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Global nonexistence and blow-up results for a quasi-linear evolution equation with variable-exponent nonlinearities

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Abstract. In this paper, we consider a class of quasi-linear parabolic equations with variable exponents,

$$a\left(x,t\right)u_{t}-\Delta_{m(.)}u=f_{p(.)}\left(u\right)$$

in which $f_{p(.)}\left(u\right)$ the source term, $a(x,t)>0$ is a nonnegative function, and the exponents of nonlinearity $m(x)$, $p(x)$ are given measurable functions. Under suitable conditions on the given data, a finite-time blow-up result of the solution is shown if the initial datum possesses suitable positive energy, and in this case, we precise estimate for the lifespan T^{*} of the solution. A blow-up of the solution with negative initial energy is also established.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^{n}$, $n\geq 1$ with a smooth boundary $\Gamma=\partial\Omega$. We consider the following initial-boundary value problem:

$$\begin{cases} a\left(x,t\right)u_{t}-\Delta_{m(.)}u=f_{p(.)}\left(u\right), \quad x\in\Omega,\ t>0 \\ u\left(x,t\right)=0 \text{ on } \Gamma,\ t\geq 0 \\ u(x,0)=u_{0}(x),\ x\in\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_{m(.)}u=\operatorname{div}\left(\left|\nabla u\right|^{m(x)-2}\nabla u\right)$$

called the $m(.)$–Laplacian operator. This operator can be extended to a monotone operator between the space $W_0^{1,m(.)}(\Omega)$ and its dual as

$$\begin{cases} -\Delta_{m(.)}u : W_0^{1,m(.)}(\Omega) \to W^{-1,m'(.)}(\Omega), \\ < -\Delta_{m(.)}u, \phi(x) >_{m(.)} = \int_\Omega |\nabla u|^{m(x)-2} \nabla u \nabla \phi(x)\,\mathrm{d}x, \\ \text{where } 2 < m_1 \le m(x) \le m_2 < \infty. \end{cases}$$

where $< .,. >_{m(.)}$ denotes the duality pairing between $W_0^{1,m(.)}(\Omega)$ and $W^{-1,m'(.)}(\Omega)$,

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1.$$

$f_{p(.)}(u)$ is a general source term depends on $p(.)$, the coefficients $a(x,.)$ is a nonnegative function, the exponents $p(.)$ and $m(.)$ are given measurable functions on $\overline{\Omega}$ such that:

$$2 < m_1 \le m(x) \le m_2 < p_1 \le p(x) \le p_2 \le m_*(x), \tag{1.2}$$

where, for any function ψ, we set

$$\psi_2 = ess \sup_{x\in\Omega} \psi(x), \quad \psi_1 = ess \inf_{x\in\Omega} \psi(x).$$

and

$$m_*(x) = \begin{cases} \frac{nm(x)}{(n-m(x))_2} & \text{if } n > m_2 \\ +\infty & \text{if } n \le m_2. \end{cases}$$

We also assume that $m(.)$ satisfies the following Zhikov–Fan uniform local continuity condition:

$$|m(x) - m(y)| \le \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0. \tag{1.3}$$

A considerable effort has been devoted to the study of problem (1.1) in the case of constant variable when $p(x) = p$ =constant and $m(x) = m$ =constant. The problem (1.1) with the usual m-Laplacian operator $\Delta_m u = \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$, ($m$ =constant ≥ 2); ($m = 2$, $\Delta_m u = \Delta u$), has been extensively studied concerning existence, nonexistence and long-time dynamics. For results of the nature and in the case when $p(x) = p$ =constant≥ 2 and $m(x) = m$ =constant> 2, we refer the reader to [14, 18, 21] related to the equation

$$a(x)u_t - \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right) = f_p(u), \ x \in \Omega,\ t > 0.$$

When $m(x) = m = 2$, $a(x,t) = 1$ and $f_{p(.)}(u) = u^{p(x)}$, problem (1.1) becomes the following

$$u_t - \Delta u = u^{p(x)},\ x \in \Omega,\ t > 0. \tag{1.4}$$

The problem (1.4) arises from many important mathematical models in engineering and physical sciences. For example, nuclear science, chemical reactions, heat transfer, population dynamics, biological sciences, etc., and have interested a great deal of attention in the research, see [1, 2, 4, 7, 12] and the references therein. For problem (1.4), Hua Wang et al. [15] established a blow-up result with positive initial energy under some suitable assumptions on the parameters $p(.)$ and u_0. In [12], the authors proved that there are non-negative solutions with a blow-up in finite time if and only if $p_2 > 1$. The authors in [20] obtained the solution of problem (1.1) blows up in a finite time when the initial energy is positive. In [8], authors based on the idea as in [5] derived the lower bounds for the time of blow-up if the solutions blow up.

