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Geometric properties of mixed operator involving Ruscheweyh derivative and Sălăgean operator

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Abstract. Operator theory is a magnificent tool for studying the geometric behaviors of holomorphic functions in the open unit disk. Recently, a combination between two well known differential operators, Ruscheweyh derivative and Sălăgean operator are suggested by Lupas in [10]. In this effort, we shall follow the same principle, to formulate a generalized differential-difference operator. We deliver a new class of analytic functions containing the generalized operator. Applications are illustrated in the sequel concerning some differential subordinations of the operator.

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1. Introduction

Differential operators in a complex domain play a significant role in functions theory and its information. They have used to describe the geometric interpolation of analytic functions in a complex domain. Also, they have utilized to generate new formulas of holomorphic functions. Lately, Lupas [10] presented a amalgamation of two well-known differential operators prearranged by Ruscheweyh [12] and Sălăgean [13]. Later, these operators are investigated by researchers considering different classes and formulas of analytic functions [5, 8].

In this note, we consider a special class of functions in the open unit disk

$$\sqcup = \{\xi \in \mathbb{C} |\, |\xi| < 1\}$$

denoting by Σ and having the series

$$\varphi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n, \quad \xi \in \sqcup.$$

Let $\varphi \in \Sigma$, then the Ruscheweyh formula is indicated by the structure formula

$$\Phi^m \varphi(\xi) = \xi + \sum_{n=2}^{\infty} C^m_{m+n-1} \varphi_n \xi^n.$$

While, the Sălăgean operator admits the construction

$$\Psi^m \varphi(\xi) = \xi + \sum_{n=2}^{\infty} n^m \varphi_n \xi^n.$$

Lupas operator is formulated by the structure

$$\lambda^m_\sigma \varphi(\xi) = \xi + \sum_{n=2}^{\infty} [\sigma n^m + (1-\sigma) C^m_{m+n-1}]\varphi_n \xi^n, \quad \xi \in \sqcup, \sigma \in [0,1].$$

Newly, Ibrahim and Darus [7] considered the next differential operator

$$\begin{aligned}
&\Theta^0_\kappa \varphi(\xi) = \varphi(\xi)\\
&\Theta^1_\kappa \varphi(\xi) = \xi\, \varphi(\xi)' + \frac{\kappa}{2}\left(\varphi(\xi) - \varphi(-\xi) - 2\xi\right), \quad \kappa \in \mathbb{R}\\
&\vdots\\
&\Theta^m_\kappa \varphi(\xi) = \Theta_\kappa(\Theta^{m-1}_\kappa \varphi(\xi))\\
&\qquad\qquad = \xi + \sum_{n=2}^{\infty} [n + \frac{\kappa}{2}(1+(-1)^{n+1})]^m \varphi_n \xi^n.
\end{aligned}$$

When $\kappa = 0$,we have $\Psi^m \varphi(\xi)$ In addition, it is a modified formula of the well-known Dunkl operator [2], where κ is known as the Dunkl order. Proceeding, we define a generalized formula of λ^m_σ, as follows:

$$\begin{aligned}
J^m_{\sigma,\kappa}\varphi(\xi) &= (1-\sigma)\Phi^m \varphi(\xi) + \sigma \Theta^m_\kappa \varphi(\xi)\\
&= \xi + \sum_{n=2}^{\infty} [(1-\sigma) C^m_{m+n-1} + \sigma\big(n + \frac{\kappa}{2}(1+(-1)^{n+1})\big)^m]\, \varphi_n \xi^n.
\end{aligned} \tag{1.1}$$

Clearly, the operator $J^m_{\sigma,\kappa}\varphi(\xi) \in \Sigma$.

Remark 1.1.

- $m = 0 \Longrightarrow J^0_{\sigma,\kappa}\varphi(\xi) = \varphi(\xi)$;
- $\kappa = 0 \Longrightarrow J^m_{\sigma,0}\varphi(\xi) = \lambda^m_\sigma \varphi(\xi)$;
- $\sigma = 0 \Longrightarrow J^m_{0,\kappa}\varphi(\xi) = \Phi^m \varphi(\xi)$;
- $\sigma = 1 \Longrightarrow J^m_{1,\kappa}\varphi(\xi) = \Theta^m_\kappa \varphi(\xi)$;
- $\kappa = 0, \sigma = 1 \Longrightarrow J^m_{1,0}\varphi(\xi) = \Psi^m \varphi(\xi)$.

Definition 1.2. Consider the following data $\epsilon \in [0,1), \sigma \in [0,1], \kappa \geq 0$, and $m \in \mathbb{N}$. Then a function $\varphi \in \Sigma$ belongs to the set $\top_m(\sigma,\kappa,\epsilon)$ if and only if

$$\Re\Big((J^m_{\sigma,\kappa}\varphi(\xi))'\Big) > \epsilon, \quad \xi \in \sqcup.$$

Observe that the set $\top_m(\sigma,\kappa,\epsilon)$ is an extension of the well known class of bounded turning functions (see [1]-[14]). Next results are requested to prove our results depending on the subordination concept (see [11]).

