}(P)$. It was generalized to the case of H -interior G -algebras, where H is normal in G and $P \leq G$, in [4]. In this situation the authors restrict the constructions presented in 3.1 through 3.4 for all automorphisms $\varphi \in \text{Aut}(P)$ that satisfy $\varphi(u) \in uH$ for all $u \in P$. Further generalizations of the extended Brauer quotient are given in [3]. We present them here. Assume H is normal in G and B is an H -interior G -algebra. Then

$$A := B \otimes_{\mathcal{O}H} \mathcal{O}G$$

is a G -interior algebra via $g \mapsto 1 \otimes g$ with the multiplication given by

$$(a \otimes g)(b \otimes h) = ab^{g^{-1}} \otimes gh \text{ for } a, b \in A \text{ and } gh \in G.$$

Obviously A is a $\bar{G} := G/H$ -graded G -algebra with $A_{\bar{g}} := B \otimes g$ for any $g \in \bar{g}$ and any $\bar{g} \in \bar{G}$. Furthermore if $x \in G$ and $\bar{g} \in \bar{G}$ then $(A_{\bar{g}})^x = A_{\bar{g}^x}$. Set $\bar{P} = PH/H$. Any element $\varphi \in \text{Aut}(P)$ determines an element $\bar{\varphi} \in \text{Aut}(\bar{P})$ with $\bar{\varphi}(u) = \overline{\varphi(u)}$ for all $u \in P$. According to [3, Paragraph 4] quotient of the groups

$$\text{Aut}_{\bar{G}}(P) = \{\varphi \in \text{Aut}(P) \mid \overline{\varphi(u)} = \bar{u}^{\bar{g}} \text{ for some } \bar{g} \in \bar{G} \text{ and for all } u \in P\}$$

and

$$\text{Aut}_{\bar{1}}(P) = \{\varphi \in \text{Aut}(P) \mid \overline{\varphi(u)} = \bar{u} \text{ for all } u \in P\}$$

determine the graded structure of $\bar{N}_A^{\text{Aut}(P)}(P)$. In the case $B = \mathcal{O}H$,

$$\bar{N}_A^{\text{Aut}(P)}(P) \simeq kN_G(P)$$

as $N_G(P)/C_H(P)$ -graded $N_G(P)$ -interior algebras.

4. A property of the generalized Brauer pairs

We use the notations above, and chose H a normal subgroup of G and b a block of $\mathcal{O}H$ that is G -invariant. Let Q denote a p -subgroup of G and consider the Brauer homomorphism with respect to Q

$$\text{Br}_Q : \mathcal{O}H^Q \rightarrow kC_H(Q).$$

Let (Q, e_Q) be a (G, H, b) -Brauer pair. That is e_Q is a block of $kC_H(Q)$ that verifies $\text{Br}_Q(b)e_Q = e_Q$. Since $C_H(Q)$ is a normal subgroup of $N_G(Q)$, the latter group acts on $Z(kC_H(Q))$ hence it makes sense to consider the stabilizer of e_Q in $N_G(Q)$, denoted $N_G(Q, e_Q)$. Consider the group homomorphism

$$N_G(Q) \rightarrow \text{Aut}(Q) \tag{4.1}$$

Let T denote a subgroup of the image of $N_G(Q, e_Q)$ in $\text{Aut}(Q)$. We denote by $N_G^T(Q)$ the inverse image of T via 4.1.

Finally we denote $N_H^T(Q) = N_G^T(Q) \cap H$ and $Q_1 = Q \cap N_H(Q)$ and we assume that Q_1 is not trivial.

Remark 4.1. As [6, 4.1] states for the case $H = G$ any block of $kC_H(Q)$ remains a block of kH where $C_G(Q) \leq H \leq N_G^T(Q)$. In our situation it might happen that a block of $kC_H(Q)$ is not primitive in $Z(kN_H^T(Q))$. It could happen that is also not primitive in $kN_H^T(Q)^{N_G^T(Q)}$, unless $Q \leq N_G^T(Q)$. In any case the injective homomorphism 4.3 can be constructed and used for its purpose in the proof.

Theorem 4.2. *There exists a point ν of $Q \cdot N_G^T(Q)$ on $\mathcal{O}Hb$ with $\text{Br}_Q(\nu) = e_Q$, and for any local pointed group R_ϵ on $\mathcal{O}Hb$ satisfying $Q \leq R$ and $R_\epsilon \leq Q \cdot N_G^T(Q)_\nu$, there is a local pointed group $N_R^T(Q)_\epsilon$ on $\mathcal{O}N_H^T(Q)e_Q$ such that there exists a $N_R^T(Q)$ -interior algebra embedding*

$$(kN_G^T(Q))_\epsilon \rightarrow \bar{N}_{(\mathcal{O}G_b)_\epsilon}^T(Q) \tag{4.2}$$

determined by the structural homomorphism

$$N_G^T(Q) \rightarrow \bar{N}_{\mathcal{O}Gb}^T(Q).$$

Moreover, R_ϵ is a defect pointed group of $Q \cdot N_G^T(Q)_\nu$ on $\mathcal{O}Hb$ if and only if $N_R^T(Q)_{\text{Br}_Q(\epsilon)} \leq N_G^T(Q)_{\{e_Q\}}$ is a maximal local pointed group with this property over $kC_H(Q)e_Q$. In particular, $N_R^T(Q)$ is a defect group of e_Q as a primitive idempotent of $(kC_H(Q))^{N_G^T(Q)}$ if and only if R is a defect group of e_Q as a primitive idempotent of $(k(Q_1 \cdot N_H^T(Q)))^{Q \cdot N_G^T(Q)}$.