This work is extend the results established in bounded domains to general problem as in (1.1) in the case, when the exponents $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ and satisfy (1.2) and $f_{p(.)}(u)$ is a more generalized source term. We note that the presence of the variable-exponent nonlinearities and the coefficient $a(x,t)$ in this problem make analysis in the paper somewhat harder than that in the related ones. The goal of the current project is to study the blow-up phenomenon of solutions to the problem (1.1) in the framework of the Lebesgue and Sobolev spaces with variable exponents, we will establish a blow-up result and give a precise estimate for the lifespan T^* of the solution in this case. The method used here is the concavity method. However, because of the presence of the variable-exponent nonlinearities in our problem, our argument is considerably different and it is more abbreviated. The present report is organized as follows. In Sections 2, the Orlicz-Sobolev function spaces are introduced, and a brief description of their main properties are presented. In Sections 3, the blow up for positive initial energy of problem (1.1) is stated. Section 4 provides proof of the blow-up for negative initial energy of problem (1.1).

2. Preliminaries

In this section, some well-known results and facts from the theory of Sobolev spaces with variable exponents are recalled and listed (for details, see [9, 10, 11, 13, 17]). Throughout the rest of this report, Ω is assumed to be a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary Γ, assuming that $p(.)$ is a measurable function on $\overline{\Omega}$ and satisfy the following Zhikov–Fan uniform local continuity condition:

$$|p(x) - p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0.$$

Let $p : \Omega \to [1, \infty]$ be a measurable function. $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions u on Ω such that

$$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x.$$

The variable-exponent space $L^{p(\cdot)}$ equipped with the Luxemburg norm

$$\|u\|_{p(.)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0,\ \ \varrho_{p(.)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

is a Banach space. In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects; see the first discussion of $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ spaces by Kovàcik and Rákosnik in [17].

Here are some properties of the space $L^{p(\cdot)}(\Omega)$, which will be used in the study of a problem (1.1).

- It follows directly from the definition of the norm that
$$\min\left(\|u\|_{p(.)}^{p_1},\|u\|_{p(.)}^{p_2}\right)\leq \varrho_{p(.)}(u)\leq \max\left(\|u\|_{p(.)}^{p_1},\|u\|_{p(.)}^{p_2}\right).$$
- The following generalized Hölder inequality
$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x\leq\left(\frac{1}{p_1}+\frac{1}{(p_1)'}\right)\|u\|_{p(x)}\|v\|_{p'(x)}\leq 2\|u\|_{p(x)}\|v\|_{p'(x)}$$
applies for all $u\in L^{p(.)}(\Omega)$, $v\in L^{p'(.)}(\Omega)$ with $p(x)\in(1,\infty)$, $p'(x)=\frac{p(x)}{p(x)-1}$.
- If condition (2.4) is fulfilled, Ω has a finite measure, and p, q are variable exponents such that $p(x)\leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(.)}(\Omega)\hookrightarrow L^{p(.)}(\Omega)$ is continuous.
- The Sobolev space $W_0^{1,p(.)}(\Omega)$ with $p(x)\in[p_1,p_2]\subset(1,\infty)$, and $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, is defined as
$$\left\{\begin{array}{c} W_0^{1,p(.)}(\Omega)=\left\{u\in L^{p(.)}(\Omega)\mid\ |\nabla u|^{p(x)}\in L^1(\Omega),\ u=0 \text{ on } \partial\Omega\right\},\\ \|u\|_{W_0^{1,p(.)}(\Omega)}=\|u\|_{1,p(.)}=\sum_i\|D_iu\|_{p(.),\Omega}+\|u\|_{p(.),\Omega}, \end{array}\right\}$$
and $W^{-1,p'(.)}(\Omega)$ is defined in the same way as the usual Sobolev spaces (see [9]).
- An equivalent norm of $W_0^{1,p(.)}(\Omega)$ is given by
$$\|u\|_{W_0^{1,p(.)}(\Omega)}=\|\nabla u\|_{p(.),\Omega}.$$

Furthermore, we set $W_0^{1,p(.)}(\Omega)$, to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined in a different way for the variable exponent case. However (see Diening et al [9]), both definitions are equivalent under (1.3). The $\left(W_0^{1,p(.)}(\Omega)\right)'$ is the dual space of $W_0^{1,p(.)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$ and is defined as $W^{-1,p'(.)}(\Omega)$, in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)}+\frac{1}{p(.)'}=1$.

