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Certain class of m-fold functions by applying Faber polynomial expansions

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Abstract. In this paper, we introduce new class $\Sigma_m(\mu,\lambda,\gamma,\beta)$ of m-fold symmetric bi-univalent functions. Furthermore, we use the Faber polynomial expansions to find upper bounds for the general coefficients $|a_{mk+1}|(k \geqq 2)$ of functions in the class $\Sigma_m(\mu,\lambda,\gamma,\beta)$. Moreover, we obtain estimates for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this class. The results presented in this paper would generalize and improve some recent works.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$.

We let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (see details [5, 7]).

Every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f), \ r_0(f) \geqq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. Let $\sigma_{\mathcal{B}}$ denote the class of bi-univalent functions in $\mathbb{U}$. In fact that this widely-cited work by Srivastava et al. [18] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [16, 17, 18, 21, 22] and others [6, 23] .

Also the coefficients of $g = f^{-1}$, the inverse map of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, are given by the Faber polynomial [9] (see also [1, 2]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{1.3}$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \\ \times [a_5 + (-n+2)a_3^2] & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] \\ & + \sum_{j \geqq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leqq j \leqq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$.

In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2} K_1^{-2} = -a_2, \ \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \ \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).$$

In general, for $n \geq 1$ and for any $\mu \in \mathbb{R}$, an expansion of K_n^{μ} is (see for details [1, 20] or [2])

$$K_n^{\mu}(a_2, \ldots, a_{n+1}) = \mu a_{n+1} + \frac{\mu(\mu-1)}{2} D_n^2 + \frac{\mu!}{(\mu-3)!3!} D_n^3 + \cdots + \frac{\mu!}{(\mu-n)!n!} D_n^n \tag{1.4}$$

where

$$D_n^m = D_n^m(a_2, a_3, \ldots, a_{n+1}) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\nu_1} \cdots (a_{n+1})^{\nu_n}}{\nu_1! \cdots \nu_n!}, \tag{1.5}$$

the sum is taken over all non negative integers $\nu_1, \nu_2, \cdots, \nu_n$ satisfying

$$\begin{cases} \nu_1 + \nu_2 + \cdots + \nu_n = m, \\ \nu_1 + 2\nu_2 + \cdots + n\nu_n = n \end{cases}$$

The polynomials D_n^m proved by Todorov [20].
It is clear that $D_n^n(a_2, a_3, \cdots, a_{n+1}) = a_2^n$ $(n \geq 1)$, [20, Page 2].

For each function $f \in \mathcal{S}$ function, the function

$$h(z) = \sqrt[m]{f(z^m)} \tag{1.6}$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [14, 15]) if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \ (z \in \mathbb{U}, m \in \mathbb{N}). \tag{1.7}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

The functions in the class $\mathcal{S}$ are said to be one-fold symmetric. The normalized form of f is given as in (1.7) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [19], is given as follows:

$$f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1} \tag{1.8}$$

$$\begin{aligned} &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ -[&\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots . \end{aligned} \tag{1.9}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. Thus, when $m = 1$, the formula (1.9) coincides with the formula (1.2). Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right] \ \textit{and} \ [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

In this work, we introduce new class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ of m-fold symmetric bi-univalent functions defined on $\mathbb{U}$ and use the Faber polynomial expansions to obtain the general coefficients $a_{mk+1} (k \geqq 2)$ of m-fold bi-univalent functions in this class. Also, we gain estimates for the general coefficients and early coefficients of functions belonging to this class. We show that the results would improve some of the previouse works like Hamidi and Jahangiri [11, 12, 13], Eker [8], Srivastava et al. [18, 19], Çağlar et al. [6], Frasin and Aouf [10] and Altinkaya and Yalçin [3].

2. Preliminary results

For finding the coefficients for functions belonging to the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$, we need the following lemmas and remarks.

Lemma 2.1. [1, 2] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. *Then for any* $\mu \in \mathbb{R}$, *there are the polynomials* K_n^μ, *such that*

$$\left(\frac{f(z)}{z}\right)^\mu = 1 + \sum_{n=1}^{\infty} K_n^\mu(a_2, \cdots, a_{n+1}) z^n,$$

where K_n^μ *given by* (1.4).