Proof. The block e_Q lies in $kC_H(Q)^{Q \cdot N_G^T(Q)}$, where it remains primitive. It is clear that it lifts to a point $\nu \subseteq (\mathcal{O}H)^{Q \cdot N_G^T(Q)}$ via Br_Q . Since $\text{Br}_Q(b)e_Q = e_Q$ we obtain $b\nu = \nu$ giving $\nu \subseteq (\mathcal{O}Hb)^{Q \cdot N_G^T(Q)}$. For any T chosen as above e_Q is a primitive idempotent in $\mathcal{O}N_H^T(Q)^{Q \cdot N_G^T(Q)}$ and in $\mathcal{O}(Q_1 \cdot N_H^T(Q))^{Q \cdot N_G^T(Q)}$ according to [8, Lemma 2.1].

Let R_ϵ be as mentioned. Since Q is normal in R and since we deal with permutation algebras we get that $\text{Br}_R^{\mathcal{O}H}(\epsilon) \neq 0$ implies $\text{Br}_Q^{\mathcal{O}H}(\epsilon) \neq 0$. Furthermore $\text{Br}_Q(\epsilon)e_Q = \text{Br}_Q(\epsilon)$, we set $\tilde{\epsilon} = \text{Br}_Q(\epsilon)$ and since $\text{Br}_R^{kC_H(Q)}(\tilde{\epsilon}) \neq 0$ it is clear that $N_R^T(Q)_{\tilde{\epsilon}}$ is a local pointed group on $kC_H(Q)e_Q$. The injective algebra homomorphism

$$(kC_H(Q)e_Q)(N_R^T(Q)) \rightarrow (kN_H^T(Q)e_Q)(N_R^T(Q)) \tag{4.3}$$

determines a point $\bar{\epsilon} \subseteq (kN_H^T(Q)e_Q)^{N_R^T(Q)}$ satisfying

$$\tilde{\epsilon} \cdot \bar{\epsilon} = \bar{\epsilon} \text{ and } \text{Br}_{N_R^T(Q)}^{kN_H^T(Q)}(\bar{\epsilon}) \neq 0. \tag{4.4}$$

The construction of the extended Brauer quotient and [4, Theorem 3.1] give the $N_R^T(Q)$ -interior algebra isomorphism

$$\bar{N}_{(\mathcal{O}Gb)_\epsilon}^T(Q) \simeq (\bar{N}_{\mathcal{O}Gb}^T(Q))_{\bar{\epsilon}}.$$

We apply [3, Remark 2.4] and 4.4 to obtain that the group homomorphism

$$N_G^T(Q) \rightarrow (\bar{N}_{\mathcal{O}Gb}^T(Q))^*,$$

given by

$$g \mapsto \overline{g \cdot b} \in \overline{((\mathcal{O}Gb)^{\Delta_{\varphi_g}}(Q))^*}$$

induces the mentioned $N_R^T(Q)$ -interior algebra embedding.

First assume that R_ϵ is a defect pointed group of $Q \cdot N_G^T(Q)_\nu$ on $\mathcal{O}Hb$.

Then $\nu \subseteq (\mathcal{O}Hb)_R^{Q \cdot N_G^T(Q)}$ and $\nu \cdot \epsilon = \epsilon$. Applying Br_Q on the last equality we get $e_Q \cdot \text{Br}_Q(\epsilon) = \text{Br}_Q(\epsilon)$ and $e_Q \in kC_H(Q)_{N_R^T(Q)}^{N_G^T(Q)}$. According to 4.4 we obtain $\text{Br}_{N_R^T(Q)}^{kC_H(Q)}(e_Q) \neq 0$.

Conversely, by our assumptions we have $e_Q \cdot \text{Br}_Q(\epsilon) = \text{Br}_Q(\epsilon)$, $e_Q \in kC_H(Q)_{N_R^T(Q)}^{N_G^T(Q)}$, $\text{Br}_{N_R^T(Q)}^{kC_H(Q)}(e_Q) \neq 0$, for some local point $\epsilon \subseteq (\mathcal{O}Hb)^{Q \cdot N_G^T(Q)}$. Using the \mathcal{O} -algebra epimorphism

$$(\mathcal{O}Hb)_{Q \cdot N_R^T(Q)}^{Q \cdot N_G^T(Q)} \rightarrow (kC_H(Q)e_Q)_{Q \cdot N_R^T(Q)}^{Q \cdot N_G^T(Q)}$$

we see that ν in relatively $R = Q \cdot N_R^T(Q)$ -projective and $\nu \cdot \epsilon = \epsilon$. This concludes the proof of the first equivalence.