- If $p\in C\left(\overline{\Omega}\right)$, $q:\Omega\to[1,+\infty)$ is a measurable function and $\operatorname{ess}\inf\limits_{x\in\Omega}(p^*(x)-q(x))>0$ with $p^*(x)=\frac{np(x)}{(n-p(x))_2}$, then $W_0^{1,p(.)}(\Omega)\hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemma 2.1. ([9]) *Let Ω be a bounded domain of $\mathbb{R}^n$, $p(.)$ and $m(.)$ satisfy* (1.2) *and* (1.3), *then*
$$B_0\|\nabla u\|_{m(.)}\geq\|u\|_{p(.)},\ \textit{for all } u\in W_0^{1,m(.)}(\Omega), \tag{2.1}$$
where the optimal constant of Sobolev embedding B_0 is depend on $p_{1,2}$ and $|\Omega|$.

Lemma 2.2 (Poincaré's Inequality). ([9]) *Let Ω be a bounded domain of $\mathbb{R}^n$ and $m(.)$ satisfies* (1.3), *then*
$$D_0\|\nabla u\|_{m(.)}\geq\|u\|_{m(.)},\ \textit{for all } u\in W_0^{1,m(.)}(\Omega), \tag{2.2}$$
where the optimal constant of Sobolev embedding D_0 is depend on $m_{1,2}$ and $|\Omega|$.

2.1. Mathematical assumptions

In this section, we establish the blow-up result for solutions with positive energy. Let the function $f_{p(.)} \in C^0(\mathbb{R}, \mathbb{R}^+)$, with the primitive

$$F(u) = \int_0^u f_{p(.)}(\eta)\, \mathrm{d}\eta, \tag{2.3}$$

satisfies

$$\left|f_{p(.)}(s)\right| \le C_0 |s|^{p(.)-1}, \quad p(x) F(s) \le s f_{p(.)}(s), \; s \in \mathbb{R}, \; C_0 > 0. \tag{2.4}$$

A simple typical example of these functions is

$$f_{p(.)}(s) = |s|^{p(x)-2} s.$$

Assume that $a(x,t)$ is a positive function which belongs to the space $W^{1,\infty}(0,\infty; L^\infty(\Omega))$ and that $a_t(x,t) \le 0$ a.e. for $t \ge 0$. Let

$$B_1 = \max\left(1, B_0, \left(\frac{1}{C_0}\right)^{\frac{1}{p_1}}\right), \quad \alpha_1 = \left(\frac{1}{B_1^{p_1} C_0}\right)^{\frac{m_2}{p_1 - m_2}}, \quad \alpha_0 = \|\nabla u_0\|_{m(.)}^{m_2}, \tag{2.5}$$

and

$$E_0 = \left(\frac{1}{B_1^{p_1} C_0}\right)^{\frac{m_2}{p_1 - m_2}} \left(\frac{1}{m_2} - \frac{1}{p_1}\right) = \left(\frac{1}{m_2} - \frac{1}{p_1}\right) \alpha_1. \tag{2.6}$$

3. Main result

In this section, we present our main blow-up result. We start with a local existence result for the problem (1.1), which can be established by combining the arguments of [3, 6], the following theorem, which confirms the existence of a local solution is a direct result.

Theorem 3.1. *For all $u_0 \in W_0^{1,m(.)}(\Omega)$, there exists a number $T_0 \in (0,T]$ such that the problem (1.1) has a strong solution u on $[0,T_0]$ satisfying*

$$u \in C([0,T_0]; W_0^{1,m(.)}(\Omega)) \cap C([0,T_0]; L^{p(.)}(\Omega)) \cap W^{1,2}([0,T_0]; L^2(\Omega)).$$

4. Blow up for positive initial energy

This section first presents our main blow-up result and its proof for the problem (1.1). For this purpose, we start by the following lemma defining the energy of the solution.