In particular

$$K_1^\mu(a_2) = \mu a_2, \quad K_2^\mu(a_2, a_3) = \mu a_3 + \frac{\mu(\mu-1)}{2} a_2^2$$

and

$$K_3^\mu(a_2, a_3, a_4) = \mu a_4 + \mu(\alpha - 1) a_2 a_3 + \frac{\mu(\alpha-1)(\mu-2)}{3!} a_2^3.$$

Remark 2.2. Let $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$. Then for any $\mu \in \mathbb{R}$, there are the polynomials K_k^μ, such that

$$\left(\frac{f(z)}{z}\right)^\mu = 1 + \sum_{k=1}^{\infty} K_k^\mu(a_{m+1}, \cdots, a_{mk+1}) z^{mk}.$$

Proof. The proof has been satisfied from $f(z) \in \mathcal{S}$, and Lemma 2.1. □

Case 2.3. *In special case, if* $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, *then*

$$K_i^\mu(a_{m+1}, \cdots, a_{mi+1}) = 0 \ ; \ 1 \leqq i \leqq k-1$$

and

$$K_k^\mu(a_{m+1}, \cdots, a_{mk+1}) = \mu a_{mk+1}.$$

Lemma 2.4. [4] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. *Then*

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} F_k(a_2, \cdots, a_{k+1}) z^k,$$

where $F_k(a_2, a_3, \cdots, a_{k+1})$ *is a Faber polynomial of degree* k,

$$F_k(a_2, a_3, \cdots, a_{k+1}) = \sum_{i_1 + 2i_2 + \cdots + k i_k = k} A_{(i_1, i_2, \cdots, i_k)} a_2^{i_1} a_3^{i_2} \cdots a_{k+1}^{i_k} \tag{2.1}$$

and

$$A_{(i_1, i_2, \cdots, i_k)} := (-1)^{k + 2i_1 + 3i_2 + \cdots + (k+1) i_k} \frac{(i_1 + i_2 + \cdots + i_k - 1)! k}{i_1! i_2! \cdots i_k!}. \tag{2.2}$$

The first Faber polynomials $F_k(a_2, a_3, \cdots, a_{k+1})$ *are given by:*

$$F_1(a_2) = -a_2, \ F_2(a_2, a_3) = a_2^2 - 2a_3 \text{ and } F_3(a_2, a_3, a_4) = -a_2^3 + 3a_2 a_3 - 3a_4.$$

Lemma 2.5. *Let* $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$. *Then we can write,*

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} T_{mk}(a_{m+1}, \cdots, a_{mk+1}) z^{mk},$$

where

$$T_{mk}(a_{m+1}, \cdots, a_{mk+1}) = F_{mk}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{mk+1}}_{mk})$$

$$= \sum_{mi_m + 2mi_{2m} + \cdots + mki_{mk} = mk} A_{(i_1, i_2, \ldots, i_{mk})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{mk+1}^{i_{mk}}.$$

Proof. By using Lemma 2.4 for function $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$, we have

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k \geqq 1} F_k z^k.$$

Suppose that $t, j \in \mathbb{N}$ and $1 \leqq j \leq m-1$. We consider three cases for k.

(i) If $1 \leqq k \leq m-1$, then $F_k(\underbrace{0, \cdots, 0}_{k}) = 0$.

