

Classes of an univalent integral operator

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Abstract. In this paper we introduce a new general integral operator for analytic functions in the open unit disk \mathbf{U} and we obtain sufficient conditions for univalence of this integral operator.

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1. Introduction

Let \mathcal{A} be the class of the functions f which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by \mathcal{S} the subclass of \mathcal{A} consisting of functions $f \in \mathcal{A}$, which are univalent in \mathbf{U} .

We consider the integral operator

$$\mathcal{T}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i-1} \cdot (g_i(t))^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \right] dt \right\}^{\frac{1}{\delta}} \quad (1.1)$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$ and the complex numbers $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i$, with $\delta \neq 0$, $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$.

Remark 1.1. The integral operator \mathcal{T}_n defined by (1.1), is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [7].

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt,$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18].

$$\mathcal{G}_\alpha(z) = \int_0^z (f'(t))^\alpha dt,$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [2].

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [4]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n [f'_i(t)]^{\alpha_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [17].

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by Frasin [5]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\alpha_i} (f'_i(t))^{\beta_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Ovesea in [9].

vi) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by Pescar [13].

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)} \right)^{\gamma_i} \left(\frac{f'_i(t)}{g'_i(t)} \right)^{\delta_i} dt \right]^{\frac{1}{\delta}},$$

Thus, the integral operator \mathcal{T}_n , introduced here by the formula (1.1), can be considered as an extension and a generalization of these operators above mentioned.

We need the following lemmas.

Lemma 1.2. [11] *Let γ, δ be complex numbers, $Re\gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then for any complex number δ , $Re\delta \geq Re\gamma$, the function F_δ defined by

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t) dt \right)^{\frac{1}{\delta}},$$

is regular and univalent in \mathbb{U} .

Lemma 1.3. [14] Let δ be complex number, $\text{Re}\delta > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \dots$. If

$$\left| c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zf''(z)}{\delta f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then the function F_δ defined by

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t) dt \right)^{\frac{1}{\delta}},$$

is regular and univalent in \mathbb{U} .

Lemma 1.4. [8] Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then

$$|f(z)| \leq \frac{M}{R^m} z^m,$$

the equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2. Main results

Theorem 2.1. Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = \text{Re}\gamma > 0$, $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$,

$$\begin{aligned} f_i(z) &= z + a_{2i}z^2 + a_{3i}z^3 + \dots, \\ g_i(z) &= z + b_{2i}z^2 + b_{3i}z^3 + \dots, \\ h_i(z) &= z + c_{2i}z^2 + c_{3i}z^3 + \dots, \\ k_i(z) &= z + d_{2i}z^2 + d_{3i}z^3 + \dots, \quad i = \overline{1, n}. \end{aligned}$$

If

$$\begin{aligned} \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| &\leq M_i, & \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| &\leq N_i, & \left| \frac{zh'_i(z)}{h_i(z)} - 1 \right| &\leq P_i, \\ \left| \frac{zk'_i(z)}{k_i(z)} - 1 \right| &\leq Q_i, & \left| \frac{zh''_i(z)}{h'_i(z)} \right| &\leq R_i, & \left| \frac{zk''_i(z)}{k'_i(z)} \right| &\leq S_i, \end{aligned}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n [|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)] \leq \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2}, \quad (2.1)$$

then, for all δ complex numbers, $\text{Re}\delta \geq \text{Re}\gamma$, the integral operator \mathcal{T}_n , given by (1.1) is in the class \mathcal{S} .

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i-1} \cdot (g_i(t)')^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \right] dt,$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$, $i = \overline{1, n}$.

The function H_n is regular in \mathbb{U} and satisfy the following usual normalization conditions $H_n(0) = H_n'(0) - 1 = 0$.

Now

$$H_n'(z) = \prod_{i=1}^n \left[\left(\frac{f_i(z)}{z} \right)^{\alpha_i-1} \cdot (g_i(z)')^{\beta_i} \cdot \left(\frac{h_i(z)}{k_i(z)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(z)}{k_i'(z)} \right)^{\delta_i} \right].$$

We have

$$\begin{aligned} \frac{zH_n''(z)}{H_n'(z)} &= \sum_{i=1}^n \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right] \\ &+ \sum_{i=1}^n \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right], \end{aligned}$$

for all $z \in \mathbb{U}$.

Thus, we have

$$\begin{aligned} \frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| &= \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right] \\ &+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right], \end{aligned}$$

for all $z \in \mathbb{U}$.