Lemma 4.1. *The corresponding energy to problem (1.1) is given by*

$$E(t) = \int_\Omega \frac{1}{m(x)} |\nabla u(x,t)|^{m(x)} \mathrm{d}x - \int_\Omega F(u(x,t))\, \mathrm{d}x, \tag{4.1}$$

furthermore, by the easily verified formula

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_\Omega a(x,t)\, u_t^2(x,t)\, \mathrm{d}x \le 0, \tag{4.2}$$

the inequality $E(t) \leq E(0)$ is obtained.

Now, we are in a position to state our main theorem results.

Theorem 4.2. *If the initial data $u_0 \in W^{1,m(.)}(\Omega)$ are such that $u_0 \neq 0$,*

$$E(0) = \int_\Omega \frac{1}{m(x)} |\nabla u_0(x)|^{m(x)} \mathrm{d}x - \int_\Omega F(u_0(x)) \mathrm{d}x \leq E_0, \tag{4.3}$$

then there exists T^* such that $\limsup_{t\to T^*} \|u(.,t)\|_2 = +\infty$. Moreover, if $E(0) < E_0$, then the T^* can be bounded above as:

$$T^* \leq \frac{8\left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}}{(p_1-2)^2(E_0 - E(0))}, \tag{4.4}$$

where $a(x,0) := a_0$ and $u(x,0) := u_0$.

In order to prove the main theorem, we recall the following lemmas.

Lemma 4.3. ([16, Lemma1.1] and [19, Logarithmic convexity methods]) *Assume that $\varphi \in C^2([0,T))$ satisfying:*

$$\varphi''\varphi - (1+\alpha)(\varphi')^2 \geq 0, \quad \alpha > 0,$$

and

$$\varphi(0) > 0, \quad \varphi'(0) > 0,$$

then

$$\varphi \to \infty \text{ as } t \to t_1 \leq t_2 = \frac{\varphi(0)}{\alpha\varphi'(0)}.$$

Lemma 4.4. *Suppose $E(0) < E_0$ and $\alpha_1 < \alpha_0 \leq B_1^{-m_2}$. Then it exists a constant $\alpha_2 > \alpha_1$ such that:*

$$\|\nabla u\|^{m_2}_{m(.)} \geq \alpha_2 > \alpha_1 \quad \text{for all } t \geq 0.$$

Proof. Thanks to (2.3) and (2.1), we have for any $t \geq 0$

$$\begin{aligned} E(t) &= \int_\Omega \frac{1}{m(x)} |\nabla u(x,t)|^{m(x)} \mathrm{d}x - \int_\Omega F(u(x,t)) \mathrm{d}x \\ &\geq \frac{1}{m_2} \min\left(\|\nabla u\|^{m_1}_{m(.)}, \|\nabla u\|^{m_2}_{m(.)}\right) - \int_\Omega \frac{C_0}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x \\ &\geq \frac{1}{m_2} \min\left(\|\nabla u\|^{m_1}_{m(.)}, \|\nabla u\|^{m_2}_{m(.)}\right) - \frac{C_0}{p_1} \max\left(B_1^{p_1} \|\nabla u\|^{p_1}_{m(.)}, B_1^{p_2} \|\nabla u\|^{p_2}_{m(.)}\right) \qquad (4.5) \\ &= \frac{1}{m_2} \min\left(\alpha^{\frac{m_1}{m_2}}, \alpha\right) - \frac{C_0}{p_1} \max\left((\alpha B_1^{m_2})^{\frac{p_1}{m_2}}, (\alpha B_1^{m_2})^{\frac{p_2}{m_2}}\right) : = g(\alpha), \ \forall \alpha \in [0,+\infty[\end{aligned}$$

where $\alpha = \|\nabla u\|^{m_2}_{m(.)}$. Now if we let

$$h(\alpha) = \frac{1}{m_2}\alpha - \frac{C_0}{p_1}(\alpha B_1^{m_2})^{\frac{p_1}{m_2}}$$

Notice that $h(\alpha) = g(\alpha)$, for $0 < \alpha < B_1^{-m_2}$. It is easy to check that the function $h(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha \leq +\infty$.

Because $E(0) < E_0 = h(\alpha_1)$, there exists a positive constant $\alpha_2 \in (\alpha_1, +\infty)$ such that $h(\alpha_2) = E(0)$. Then we have

$$h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2).$$

It implies that $\alpha_0 \geq \alpha_2 > \alpha_1$.