(ii) If $k = tm$, then, we have

$$F_{tm}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}}_{tm})$$

$$= \sum_{mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm} A_{(i_1, i_2, \ldots, i_{tm})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

(iii) If $k = tm + j$, then

$$F_{tm+j}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, \underbrace{0, \cdots, 0}_{j}}_{tm+j})$$

$$= \sum_{mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm+j} A_{(i_1, i_2, \ldots, i_{tm+j})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

Since the equation

$$mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm + j,$$

doesn't have positive integer solution, so

$$F_{tm+j}(0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, 0, \cdots, 0) = 0. \qquad \square$$

Case 2.6. *In special case, if* $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, *then*

$$T_{mi}(a_{m+1}, \cdots, a_{mi+1}) = 0 \ ; \ 1 \leqq i \leqq k-1$$

and

$$T_{mk}(a_{m+1}, \cdots, a_{mk+1}) = -mka_{mk+1}.$$

Lemma 2.7. *Let* $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$*, then for every* $\mu \geqq 0$*, we have the following expansion,*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + \sum_{k=1}^{\infty} L_k(a_{m+1}, \cdots, a_{mk+1}) z^{mk}$$

$$= \begin{cases} \left(\frac{f(z)}{z}\right)^{\mu} + \frac{\lambda z}{\mu}\frac{d}{dz}\left(\frac{f(z)}{z}\right)^{\mu} = 1 + \sum_{k=1}^{\infty} \frac{\mu + \lambda mk}{\mu} K_k^{\mu}(a_{m+1}, \cdots, a_{mk+1}) z^{mk}; \mu > 0 \\ \\ \lambda \frac{zf'(z)}{f(z)} + 1 - \lambda = 1 - \sum_{k=1}^{\infty} \lambda T_{mk}(a_{m+1}, \cdots, a_{mk+1}) z^{mk} \ ; \ \mu = 0. \end{cases}$$

Proof. The proof has been satisfied from Remark 2.2, and Lemma 2.5. □

Lemma 2.8. [15] *If* $p \in \mathcal{P}$*, then* $|c_k| \leqq 2$ *for each* k*, where* $\mathcal{P}$ *is the family of all functions* p *analytic in* $\mathbb{U}$ *for which* $Re(p(z)) > 0$ *where*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots .$$

3. Class $\Sigma_m(\mu, \lambda, \gamma, \beta)$

In this section, we introduce and investigate class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ of m-fold symmetric bi-univalent functions defined on $\mathbb{U}$.

Definition 3.1. A function f given by (1.7) is said to be in the class Σ_m $(\mu, \lambda, \gamma, \beta)$ $(\mu \geqq 0,\ \lambda \geqq 1,\ \gamma \in \mathbb{C} - \{0\},\ 0 \leqq \beta < 1)$, if the following conditions are satisfied:

$$f \in \Sigma_m,\ \Re\left(1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right]\right) > \beta\ (z \in \mathbb{U})$$

and

$$\Re\left(1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} - 1\right]\right) > \beta\ (w \in \mathbb{U}),$$

where the function g is the inverse of f given by (1.8).

Remark 3.2. There are some options of the parameters γ, λ and μ which would provide interesting classes of m-fold symmetric bi-univalent functions. For example,

(I) By putting $\gamma = 1$; the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ reduces to the class $\mathcal{N}^{\mu}_{\Sigma_m}(\beta, \lambda)$, which was considered by Altinkaya and Yalçn [3].

(II) By putting $\gamma = 1$ and $\lambda = 1$ and $\mu = 0$; the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ reduces to the class of m-fold symmetric bi-starlike of order β, which was considered by Jahangiri and Hamidi [11].

(III) By putting $\gamma = 1$ and $\mu = 1$; the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ reduces to the class $\mathcal{A}^{\lambda}_{\Sigma,m}(\beta)$, which was considered by Eker [8].

(IV) By putting $\gamma = 1$, $\mu = 1$ and $\lambda = 1$; the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ reduces to the class $\mathcal{H}_{\Sigma,m}(\beta)$, which was considered by Srivastava et al. [19].

Remark 3.3. For one-fold symmetric bi-univalent functions, we denote the class $\Sigma_1(\mu,\lambda,\gamma,\beta)=\Sigma(\mu,\lambda,\gamma,\beta)$. Special cases of the parameters γ,λ and μ which would provide interesting classes of bi-univalent functions as follows:

(I) By putting $\gamma=1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$, which was considered by Çağlar et al. [6].