Therefore

$$\begin{aligned} \frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| &\leq \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \right] \\ &+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\gamma_i| \left(\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| + \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \right) \right] \\ &+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right], \end{aligned}$$

for all $z \in \mathbb{U}$.

By applying the General Schwarz Lemma (1.4) we obtain

$$\begin{aligned} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| &\leq M_i |z|, & \left| \frac{zg_i''(z)}{g_i'(z)} \right| &\leq N_i |z|, & \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| &\leq P_i |z|, \\ \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| &\leq Q_i |z|, & \left| \frac{zh_i''(z)}{h_i'(z)} \right| &\leq R_i |z|, & \left| \frac{zk_i''(z)}{k_i'(z)} \right| &\leq S_i |z|, \end{aligned}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$.

Using these inequalities we have

$$\begin{aligned} & \frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \\ \leq & \frac{1 - |z|^{2c}}{c} |z| \sum_{i=1}^n [|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)], \end{aligned} \tag{2.2}$$

for all $z \in \mathbb{U}$.

Since

$$\max_{|z| \leq 1} \frac{(1 - |z|^{2c}) |z|}{c} = \frac{2}{(2c + 1)^{\frac{2c+1}{2c}}},$$

from (2.2), we obtain

$$\begin{aligned} & \frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \\ \leq & \frac{2}{(2c + 1)^{\frac{2c+1}{2c}}} \sum_{i=1}^n [|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)], \end{aligned}$$

and hence, by (2.1) we have

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{2}{(2c + 1)^{\frac{2c+1}{2c}}} \cdot \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2} = 1,$$

for all $z \in \mathbb{U}$.

So,

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq 1. \tag{2.3}$$

and using (2.3), by Lemma 1.2, it results that the integral operator \mathcal{T}_n , given by (1.1) is in the class \mathcal{S} . \square

If we consider $\delta = 1$ in Theorem 2.1, obtain the next corollary:

Corollary 2.2. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < \text{Re}\gamma \leq 1$, $c = \text{Re}\gamma$, $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\begin{aligned} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| &\leq M_i, & \left| \frac{zg_i''(z)}{g_i'(z)} \right| &\leq N_i, & \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| &\leq P_i, \\ \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| &\leq Q_i, & \left| \frac{zh_i''(z)}{h_i'(z)} \right| &\leq R_i, & \left| \frac{zk_i''(z)}{k_i'(z)} \right| &\leq S_i, \end{aligned}$$

$$\sum_{i=1}^n [|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)] \leq \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator \mathcal{F}_n defined by

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot (g_i(t)')^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \right] dt, \tag{2.4}$$

is in the class \mathcal{S} .

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = \dots = \delta_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.3. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$, $c = \operatorname{Re}\gamma$, M_i, N_i, P_i, Q_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zg''_i(z)}{g'_i(z)} \right| \leq N_i,$$

$$\left| \frac{zh'_i(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{zk'_i(z)}{k_i(z)} - 1 \right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n [|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i)] \leq \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator \mathcal{S}_n defined by

$$\mathcal{S}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot (g_i(t)')^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \right] dt, \tag{2.5}$$

is in the class \mathcal{S} .

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = \dots = \beta_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.4. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$, $c = \operatorname{Re}\gamma$, M_i, P_i, Q_i, R_i, S_i real positive numbers and $f_i, h_i, k_i \in \mathcal{A}$. If*

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zh'_i(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{zk'_i(z)}{k_i(z)} - 1 \right| \leq Q_i,$$

$$\left| \frac{zh''_i(z)}{h'_i(z)} \right| \leq R_i, \quad \left| \frac{zk''_i(z)}{k'_i(z)} \right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n [|\alpha_i - 1| M_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)] \leq \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator \mathcal{X}_n defined by

$$\mathcal{X}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h'_i(t)}{k'_i(t)} \right)^{\delta_i} \right] dt, \tag{2.6}$$

is in the class \mathcal{S} .

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.5. Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$, $c = \operatorname{Re}\gamma$, N_i, P_i, Q_i, R_i, S_i real positive numbers and $g_i, h_i, k_i \in \mathcal{A}$. If

$$\begin{aligned} \left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq N_i, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \leq Q_i, \\ \left| \frac{zh_i''(z)}{h_i'(z)} \right| \leq R_i, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \leq S_i, \end{aligned}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n [|\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)] \leq \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator \mathcal{D}_n defined by

$$\mathcal{D}_n(z) = \int_0^z \prod_{i=1}^n \left[(g_i(t)')^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \right] dt, \tag{2.7}$$

is in the class \mathcal{S} .