We use [8, Lemma 2.1] again to obtain that $e_Q \in k(Q_1 \cdot N_H^T(Q))^{Q \cdot N_G^T(Q)}$ is still a primitive idempotent. The last assertion follows immediately from the above statements by using $Q_1 \cdot N_H^T(Q)$ in place of H and e_Q in place of ν . \square

Remark 4.3. Although the block e_Q is not necessarily primitive in $Z(kN_H^T(Q))$ or in $kN_H^T(Q)^{N_G^T(Q)}$, the theorem above covers [6, Lemma 4.18] when applied in the case $H = G$, which gives $Q_1 = Q$, and by substituting $C_G(Q)$ for $N_G^T(Q)$ in the second statemen of the theorem. We can obtain another generalization as it follows from the next corollary.

Corollary 4.4. *There exists a point ν of $Q \cdot N_G^T(Q)$ on $\mathcal{O}Hb$ with $\text{Br}_Q(\nu) = e_Q$ and for any local pointed group R_ϵ on $\mathcal{O}Hb$ satisfying $Q \leq R$ and $R_\epsilon \leq Q \cdot N_G^T(Q)_\nu$, there is a local pointed group $N_R^T(Q)_{\bar{\epsilon}}$ on $kC_H(Q)e_Q$ such that there exists a $N_R^T(Q)$ -interior $N_G^T(Q)/C_H(Q)$ -graded algebra isomorphism*

$$(kN_G^T(Q))_{\bar{\epsilon}} \rightarrow \bar{N}_{(\mathcal{O}Gb)_\epsilon}^T(Q) \tag{4.5}$$

determined by the structural homomorphism

$$N_G^T(Q) \rightarrow \bar{N}_{\mathcal{O}Gb}^T(Q).$$

Moreover, R_ϵ is a defect pointed group of $Q \cdot N_G^T(Q)_\nu$ on $\mathcal{O}Hb$ if and only if $N_R^T(Q)_{\text{Br}_Q(\epsilon)} \leq N_G^T(Q)_{\{e_Q\}}$ is a maximal local pointed group with this property over $kC_H(Q)e_Q$. In particular, $N_R^T(Q)$ is a defect group of e_Q as a primitive idempotent of $(kC_H(Q))^{N_G^T(Q)}$ if and only if R is a defect group of e_Q as a primitive idempotent of $(kQ_1 \cdot C_H(Q))^{Q \cdot N_G^T(Q)}$.

Remark 4.5. 1. Notice that we can also consider T such that $N_G^T(Q) = N_G(Q, e_Q)$, but in this situation the statement of Corollary 4.4 only points out the $N_G(Q, e_Q)/C_H(Q)$ -graded isomorphism 4.5 and the fact that the primitive idempontents of $kC_H(Q)^{N_G(Q, e_Q)}$ are the primitive idempontents of $(kQ_1 C_H(Q))^{N_G(Q, e_Q)}$ admitting the same defect group R that includes Q .

2. [2, Lemma 6.6] is an important ingredient of the proof of the main result [2, Theorem 6.9], and it can be deduced from the above corollary when applied in the general situation. Indeed, if T is a defect group of e_Q as a primitive idempotent of $kC_H(Q)^{N_G^K(Q)}$ then $Q \cdot T$ is a defect group of ν as a primitive idempotent of $\mathcal{O}H^{Q \cdot N_G^K(Q)}$. At this point we denote by \tilde{e}_Q the primitive idempotent of $\mathcal{O}C_H(Q)^{N_G^K(Q)}$ corresponding to e_Q via 2.1. Note that \tilde{e}_Q is primitive in $\mathcal{O}C_H(Q)^{Q \cdot N_G^K(Q)}$ and $\text{Br}_Q(\tilde{e}_Q) = e_Q$. Consider the decomposition

$$\mathcal{O}H^{Q \cdot N_G^K(Q)} = \mathcal{O}C_H(Q)^{Q \cdot N_G^K(Q)} \oplus \left(\sum_{Q' < Q} (\mathcal{O}H)_{Q'}^Q \right)^{Q \cdot N_G^K(Q)}.$$

Since the sum on the right lies in the radical of $\mathcal{O}H$ it follows that $\nu \subseteq \mathcal{O}C_H(Q)^{Q \cdot N_G^K(Q)}$ and then $\nu = \tilde{e}_Q$. This means that e_Q has defect group $Q \cdot T$

as a primitive idempotent of $kC_H(Q)^{Q \cdot N_G^K(Q)}$. At last, if $Q \subseteq N_G^K(Q)$ then up to a $N_G(Q)$ -conjugate we have $Q \cdot T = T$ forcing $Q \subseteq T$.

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