(II) By putting $\gamma=1$, $\lambda=1$ and $\mu=0$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}(\beta)$, which was studied by Brannan and Taha [5].

(III) By putting $\gamma=1$ and $\mu=1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{B}_{\Sigma}(\beta,\lambda)$, which was studied by Frasin, Aouf [10] and Hamidi, Jahangiri [13].

(IV) By putting $\gamma=1$, $\lambda=1$ and $\mu=1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{H}_{\Sigma}(\beta)$, which was studied by Srivastava et al. [18].

(V) By putting $\gamma=1$, $\lambda=1$ and $0\leqq\mu<1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class of bi-Bazilevic of order β and type μ, which was studied by Jahangiri and Hamidi [12].

Theorem 3.4. *Let f given by* (1.7) *be in the class* $\Sigma_m(\mu,\lambda,\gamma,\beta)$ $(\mu\geqq 0,\ \lambda\geqq 1,$ $\gamma\in\mathbb{C}-\{0\},\ 0\leqq\beta<1)$. *If* $a_{m+1}=\cdots=a_{m(k-1)+1}=0$, *then*

$$|a_{mk+1}|\leqq\frac{2|\gamma|(1-\beta)}{\mu+\lambda mk}\ ;\ (k\geqq 2).$$

Proof. By using Lemma 2.7, for m-fold symmetric bi-univalent functions f of the form (1.7), we have:

$$\begin{aligned}1+\frac{1}{\gamma}&\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right]\\&=1+\sum_{k=1}^{\infty}\frac{L_k(a_{m+1},\cdots,a_{mk+1})}{\gamma}z^{mk}.\end{aligned}\tag{3.1}$$

Similarly for its inverse map, $g(w)=f^{-1}(w)=w+\sum\limits_{k=1}^{\infty}A_{mk+1}w^{mk+1}$, we have:

$$\begin{aligned}1+\frac{1}{\gamma}&\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}-1\right]\\&=1+\sum_{k=1}^{\infty}\frac{L_k(A_{m+1},\cdots,A_{mk+1})}{\gamma}w^{mk}.\end{aligned}\tag{3.2}$$

Furthermore, since $f\in\Sigma_m(\mu,\lambda,\gamma,\beta)$, by definition, there exist two positive real-part functions

$$p(z)=1+\sum_{k=1}^{\infty}c_{mk}z^{mk}$$

and

$$q(w)=1+\sum_{k=1}^{\infty}d_{mk}w^{mk},$$

where $Re\ p(z) > 0$ and $Re\ q(w) > 0$ in $\mathbb{U}$ so that:

$$\begin{aligned}1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right] \\ = 1 + (1-\beta)\sum_{k=1}^{\infty} K_k^1(c_m, \cdots, c_{mk}) z^{mk}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} - 1\right] \\ = 1 + (1-\beta)\sum_{k=1}^{\infty} K_k^1(d_m, \cdots, d_{mk}) w^{mk}.\end{aligned} \tag{3.4}$$

Equating the corresponding coefficients of (3.1) and (3.3), we have:

$$\frac{L_k(a_{m+1}, \cdots, a_{mk+1})}{\gamma} = (1-\beta) K_k^1(c_m, \cdots, c_{mk}). \tag{3.5}$$

Similarly, from (3.2) and (3.4) we obtain:

$$\frac{L_k(A_{m+1}, \cdots, A_{mk+1})}{\gamma} = (1-\beta) K_k^1(d_m, \cdots, d_{mk}). \tag{3.6}$$

Note that for $a_{mi+1} = 0$ $(1 \leqq i \leqq k-1)$, we have $A_{mi+1} = 0$ $(1 \leqq i \leqq k-1)$ and $A_{mk+1} = -a_{mk+1}$.