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = \dots = \gamma_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.6. Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$, $c = \operatorname{Re}\gamma$, M_i, N_i, R_i, S_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If

$$\begin{aligned} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq N_i, \\ \left| \frac{zh_i''(z)}{h_i'(z)} \right| \leq R_i, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \leq S_i, \end{aligned}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n [|\alpha_i - 1| M_i + |\beta_i| N_i + |\delta_i| (R_i + S_i)] \leq \frac{(2c + 1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator \mathcal{Y}_n defined by

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot (g_i(t)')^{\beta_i} \cdot \left(\frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \right] dt, \tag{2.8}$$

is in the class \mathcal{S} .

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.1, obtain the next corollary:

Corollary 2.7. Let α be complex number, $\operatorname{Re}\alpha > 0$, M, N, P, Q, R, S real positive numbers and $f, g, h, k \in \mathcal{A}$. If

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq M, \quad \left| \frac{zg''(z)}{g(z)'} \right| \leq N, \quad \left| \frac{zh'(z)}{h(z)} - 1 \right| \leq P, \\ \left| \frac{zk'(z)}{k(z)} - 1 \right| \leq Q, \quad \left| \frac{zh''(z)}{h'(z)} \right| \leq R, \quad \left| \frac{zk''(z)}{k'(z)} \right| \leq S, \end{aligned}$$

for all $z \in \mathbb{U}$, and

$$|\alpha - 1|(M + N + P + Q + R + S) \leq \frac{(2\operatorname{Re}\alpha + 1)^{\frac{2\operatorname{Re}\alpha + 1}{2\operatorname{Re}\alpha}}}{2},$$

then the integral operator \mathcal{T} defined by

$$\mathcal{T}(z) = \left[\alpha \int_0^z \left(f(t) \cdot g'(t) \cdot \frac{h(t)}{k(t)} \cdot \frac{h'(t)}{k'(t)} \right)^{\alpha-1} dt \right]^{\frac{1}{\alpha}}, \tag{2.9}$$

is in the class \mathcal{S} .

Theorem 2.8. Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = \operatorname{Re}\gamma > 0$ and $f_i, h_i, k_i \in \mathcal{S}$, $g_i', h_i', k_i' \in \mathcal{P}$. If

$$4 \sum_{i=1}^n |\alpha_i - 1| + 2 \sum_{i=1}^n |\beta_i| + 8 \sum_{i=1}^n |\gamma_i| + 4 \sum_{i=1}^n |\delta_i| \leq \frac{c}{2}, \quad \text{for } 0 < c < 1 \tag{2.10}$$

or

$$4 \sum_{i=1}^n |\alpha_i - 1| + 2 \sum_{i=1}^n |\beta_i| + 8 \sum_{i=1}^n |\gamma_i| + 4 \sum_{i=1}^n |\delta_i| \leq \frac{1}{2}, \quad \text{for } c \geq 1 \tag{2.11}$$

then, for any complex numbers δ , $\operatorname{Re}\delta \geq c$, the integral operator \mathcal{T}_n defined in (1.1) is in the class \mathcal{S} .

Proof. After the same steps as in the proof of Theorem 2.1., we get

$$\begin{aligned} \frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| &\leq \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\alpha_i - 1| \left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \right] \\ &+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\gamma_i| \left(\left| \frac{zh_i'(z)}{h_i(z)} \right| + 1 + \left| \frac{zk_i'(z)}{k_i(z)} \right| + 1 \right) \right] \\ &+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right], \end{aligned}$$

for all $z \in \mathbb{U}$.

Since $f_i, h_i, k_i \in \mathcal{S}$ we have

$$\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq \frac{1 + |z|}{1 - |z|}, \quad \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq \frac{1 + |z|}{1 - |z|}, \quad \left| \frac{zk_i'(z)}{k_i(z)} \right| \leq \frac{1 + |z|}{1 - |z|},$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$.

For $g_i', h_i', k_i' \in \mathcal{P}$ we have

$$\left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq \frac{2|z|}{1 - |z|^2}, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \leq \frac{2|z|}{1 - |z|^2}, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \leq \frac{2|z|}{1 - |z|^2},$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$.