To show that $\|\nabla u(x,t)\|_{m(.)}^{m_2} \geq \alpha_2$ we reason by absurd while supposing that

$$\|\nabla u(x,t^*)\|_{m(.)}^{m_2} < \alpha_2$$

for a some t^*. Then by the continuity of $\|\nabla u(.,t)\|_{m(.)}$-norm with respect to time variable, one can choose t^* such that

$$\alpha_2 > \|\nabla u(x,t^*)\|_{m(.)}^{m_2} > \alpha_1.$$

The monotonicity of $h(\alpha)$, gives

$$E(t^*) \geq h(\|\nabla u(x,t)\|_{m(.)}^{m_2}) > h(\alpha_2) = E(0)$$

it is impossible because $E(0) \geq E(t)$ for all $t \geq 0$. Then, for all time $t \geq 0$:

$$\|\nabla u\|_{m(.)}^{m_2} \geq \alpha_2 > \alpha_1. \tag{4.6}$$

□

Proof of Theorem 1. **Case 1:** $E(0) < E_0$. The goal is to construct a suitable function which satisfies the conditions in Lemma (4.3). Following the arguments of [22, 23], for our purpose, we define the following suitable function

$$\begin{aligned}\varphi(t) = &\int_0^t \int_\Omega a(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega (s-t)\, a_t(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s \\ &+ (T_0 - t) \int_\Omega a_0(x)\, u_0^2(x)\, \mathrm{d}x + \beta (t+t_0)^2, \ t < T_0\end{aligned} \tag{4.7}$$

where t_0, T_0 and β are positive constants to be determined later. Then using equation (1.1) and integration by parts, to obtains

$$\begin{aligned}\varphi'(t) &= \int_\Omega a(x,t)\, u^2(x,t)\, \mathrm{d}x - \int_0^t \int_\Omega a_t(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s \\ &\quad - \int_\Omega a_0(x)\, u_0^2(x)\, \mathrm{d}x + 2\beta(t+t_0) \\ &= 2\int_0^t \int_\Omega a(x,s)\, u(x,s)\, u_t(x,s)\, \mathrm{d}x\mathrm{d}s + 2\beta(t+t_0),\end{aligned} \tag{4.8}$$

and

$$\varphi''(t) = 2\int_\Omega a(x,t)\, u(x,t)\, u_t(x,t)\, \mathrm{d}x + 2\beta. \tag{4.9}$$

Then, due to (2.4) and (4.6), the following is obtained

$$\varphi''(t) \geq -2\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x + 2\int_\Omega p(x)\,F(u)\,\mathrm{d}x + 2\beta$$

$$\geq -2\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x + 2p_1\left(\int_\Omega \frac{1}{m(x)}|\nabla u(x,t)|^{m(x)}\,\mathrm{d}x - E(t)\right) + 2\beta$$

$$\geq \left(\frac{2p_1}{m_2} - 2\right)\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x - 2p_1E(t) + 2\beta$$

$$\geq \left(\frac{2p_1}{m_2} - 2\right)\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x$$

$$+2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s - 2p_1E(0) + 2\beta$$

$$\geq \left(\frac{2p_1}{m_2} - 2\right)\min\left(\|\nabla u\|_{m(.)}^{m_1}, \|\nabla u\|_{m(.)}^{m_2}\right)$$

$$+2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s - 2p_1E(0) + 2\beta$$

$$\geq \left(\frac{2p_1}{m_2} - 2\right)\min\left(\alpha_2^{\frac{m_1}{m_2}}, \alpha_2\right)$$

$$+2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s - 2p_1E(0) + 2\beta$$

$$\geq 2p_1\left(\frac{1}{m_2} - \frac{1}{p_1}\right)\min\left(\alpha_1^{\frac{m_1}{m_2}}, \alpha_1\right)$$

$$-2p_1E(0) + 2\beta + 2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s$$

$$= 2p_1\left(\frac{1}{m_2} - \frac{1}{p_1}\right)\alpha_1 - 2p_1E(0) \quad \text{(by (2.5))}$$

$$+2\beta + 2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s$$

$$= 2p_1(E_0 - E(0)) + 2\beta + 2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s$$

Now, let $\beta = 2(E_0 - E(0)) > 0$, and note that $p_1 > 2$, then

$$\varphi''(t) \geq (p_1+2)\,\beta + (p_1+2)\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s \tag{4.10}$$