For $\mu > 0$, by using Case 2.3 and Lemma 2.7 the equalities (3.5), (3.6) can be written as follows:

$$\frac{\mu + \lambda mk}{\gamma} a_{mk+1} = (1-\beta) c_{mk},$$

$$-\frac{\mu + \lambda mk}{\gamma} a_{mk+1} = (1-\beta) d_{mk}.$$

By getting the absolute values of either of the above two equations and applying the Lemma 2.8, we get:

$$|a_{mk+1}| = \frac{|\gamma|(1-\beta)|c_{mk}|}{\mu + \lambda mk} = \frac{|\gamma|(1-\beta)|d_{mk}|}{\mu + \lambda mk} \leqq \frac{2|\gamma|(1-\beta)}{\mu + \lambda mk}.$$

For $\mu = 0$, by using Case 2.6 and Lemma 2.7 the equalities (3.5), (3.6) can be written as follows:

$$\frac{\lambda mk}{\gamma} a_{mk+1} = (1-\beta) c_{mk},$$

$$-\frac{\lambda mk}{\gamma} a_{mk+1} = (1-\beta) d_{mk}.$$

By getting the absolute values of either of the above two equations and applying the Lemma 2.8, we get:

$$|a_{mk+1}| = \frac{|\gamma|(1-\beta)|c_{mk}|}{\lambda mk} = \frac{|\gamma|(1-\beta)|d_{mk}|}{\lambda mk} \leqq \frac{2|\gamma|(1-\beta)}{\lambda mk}. \qquad \square$$

We obtain estimates for the initial coefficients of functions $f \in \Sigma_m(\mu, \lambda, \gamma, \beta)$.

Theorem 3.5. *Let f given by* (1.7) *be in the class* $\Sigma_m(\mu,\lambda,\gamma,\beta)$. *Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2|\gamma|(1-\beta)}{\mu+\lambda m}, 2\sqrt{\frac{|\gamma|(1-\beta)}{(\mu+2\lambda m)(m+\mu)}}\right\}$$

and

$$|a_{2m+1}| \leqq \Big\{\frac{2|\gamma|^2(1-\beta)^2(m+1)}{(\mu+\lambda m)^2}+\frac{2|\gamma|(1-\beta)}{\mu+2\lambda m},$$
$$\frac{|\gamma|(1-\beta)}{\mu+2\lambda m}\left[\frac{2m+\mu+1+|1-\mu|}{m+\mu}\right]\Big\}.$$

Proof. For $f(z)=z+a_{m+1}z^{m+1}+a_{2m+1}z^{2m+1}+\cdots$, we get

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right] \tag{3.7}$$
$$=1+\frac{(\mu+\lambda m)}{\gamma}a_{m+1}z^{m+1}+\frac{(\mu+2\lambda m)}{\gamma}\left(a_{2m+1}+\frac{(\mu-1)}{2}\right)a_{m+1}z^{2m+1}+\cdots$$

and for

$$g(w)=f^{-1}(w)=w-a_{m+1}w^{m+1}+[(m+1)a_{m+1}^2-a_{2m+1}]w^{2m+1}+\cdots,$$

we get

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}-1\right]=1-\frac{(\mu+\lambda m)}{\gamma}a_{m+1}w^{m+1}$$
$$+\frac{(\mu+2\lambda m)}{\gamma}\left(-a_{2m+1}+\frac{(2m+\mu+1)}{2}a_{m+1}^2\right)w^{2m+1}+\cdots. \tag{3.8}$$

Comparing the corresponding coefficients of (3.3) and (3.7), we have

$$(\mu+\lambda m)a_{m+1}=\gamma(1-\beta)c_m, \tag{3.9}$$

$$(\mu+2\lambda m)\left(a_{2m+1}+\frac{\mu-1}{2}a_{m+1}^2\right)=\gamma(1-\beta)c_{2m}. \tag{3.10}$$

Similarly, by comparing the corresponding coefficients of (3.4) and (3.8), we have

$$-(\mu+\lambda m)a_{m+1}=\gamma(1-\beta)d_m, \tag{3.11}$$

$$(\mu+2\lambda m)\left(-a_{2m+1}+\frac{(2m+\mu+1)}{2}a_{m+1}^2\right)=\gamma(1-\beta)d_{2m}. \tag{3.12}$$