Using these relations we get

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{1 - |z|^{2c}}{c} \left(\frac{1 + |z|}{1 - |z|} + 1 \right) \sum_{i=1}^n |\alpha_i - 1|$$

$$\begin{aligned}
 & + \frac{1 - |z|^{2c}}{c} \cdot \frac{2|z|}{1 - |z|^2} \sum_{i=1}^n |\beta_i| + \frac{1 - |z|^{2c}}{c} \left(\frac{1 + |z|}{1 - |z|} + 1 + \frac{1 + |z|}{1 - |z|} + 1 \right) \sum_{i=1}^n |\gamma_i| \\
 & \quad + \frac{1 - |z|^{2c}}{c} \left(\frac{2|z|}{1 - |z|^2} + \frac{2|z|}{1 - |z|^2} \right) \sum_{i=1}^n |\delta_i| \\
 & \leq \frac{1 - |z|^{2c}}{c} \cdot \frac{2}{1 - |z|} \sum_{i=1}^n |\alpha_i - 1| + \frac{1 - |z|^{2c}}{c} \cdot \frac{2|z|}{1 - |z|^2} \sum_{i=1}^n |\beta_i| \\
 & \quad + \frac{1 - |z|^{2c}}{c} \cdot \frac{4}{1 - |z|} \sum_{i=1}^n |\gamma_i| + \frac{1 - |z|^{2c}}{c} \cdot \frac{4|z|}{1 - |z|^2} \sum_{i=1}^n |\delta_i|, \tag{2.12}
 \end{aligned}$$

for all $z \in \mathbb{U}$.

For $0 < c < 1$, we have $1 - |z|^{2c} \leq 1 - |z|^2$, $z \in \mathbb{U}$ and by (2.12), we obtain

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{4}{c} \sum_{i=1}^n |\alpha_i - 1| + \frac{2}{c} \sum_{i=1}^n |\beta_i| + \frac{8}{c} \sum_{i=1}^n |\gamma_i| + \frac{4}{c} \sum_{i=1}^n |\delta_i|, \tag{2.13}$$

for all $z \in \mathbb{U}$.

From (2.10) and (2.13) we have

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq 1. \tag{2.14}$$

for all $z \in \mathbb{U}$ and $0 < c < 1$.

For $c \geq 1$ we have $\frac{1 - |z|^{2c}}{c} \leq 1 - |z|^2$, for all $z \in \mathbb{U}$ and by (2.12), we obtain

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq 4 \sum_{i=1}^n |\alpha_i - 1| + 2 \sum_{i=1}^n |\beta_i| + 8 \sum_{i=1}^n |\gamma_i| + 4 \sum_{i=1}^n |\delta_i|, \tag{2.15}$$

for all $z \in \mathbb{U}$ and $c \geq 1$.

From (2.11) and (2.15) we obtain

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq 1. \tag{2.16}$$

for all $z \in \mathbb{U}$ and $c \geq 1$.

And by (2.14), (2.16) and Lemma 1.2, it results that the integral operator \mathcal{T}_n , defined by (1.1) is in the class \mathcal{S} . □

If we consider $\delta = 1$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.9. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, $g_i', h_i', k_i' \in \mathcal{P}$. If*

$$4 \sum_{i=1}^n |\alpha_i - 1| + 2 \sum_{i=1}^n |\beta_i| + 8 \sum_{i=1}^n |\gamma_i| + 4 \sum_{i=1}^n |\delta_i| \leq \frac{Re\gamma}{2}, \text{ for } 0 < c < 1$$

then the integral operator \mathcal{F}_n defined by (2.4) belongs to the class \mathcal{S} .

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = \dots = \beta_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.10. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, $h_i', k_i' \in \mathcal{P}$. If*

$$4 \sum_{i=1}^n |\alpha_i - 1| + 8 \sum_{i=1}^n |\gamma_i| + 4 \sum_{i=1}^n |\delta_i| \leq \frac{\operatorname{Re}\gamma}{2}, \quad \text{for } 0 < c < 1$$

then the integral operator \mathcal{X}_n defined by (2.6) belongs to the class \mathcal{S} .

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = \dots = \gamma_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.11. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$ and $f_i \in \mathcal{S}$, $g_i', h_i', k_i' \in \mathcal{P}$. If*

$$4 \sum_{i=1}^n |\alpha_i - 1| + 2 \sum_{i=1}^n |\beta_i| + 4 \sum_{i=1}^n |\delta_i| \leq \frac{\operatorname{Re}\gamma}{2}, \quad \text{for } 0 < c < 1$$

then the integral operator \mathcal{Y}_n defined by (2.8) belongs to the class \mathcal{S} .