From (4.7), (4.8), (4.9) and (4.10), we have

$$\begin{cases} \varphi(0) = T_0\int_\Omega a_0(x)\,u_0^2(x)\,\mathrm{d}x + \beta t_0^2 > 0; \\ \varphi'(0) = 2\beta t_0 > 0; \\ \varphi''(t) \geq (p_1+2)\,\beta > 0 \ \forall t \geq 0. \end{cases}$$

Therefore φ and φ' are both positive. Since $a_t(x,t) \leq 0$, for all $x \in \Omega$ and $t \geq 0$, we have

$$\varphi(t) \geq \int_0^t \int_\Omega a(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s + \beta (t+t_0)^2, \tag{4.11}$$

Thus, from (4.7)-(4.10) and (4.11), the following inferred for all $(\zeta, \eta) \in \mathbb{R}^2$

$$\begin{aligned}
&\varphi(t)\,\zeta^2 + \varphi'(t)\,\zeta\eta + \frac{\eta^2}{p_1+2}\varphi''(t) \\
&\geq \left(\int_0^t \int_\Omega a(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s + \beta (t+t_0)^2 \right) \zeta^2 \\
&+2\zeta\eta \int_0^t \int_\Omega a(x,s)\, u(x,s)\, u_t(x,s)\, \mathrm{d}x\mathrm{d}s + 2\zeta\eta\beta(t+t_0) \\
&+\beta\eta^2 + \eta^2 \int_0^t \int_\Omega a(x,s)\, u_t^2(x,s)\, \mathrm{d}x\mathrm{d}s \geq 0,
\end{aligned}$$

which implies that

$$\varphi(t)\frac{\varphi''(t)}{p_1+2} - \left(\frac{\varphi'(t)}{2}\right)^2 \geq 0,$$

subsequently

$$\varphi(t)\,\varphi''(t) - \frac{p_1+2}{4}\left(\varphi'(t)\right)^2 \geq 0. \tag{4.12}$$

Then using Lemma (4.3), to infer $\varphi(t) \to \infty$ as $t \to T^*$, where,

$$T^* \leq \frac{\varphi(0)}{\left(\frac{p_1-2}{4}\right)\varphi'(0)} = \frac{2\left(T_0 \left\|\sqrt{a_0}u_0\right\|_{L^2(\Omega)}^2 + \beta t_0^2\right)}{(p_1-2)\,\beta t_0}.$$

Now we go to choose appropriate t_0 and T_0. Let t_0 be any number which depends only on p_1, $E_0 - E(0)$ and $\|u_0\|_{L^2(\Omega)}$ as

$$t_0 > \frac{\left\|\sqrt{a_0}u_0\right\|_{L^2(\Omega)}^2}{(p_1-2)\,(E_0 - E(0))}.$$

Fix t_0, then T_0 can be picking as

$$T_0 = \frac{2\left(T_0 \left\|\sqrt{a_0}u_0\right\|_{L^2(\Omega)}^2 + \beta t_0^2\right)}{(p_1-2)\,\beta t_0},$$

so that

$$T_0 = \frac{2\,(E_0 - E(0))\, t_0^2}{(p_1-2)\,(E_0 - E(0))\, t_0 - \left\|\sqrt{a_0}u_0\right\|_{L^2(\Omega)}^2},$$

Therefore the lifespan of the solution $u(x,t)$ is bounded by

$$T^* \leq \inf_{t\geq t_0} \frac{2\left(E_0 - E\left(0\right)\right)t^2}{\left(p_1-2\right)\left(E_0 - E\left(0\right)\right)t - \left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}},$$
$$= \frac{8\left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}}{\left(p_1-2\right)^2\left(E_0 - E\left(0\right)\right)}.$$

Case 2: $E(0) = E_0$. For this case, actually we consider the following claim

Claim 4.5. There exists $t^* > 0$ such that $E(t^*) < E_0$.