From (3.9) and (3.11), we get

$$c_m=-d_m \tag{3.13}$$

and

$$a_{m+1}^2=\frac{\gamma^2(1-\beta)^2(c_m^2+d_m^2)}{2(\mu+\lambda m)^2}. \tag{3.14}$$

Adding (3.10) and (3.12), we get

$$a_{m+1}^2=\frac{\gamma(1-\beta)(c_{2m}+d_{2m})}{(\mu+2\lambda m)(m+\mu)}. \tag{3.15}$$

Therefore, we find from the equations (3.14), (3.15) and Lemma 2.8 that

$$|a_{m+1}| \leqq \frac{2|\gamma|(1-\beta)}{\mu+\lambda m}$$

and

$$|a_{m+1}| \leqq 2\sqrt{\frac{|\gamma|(1-\beta)}{(\mu+2\lambda m)(m+\mu)}}.$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.
Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (3.12) from (3.10). We thus get

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{\gamma(1-\beta)(c_{2m}-d_{2m})}{2(\mu+2\lambda m)}. \tag{3.16}$$

Upon substituting the value of a_{m+1}^2 from (3.14) into (3.16), it follows that

$$a_{2m+1} = \frac{\gamma^2(1-\beta)^2(m+1)(c_m^2+d_m^2)}{4(\mu+\lambda m)^2} + \frac{\gamma(1-\beta)(c_{2m}-d_{2m})}{2(\mu+2\lambda m)}. \tag{3.17}$$

We thus find that

$$|a_{2m+1}| \leqq \frac{2|\gamma|^2(1-\beta)^2(m+1)}{(\mu+\lambda m)^2} + \frac{2|\gamma|(1-\beta)}{\mu+2\lambda m}.$$

On the other hand, upon substituting the value of a_{m+1}^2 from (3.15) into (3.16), it follows that

$$a_{2m+1} = \frac{\gamma(1-\beta)}{2(\mu+2\lambda m)}\left[\frac{(2m+\mu+1)c_{2m}+(1-\mu)d_{2m}}{m+\mu}\right]. \tag{3.18}$$

Consequently, we have

$$|a_{2m+1}| \leqq \frac{|\gamma|(1-\beta)}{\mu+2\lambda m}\left[\frac{2m+\mu+1+|1-\mu|}{m+\mu}\right].$$

This evidently completes the proof of Theorem 3.5. □

4. Corollaries and consequences

By setting $\gamma = 1$ in Theorem 3.4, we conclude the following result.

Corollary 4.1. *Let f given by* (1.7) *be in the class $\mathcal{N}_{\Sigma_m}^{\mu}(\beta,\lambda)$.
If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{\mu+\lambda mk}\ ;\ (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.1, we conclude the following result.

Corollary 4.2. *Let f given by* (1.1) *be in the class $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{\mu+\lambda k}\ ;\ (k \geqq 2).$$

By setting $\lambda = 1$ and $\mu = 0$ in Corollary 4.1, we conclude the following result.

Corollary 4.3. *Let f given by (1.7) be m-fold symmetric bi-starlike of order β. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{mk} \ ; \ (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.3, we conclude the following result.

Corollary 4.4. *Let f given by (1.1) be in the class $\mathcal{S}^*_\Sigma(\beta)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{k} \ ; \ (k \geqq 2).$$

By setting $\mu = 1$ in Corollary 4.1, we conclude the following result.

Corollary 4.5. *Let f given by (1.7) be in the class $\mathcal{A}^\lambda_{\Sigma,m}(\beta)$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{1+\lambda mk} \ ; \ (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.5, we conclude the following result.

Corollary 4.6. *Let f given by (1.1) be in the class $\mathcal{B}_\Sigma(\beta, \lambda)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{1+\lambda k} \ ; \ (k \geqq 2).$$

By setting $\lambda = 1$ in Corollary 4.5, we conclude the following result.

Corollary 4.7. *Let f given by (1.7) be in the class $\mathcal{H}_{\Sigma,m}(\beta)$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{1+mk} \ ; \ (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.7, we conclude the following result.