Lemma 1.3. *Suppose that $\hbar$ is convex function such that $\hbar(0) = \flat$, and there is a complex number with a positive real part μ. If $\flat \in \mathfrak{H}[\flat, n]$, where*

$$\mathfrak{H}[\flat, n] = \{\flat \in \mathfrak{H} : \flat(\xi) = \flat + \flat_n \xi^n + \flat_{n+1}\xi^{n+1} + ...\}$$

(the space of holomorphic functions) and

$$\flat(\xi) + \frac{1}{\mu}\xi\flat'(\xi) \prec \hbar(\xi), \quad \xi \in \sqcup,$$

then

$$\flat(\xi) \prec \iota(\xi) \prec \hbar(\xi),$$

with

$$\iota(\xi) = \frac{\mu}{n\,\xi^{\mu/n}} \int_0^\xi \hbar(\tau)\tau^{\overline{(n-1)}^{\mu}}\, d\tau, \quad \xi \in \sqcup.$$

Lemma 1.4. *Suppose that the convex function $\flat(\xi)$ satisfies the functional*

$$\hbar(\xi) = \flat(\xi) + n\mu(\xi\,\flat'(\xi))$$

for $\mu > 0$ and n is a positive integer. If $\flat \in \mathfrak{H}[\hbar(0), n]$, and $\flat(\xi) + \mu\xi\flat'(\xi) \prec \hbar(\xi)$, $\xi \in \sqcup$ then $\flat(\xi) \prec \hbar(\xi)$, and this outcome is sharp.

Lemma 1.5. *(i) If $\lambda > 0, \gamma > 0$, $\beta = \beta(\gamma, \lambda, n)$ and $\flat \in \mathfrak{H}[1, n]$ then*

$$\flat(\xi) + \lambda\xi\flat'(\xi) \prec \left[\frac{1+\xi}{1-\xi}\right]^\beta \Rightarrow \flat(\xi) \prec \left[\frac{1+\xi}{1-\xi}\right]^\gamma.$$

(ii) If $\epsilon \in [0, 1)$, $\lambda = \lambda(\epsilon, n)$ and $\flat \in \mathfrak{H}[1, n]$ then

$$\Re\Big(\flat^2(\xi) + 2\flat(\xi).\xi\flat'(\xi)\Big) > \epsilon \Rightarrow \Re(\flat(\xi)) > \lambda.$$

2. Results

In this section, we investigate some geometric conducts of the operator (1.1).

Theorem 2.1. *The set $\top_m(\sigma,\kappa,\epsilon)$ is convex.*

Proof. Suppose that $\varphi_i, i = 1, 2$ are two functions belonging to $\top_m(\sigma,\kappa,\epsilon)$ satisfying

$$\varphi_1(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n$$

and

$$\varphi_2(\xi) = \xi + \sum_{n=2}^{\infty} \phi_n \xi^n.$$

It is sufficient to prove that the function

$$\Pi(x_1) = \wp_1\varphi_1(\xi) + \wp_2\varphi_2(\xi), \quad \xi \in \sqcup$$

is in $\top_m(\sigma,\kappa,\epsilon)$, where $\wp_1 > 0, \wp_2 > 0$ and $\wp_1 + \wp_2 = 1$. The formula of $\Pi(z)$ yields

$$\Pi(\xi) = \xi + \sum_{n=2}^{\infty}(\wp_1\varphi_n + \wp_2\phi_n)\xi^n.$$

Thus, under the operator (1.1), we get

$$J^m_{\sigma,\kappa}\Pi(\xi) = \xi + \sum_{n=2}^{\infty}(\wp_1\varphi_n + \wp_2\phi_n)[(1-\sigma)C^m_{m+n-1} + \sigma\big(n + \frac{\kappa}{2}(1+(-1)^{n+1})\big)^m]\xi^n.$$

By making a differentiation, we obtain

$$\begin{aligned}&\Re\{(J^m_{\alpha,\kappa}\Pi(\xi))'\}\\ &= 1 + \wp_1\Re\left\{\sum_{n=2}^{\infty} n[(1-\sigma)C^m_{m+n-1} + \sigma\left(n + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m]\varphi_n\xi^{n-1}\right\}\\ &+\wp_2\Re\left\{\sum_{n=2}^{\infty} n[(1-\sigma)C^m_{m+n-1} + \sigma\left(n + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m]\phi_n\,\xi^{n-1}\right\} = \epsilon. \qquad \square\end{aligned}$$

Theorem 2.2. *Define the following functions: $\varphi \in \top_m(\sigma,\kappa,\epsilon)$, ϕ be convex and*

$$F(\xi) = \frac{2+c}{\xi^{1+c}}\int_0^{\xi} t^c\,\varphi(t)dt, \quad \xi \in \sqcup.$$

Then

$$\left(J^m_{\alpha,\kappa}\varphi(\xi)\right)' \prec \phi(\xi) + \frac{(\xi\,\phi'(\xi))}{2+c}, \quad c > 0,$$

yields

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)' \prec \phi(\xi),$$

and this outcome is sharp.