Suppose Claim is not true which means that $E(t) = E_0$ for all $t \geq 0$. Then by the continuity of $\left\|\nabla u\left(.,t\right)\right\|_{m(.)}$ there exists a t_0 small enough, such that

$$E(t) = E_0 \text{ and } \left\|\nabla u\left(.,t\right)\right\|^{m_2}_{m(.)} \geq \alpha_2 > \alpha_1 \text{ for all } t \in [0,t_0]$$

Then we consider the solution of (1.1) on $[0,t_0]$,

$$0 = E(t) - E_0 = -\int_0^{t_0}\int_\Omega a\left(x,t\right)u_t^2\left(x,t\right)\mathrm{d}x\mathrm{d}t$$

which turns out to be

$$\int_\Omega a\left(x,t\right)u_t\left(x,t\right)u\left(x,t\right)\mathrm{d}x = 0 \text{ a.e. on } [0,t_0]$$

And consequently, due to the equation (1.1),

$$\int_\Omega a\left(x,t\right)u_t\left(x,t\right)u\left(x,t\right)\mathrm{d}x \tag{4.13}$$
$$= -\int_\Omega \left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x + \int_\Omega u\left(x,t\right)f_{p(.)}\left(u\left(x,t\right)\right)\mathrm{d}x = 0 \text{ a.e.on } (0,t_0].$$

On the other hand,

$$E_0 = E\left(t\right) = \int_\Omega \frac{1}{m\left(x\right)}\left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x - \int_\Omega F\left(u\left(x,t\right)\right)\mathrm{d}x$$
$$\geq \frac{1}{m_2}\int_\Omega \left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x - \frac{1}{p_1}\int_\Omega u\left(x,t\right)f_{p(.)}\left(u\left(x,t\right)\right)\mathrm{d}x$$
$$= \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x \text{ (by (4.13))}$$
$$> \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\min\left(\alpha_1^{\frac{m_1}{m_2}}, \alpha_1\right) \text{ (by (4.6))}$$
$$= \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\alpha_1 = E_0 \text{ (by (2.5) and (2.6))}$$

which is a contradiction.

The proof of Theorem (4.2) is complete since one can apply the previous case (**Case 1**) after changing the time origin to t^*. □

5. Blow up for negative initial energy

This section is devoted to the main blow-up result and its proof in the case when $E(0) \leq 0$.

Assume that $a(x,t)$ is a positive function which belongs to the space $W^{1,\infty}(0,\infty; L^{\infty}(\Omega))$ and that $a_t(x,t) \geq 0$ a.e. for $t \geq 0$.

The next Lemma gives the desired blow-up result.

Lemma 5.1. *Let* $u_0 \in W_0^{1,m(.)}(\Omega)$ *such that* $\int_\Omega u_0^2 \mathrm{d}x > 0$, $f_{p(.)}$ *satisfies* (2.4) *and* $E(0) \leq 0$. *Then there exists a finite time* $T_{\max} < \infty$ *such that*

$$\int_\Omega |u(t)|^2 \,\mathrm{d}x \to \infty \text{ if } t \to T_{\max}.$$

Proof of Lemma (5.1). We then define

$$\phi(t) = \frac{1}{2}\int_\Omega a(x,t)\,|u(t)|^2 \,\mathrm{d}x$$

Differentiating ϕ with respect to t, gets

$$\begin{aligned}
\phi'(t) &= \int_\Omega a(x,t)\, u u_t \mathrm{d}x + \frac{1}{2}\int_\Omega a_t(x,t)\,|u(t)|^2 \,\mathrm{d}x \\
&\geq -\int_\Omega \left(|\nabla u|^{m(x)} - u f_{p(.)}(u)\right) \mathrm{d}x \quad \text{(by (1.1))} \\
&\geq -\int_\Omega \left(|\nabla u|^{m(x)} - p(x)\,F(u)\right) \mathrm{d}x \quad \text{(by (2.4))} \\
&\geq -\int_\Omega |\nabla u|^{m(x)} \,\mathrm{d}x + p_1 \int_\Omega F(u)\,\mathrm{d}x \\
&= -\int_\Omega |\nabla u|^{m(x)} \,\mathrm{d}x + p_1 \int_\Omega \frac{1}{m(x)} |\nabla u(x,t)|^{m(x)} \,\mathrm{d}x - p_1 E(t) \quad \text{(by (4.1))} \\
&\geq \left(\frac{p_1}{m_2} - 1\right) \int_\Omega |\nabla u|^{m(x)} \,\mathrm{d}x - p_1 E(0) \quad \text{(by (4.2))} \\
&\geq \left(\frac{p_1}{m_2} - 1\right) \int_\Omega |\nabla u|^{m(x)} \,\mathrm{d}x = c_0 \int_\Omega |\nabla u|^{m(x)} \,\mathrm{d}x, \quad (c_0 > 0)
\end{aligned}$$