Corollary 4.8. *Let f given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{1+k} \ ; \ (k \geqq 2).$$

By setting $m = 1$, $\lambda = 1$ and $0 \leqq \mu < 1$ in Corollary 4.1, we conclude the following result.

Corollary 4.9. *Let f given by (1.1) be bi-Bazilevic of order β and type μ. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{\mu+k} \ ; \ (k \geqq 2).$$

By setting $\gamma = 1$ in Theorem 3.5, we conclude the following result.

Corollary 4.10. *Let f given by (1.7) be in the class $\mathcal{N}^{\mu}_{\Sigma_m}(\beta,\lambda)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)}{\mu+\lambda m}, 2\sqrt{\frac{1-\beta}{(\mu+2\lambda m)(m+\mu)}}\right\}$$

and

$$|a_{2m+1}| \leqq \left\{\frac{2(1-\beta)^2(m+1)}{(\mu+\lambda m)^2}+\frac{2(1-\beta)}{\mu+2\lambda m}, \frac{(1-\beta)}{\mu+2\lambda m}\left[\frac{2m+\mu+1+|1-\mu|}{m+\mu}\right]\right\}.$$

By setting $m=1$ in Corollary 4.10, we conclude the following result.

Corollary 4.11. *Let f given by (1.1) be in the class $\mathcal{N}^{\mu}_{\Sigma}(\beta,\lambda)$. Then*

$$|a_2| \leqq \min\left\{\frac{2(1-\beta)}{\mu+\lambda}, 2\sqrt{\frac{1-\beta}{(\mu+2\lambda)(1+\mu)}}\right\}$$

and

$$|a_3| \leqq \left\{\frac{4(1-\beta)^2}{(\mu+\lambda)^2}+\frac{2(1-\beta)}{\mu+2\lambda}, \frac{(1-\beta)}{\mu+2\lambda}\left[\frac{3+\mu+|1-\mu|}{1+\mu}\right]\right\}.$$

By setting $\lambda=1$ and $\mu=0$ in Corollary 4.10, we conclude the following result.

Corollary 4.12. *Let f given by (1.7) be m-fold symmetric bi-starlike of order β. Then*

$$|a_{m+1}| \leqq \begin{cases} \frac{1}{m}\sqrt{2(1-\beta)}\ ;\ 0\leqq\beta\leqq\frac{1}{2} \\ \\ \frac{2(1-\beta)}{m}\ ;\ \frac{1}{2}\leqq\beta<1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \begin{cases} \frac{(1-\beta)(m+1)}{m^2}\ ;\ 0\leqq\beta\leqq\frac{2m+1}{2(m+1)} \\ \\ \frac{2(1-\beta)^2(m+1)}{m^2}+\frac{1-\beta}{m}\ ;\ \frac{2m+1}{2(m+1)}\leqq\beta<1. \end{cases}$$

By setting $m=1$ in Corollary 4.12, we conclude the following result.

Corollary 4.13. *Let f given by (1.1) be in the class $\mathcal{S}^*_{\Sigma}(\beta)$, then*

$$|a_2| \leqq \begin{cases} \sqrt{2(1-\beta)}\ ;\ 0\leqq\beta\leqq\frac{1}{2} \\ \\ 2(1-\beta)\ ;\ \frac{1}{2}\leqq\beta<1 \end{cases}$$

and

$$|a_3| \leqq \begin{cases} 2(1-\beta)\ ;\ 0\leqq\beta\leqq\frac{3}{4} \\ \\ (1-\beta)(5-4\beta)\ ;\ \frac{3}{4}\leqq\beta<1. \end{cases}$$

By setting $\mu=1$ in Corollary 4.10, we conclude the following result.