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.12. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$ and $h_i, k_i \in \mathcal{S}$, $g_i', h_i', k_i' \in \mathcal{P}$. If*

$$2 \sum_{i=1}^n |\beta_i| + 8 \sum_{i=1}^n |\gamma_i| + 4 \sum_{i=1}^n |\delta_i| \leq \frac{\operatorname{Re}\gamma}{2}, \quad \text{for } 0 < c < 1$$

then the integral operator \mathcal{D}_n defined by (2.7) belongs to the class \mathcal{S} .

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = \dots = \delta_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.13. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, $g_i' \in \mathcal{P}$. If*

$$4 \sum_{i=1}^n |\alpha_i - 1| + 2 \sum_{i=1}^n |\beta_i| + 8 \sum_{i=1}^n |\gamma_i| \leq \frac{\operatorname{Re}\gamma}{2}, \quad \text{for } 0 < c < 1$$

then the integral operator \mathcal{S}_n defined by (2.5) belongs to the class \mathcal{S} .

Theorem 2.14. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $\operatorname{Re}\gamma > 0$, M_i, N_i, P_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq 1, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \leq N_i,$$

$$\left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \leq 1, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$|c| \leq 1 - \frac{1}{|\delta|} \left[(2 + M_i) \sum_{i=1}^n |\alpha_i - 1| + \sum_{i=1}^n |\beta_i| \right] - \frac{1}{|\delta|} \left[(N_i + P_i + 4) \sum_{i=1}^n |\gamma_i| + 2 \sum_{i=1}^n |\delta_i| \right], \tag{2.17}$$

where $c \in \mathbb{C}$, $c \neq -1$, then the integral operator \mathcal{T}_n , defined by (1.1) is in the class \mathcal{S} .

Proof. Also, a simple computation yields

$$\begin{aligned} & \left| |c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zH_n''(z)}{\delta H_n'(z)} \right| \\ & \leq |c| + \frac{1}{|\delta|} \sum_{i=1}^n |\alpha_i - 1| \left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + \frac{1}{|\delta|} \sum_{i=1}^n |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \\ & \quad + \frac{1}{|\delta|} \sum_{i=1}^n |\gamma_i| \left[\left(\left| \frac{zh_i'(z)}{h_i(z)} \right| + 1 \right) + \left(\left| \frac{zk_i''(z)}{k_i(z)} \right| + 1 \right) \right] \\ & \quad + \frac{1}{|\delta|} \sum_{i=1}^n |\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right), \end{aligned} \tag{2.18}$$

for all $z \in \mathbb{U}$.

Using these inequalities from hypothesis we have

$$\begin{aligned} \left| |c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zH_n''(z)}{\delta H_n'(z)} \right| & \leq |c| + \frac{1}{|\delta|} \left[(2 + M_i) \sum_{i=1}^n |\alpha_i - 1| + \sum_{i=1}^n |\beta_i| \right] \\ & \quad + \frac{1}{|\delta|} \left[(N_i + P_i + 4) \sum_{i=1}^n |\gamma_i| + 2 \sum_{i=1}^n |\delta_i| \right], \end{aligned}$$

for all $z \in \mathbb{U}$. and hence, by inequality (2.17) we have

$$\left| |c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zH_n''(z)}{\delta H_n'(z)} \right| \leq 1, \tag{2.19}$$

for all $z \in \mathbb{U}$.

Applying Lemma 1.3, we conclude that the integral operator \mathcal{T}_n , given by (1.1) is in the class \mathcal{S} . □

If we consider $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ and $n = 1$ in Theorem 2.14, we obtain the next corollary:

Corollary 2.15. *Let α be complex number, $\text{Re}\alpha > 0$ M, N, P real positive numbers, and $f, g, h, k \in \mathcal{A}$. If*

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| & \leq M, & \left| \frac{zg''(z)}{g'(z)} \right| & \leq 1, & \left| \frac{zh'(z)}{h(z)} - 1 \right| & \leq N, \\ \left| \frac{zk'(z)}{k(z)} - 1 \right| & \leq P, & \left| \frac{zh''(z)}{h'(z)} \right| & \leq 1, & \left| \frac{zk''(z)}{k'(z)} \right| & \leq 1, \end{aligned}$$

for all $z \in \mathbb{U}$ and

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M_i + N_i + P_i + 8), \quad c \in \mathbb{C}, \quad c \neq -1,$$

then the integral operator \mathcal{T} , given by (2.9) is in the class \mathcal{S} .

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