We define the sets

$$\Omega_2 = \{x \in \Omega \mid |\nabla u| \geq 1\} \quad \text{and} \quad \Omega_1 = \{x \in \Omega \mid |\nabla u| < 1\}.$$

So

$$\begin{aligned}
\phi'(t) &\geq c_0 \int_{\Omega_2} |\nabla u|^{m_1} \,\mathrm{d}x + c_0 \int_{\Omega_1} |\nabla u|^{m_2} \,\mathrm{d}x \\
&\geq C_1 \left(\left(\int_{\Omega_2} |\nabla u|^2 \,\mathrm{d}x \right)^{\frac{m_1}{2}} + \left(\int_{\Omega_1} |\nabla u|^2 \,\mathrm{d}x \right)^{\frac{m_2}{2}} \right),
\end{aligned}$$

Using the fact that $\|\nabla u\|_2 \leq C\|\nabla u\|_q$, for all $q \geq 2$, to obtain

$$\begin{cases} (\phi'(t))^{\frac{2}{m_2}} \geq C_2 \int_{\Omega_1} |\nabla u|^2 \, \mathrm{d}x; \\ (\phi'(t))^{\frac{2}{m_1}} \geq C_3 \int_{\Omega_2} |\nabla u|^2 \, \mathrm{d}x. \end{cases}$$

By addition, leads to

$$(\phi'(t))^{\frac{2}{m_2}} + (\phi'(t))^{\frac{2}{m_1}} \geq C_4 \int_\Omega |\nabla u|^2 \, \mathrm{d}x \tag{5.1}$$

$$\geq C_5 \int_\Omega |u|^2 \, \mathrm{d}x \geq \frac{C_5}{\sup a(x,t)} \phi(t), \ \forall t \geq 0.$$

or

$$(\phi'(t))^{\frac{2}{m_1}} \left(1 + (\phi'(t))^{\frac{2}{m_2} - \frac{2}{m_1}}\right) \geq C_6 \phi(t), \ \forall t \geq 0. \tag{5.2}$$

By (5.1) and the fact that $\phi(t) \geq \phi(0) > 0$ ($\phi'(t) \geq 0$), we have, for each $t > 0$, either

$$\begin{cases} (\phi'(t))^{\frac{2}{m_1}} \geq \frac{C_6}{2}\phi(t) \geq \frac{C_6}{2}\phi(0); \\ \text{or } (\phi'(t))^{\frac{2}{m_2}} \geq \frac{C_6}{2}\phi(t) \geq \frac{C_6}{2}\phi(0) \end{cases}$$

which gives, in turn

$$\begin{cases} \phi'(t) \geq C_7 (\phi(0))^{\frac{m_2}{2}}; \\ \text{or } \phi'(t) \geq C_8 (\phi(0))^{\frac{m_1}{2}}, \end{cases}$$

hence

$$\phi'(t) \geq \alpha = \min\left(C_7 (\phi(0))^{\frac{m_2}{2}}, C_8 (\phi(0))^{\frac{m_1}{2}}\right),$$

since $\frac{1}{p_2} - \frac{1}{p_1} \leq 0$, (5.2) yields

$$(\phi'(t))^{\frac{2}{m_1}} (1+\alpha)^{\frac{2}{m_2} - \frac{2}{m_1}} \geq C_4 \phi(t), \ \forall t \geq 0.$$

therefore

$$\phi'(t) \geq \beta \phi^{\frac{m_1}{2}}(t), \ \forall t \geq 0.$$

simple integrating then leads to

$$(\phi(t))^{1-\frac{m_1}{2}} \leq (\phi(0))^{1-\frac{m_1}{2}} - \frac{m_1 - 2}{2}\beta t, \ \forall t \geq 0.$$

which implies that

$$\phi(t) \geq \frac{1}{\left((\phi(0))^{1-\frac{m_1}{2}} - \frac{m_1-2}{2}\beta t\right)^{\frac{2}{m_1-2}}}$$

This show that ϕ blows up in finite time $T_{\max}$ given by the estimate

$$T_{\max} \leq \frac{2(\phi(0))^{1-\frac{m_1}{2}}}{(m_1 - 2)\beta}. \qquad \square$$

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