Corollary 4.14. *Let f given by (1.7) be in the class $\mathcal{A}^{\lambda}_{\Sigma,m}(\beta)$, then*

$$|a_{m+1}| \leqq \begin{cases} 2\sqrt{\frac{1-\beta}{(1+2\lambda m)(1+m)}}\ ;\ 0\leqq\beta\leqq 1-\frac{(1+\lambda m)^2}{(1+2\lambda m)(1+m)} \\ \\ \frac{2(1-\beta)}{1+\lambda m}\ ;\ 1-\frac{(1+\lambda m)^2}{(1+2\lambda m)(1+m)}\leqq\beta<1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \min\left\{\frac{2(1-\beta)^2(m+1)}{(1+\lambda m)^2}+\frac{2(1-\beta)}{1+2\lambda m}, \frac{2(1-\beta)}{1+2\lambda m}\right\} = \frac{2(1-\beta)}{1+2\lambda m}.$$

Remark 4.15. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.14 are better than those given by Eker [8, Theorem 2].

By setting $m = 1$ in Corollary 4.14, we conclude the following result.

Corollary 4.16. *Let f given by* (1.1) *be in the class $\mathcal{B}_\Sigma(\beta,\lambda)$, then*

$$|a_2| \leqq \begin{cases} \sqrt{\frac{2(1-\beta)}{1+2\lambda}} \ ; \ 0 \leqq \beta \leqq \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \\ \frac{2(1-\beta)}{1+\lambda} \ ; \ \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \min\left\{\frac{4(1-\beta)^2}{(1+\lambda)^2}+\frac{2(1-\beta)}{1+2\lambda}, \frac{2(1-\beta)}{1+2\lambda}\right\} = \frac{2(1-\beta)}{1+2\lambda}.$$

Remark 4.17. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 4.16 are better than those given by Frasin and Aouf [10, Theorem 3.2].

By setting $\lambda = 1$ in Corollary 4.14, we conclude the following result.

Corollary 4.18. *Let f given by* (1.7) *be in the class $\mathcal{H}_{\Sigma,m}(\beta)$, then*

$$|a_{m+1}| \leqq \begin{cases} 2\sqrt{\frac{1-\beta}{(1+2m)(1+m)}} \ ; \ 0 \leqq \beta \leqq \frac{m}{2m+1} \\ \\ \frac{2(1-\beta)}{1+m} \ ; \ \frac{m}{2m+1} \leqq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \min\left\{\frac{2(1-\beta)^2}{1+m}+\frac{2(1-\beta)}{1+2m}, \frac{2(1-\beta)}{1+2m}\right\} = \frac{2(1-\beta)}{1+2m}.$$

Remark 4.19. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.18 are better than those given by Srivastava et al. [19, Theorem 3].

By setting $m = 1$ in Corollary 4.18, we conclude the following result.

Corollary 4.20. *Let f given by* (1.1) *be in the class $\mathcal{H}_\Sigma(\beta)$, then*

$$|a_2| \leqq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}} \ ; \ 0 \leqq \beta \leqq \frac{1}{3} \\ \\ 1-\beta \ ; \ \frac{1}{3} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \min\left\{\frac{(1-\beta)(5-3\beta)}{3}, \frac{2(1-\beta)}{3}\right\} = \frac{2(1-\beta)}{3}.$$

Remark 4.21. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 4.20 are better than those given by Srivastava et al. [18, Theorem 2].

By setting $\lambda = 1$ and $0 \leqq \mu < 1$ in Corollary 4.11, we conclude the following result.

Corollary 4.22. *Let f given by* (1.1) *be bi-Bazilevic of order β and type μ. Then*

$$|a_2| \leqq \begin{cases} 2\sqrt{\frac{1-\beta}{(\mu+2)(1+\mu)}} \ ; \ 0 \leqq \beta \leqq \frac{1}{2+\mu} \\ \\ \frac{2(1-\beta)}{\mu+1} \ ; \ \frac{1}{2+\mu} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \begin{cases} \frac{4(1-\beta)}{(\mu+2)(1+\mu)} \ ; \ 0 \leqq \beta \leqq \frac{1}{2+\mu} \\ \\ \frac{4(1-\beta)^2}{(\mu+1)^2} + \frac{2(1-\beta)}{\mu+2} \ ; \ \frac{1}{2+\mu} \leqq \beta < 1. \end{cases}$$

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