Proof. By the assumptions, we have

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)' + \frac{\left(J^m_{\sigma,\kappa}F(\xi)\right)''}{2+c} = \left(J^m_{\sigma,\kappa}\varphi(\xi)\right)'.$$

Consequently, we get

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)' + \frac{\left(J^m_{\sigma,\kappa}F(\xi)\right)''}{2+c} \prec \phi(\xi) + \frac{(\xi\phi'(\xi))}{2+c}.$$

Assuming

$$\flat(\xi) := \left(J^m_{\sigma,\kappa}F(\xi)\right)',$$

one can find

$$\flat(\xi) + \frac{(\xi\flat'(\xi))}{2+c} \prec \phi(\xi) + \frac{(\xi\phi'(\xi))}{2+c}.$$

In virtue of Lemma 1.3, we have

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)' \prec \phi(\xi),$$

and ϕ is the best dominant. $\square$

Theorem 2.3. *Assume the convex function ϕ achieving $\phi(0) = 1$ and for $\varphi \in \Sigma$*

$$\left(J^m_{\sigma,\kappa}\varphi(\xi)\right)' \prec \phi(\xi) + \xi\,\phi'(\xi), \quad \xi \in \sqcup,$$

then

$$\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \prec \phi(\xi),$$

and this outcome is sharp.

Proof. Formulate the next functional

$$\flat(z) := \frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \in \mathfrak{H}[1,1] \tag{2.1}$$

Consequently, we get

$$J^m_{\sigma,\kappa}\varphi(\xi) = \xi\,\flat(\xi) \Longrightarrow \left(J^m_{\sigma,\kappa}\varphi(\xi)\right)' = \flat(\xi) + \xi\flat'(\xi).$$

Therefore, we obtain the inequality

$$\flat(\xi) + \xi\flat'(\xi) \prec \phi(\xi) + \xi\phi'(\xi).$$

According to Lemma 1.4, we attain

$$\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \prec \phi(\xi),$$

and ϕ is the best dominant. $\square$

Theorem 2.4. *For $\varphi \in \Sigma$ if the inequality*

$$(J^m_{\sigma,\kappa}\varphi(\xi))' \prec \left(\frac{1+\xi}{1-\xi}\right)^\beta, \quad \xi \in \sqcup,\ \beta > 0,$$

achieves then

$$\Re\Big(\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi}\Big) > \epsilon$$

for some $\epsilon \in [0,1)$.

Proof. For the function $\flat(\xi)$ in (2.1), we have

$$(J^m_{\sigma,\kappa}\varphi(\xi))' = \xi\flat'(\xi) + \flat(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^\beta.$$

According to Lemma 1.5.i, there occurs a constant $\gamma > 0$ with $\beta = \beta(\gamma)$ with

$$\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^\gamma.$$

This yields $\Re(J^m_{\sigma,\kappa}\varphi(\xi)/\xi) > \epsilon$, for some $\epsilon \in [0,1)$. $\square$

Theorem 2.5. *Assume that $\varphi \in \Sigma$ achieves the inequality*

$$\Re\Big((J^m_{\sigma,\kappa}\varphi(\xi))'\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi}\Big) > \frac{\sigma}{2}, \quad \xi \in \sqcup, \sigma \in [0,1).$$

Then $J^m_{\sigma,\kappa}\varphi(\xi) \in \top_m(\sigma,\kappa,\epsilon)$ for some $\epsilon \in [0,1)$. In addition, it is univalent of bounded turning in $\sqcup$.

Proof. Assume the function $\flat(\xi)$ as in (2.1). A Calculation implies that

$$\Re\Big(\flat^2(\xi) + 2\flat(\xi).\xi\flat'(\xi)\Big) = 2\Re\Big((J^m_{\sigma,\kappa}\varphi(\xi))'\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi}\Big) \ > \ \sigma. \tag{2.2}$$

Lemma 1.5.ii, implies that there occurs a constant $\lambda(\sigma)$ satisfying $\Re(\flat(\xi)) > \lambda(\sigma)$. Thus, we obtain $\Re(\flat(\xi)\ ) > \epsilon$ for some $\epsilon \in [0,1)$. It yields from (2.2) that $\Re\Big(J^m_{\sigma,\kappa}\varphi(\xi))'\Big) > \epsilon$ and by Noshiro-Warschawski and Kaplan Theorems (see [3]), we have that $J^m_{\sigma,\kappa}\varphi(\xi)$ is univalent and of bounded turning in $\sqcup$. □

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