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Complex left Caputo fractional inequalities

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Abstract. Here we present several complex left Caputo type fractional inequalities of well known kinds, such as of Ostrowski, Poincare, Sobolev, Opial and Hilbert-Pachpatte.

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1. Introduction

We are motivated by the following result for functions of complex variable: Complex Ostrowski type inequality

Theorem 1.1. (see [3]) *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a, b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz|$$
$$\leq \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz| \right] \|f'\|_{\gamma_{u,w};\infty},$$

and

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \leq \max_{z\in\gamma_{u,v}} |z-u|\,\|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\,\|f'\|_{\gamma_{v,w};1}$$
$$\leq \max\left\{ \max_{z\in\gamma_{u,v}} |z-u|,\, \max_{z\in\gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}.$$

If $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left| f(v)(w-u) - \int_{\gamma} f(z)\,dz \right| \le \left(\int_{\gamma_{u,v}} |z-u|^q |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^q |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{v,w};p}$$
$$\le \left(\int_{\gamma_{u,v}} |z-u|^q |dz| + \int_{\gamma_{v,w}} |z-w|^q |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{u,w};p}.$$

Above $|\cdot|$ *is the complex absolute value.*

We are also motivated by the next complex Opial type inequality:

Theorem 1.2. (see [2]) *Let* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function on the domain* D *and let* $x, y, w \in D$. *Suppose* γ *is a smooth path parametrized by* $z(t)$, $t \in [a,b]$ *with* $z(a) = x$, $z(c) = y$, *and* $z(b) = w$, *where* $c \in [a,b]$ *is floating. Assume that* $f^{(k)}(x) = 0$, $k = 0, 1, ..., n$, $n \in \mathbb{Z}_+$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

1)

$$\left| \int_a^b f(z(t))\, f^{(n+1)}(z(t))\, z'(t)\,dt \right|$$
$$\le \int_a^b |f(z(t))| \left| f^{(n+1)}(z(t)) \right| |z'(t)|\,dt$$
$$\le \frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)|\,dt \right) |z'(c)|\,dc \right]^{\frac{1}{p}}$$
$$\cdot \left(\int_a^b \left| f^{(n+1)}(z(t)) \right|^q |z'(t)|\,dt \right)^{\frac{2}{q}},$$

equivalently it holds

2)

$$\left| \int_{\gamma_{x,w}} f(z)\, f^{(n+1)}(z)\,dz \right| \le \int_{\gamma_{x,w}} |f(z)| \left| f^{(n+1)}(z) \right| |dz|$$
$$\le \frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^{pn} |dz| \right) |z'(c)|\,dc \right]^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} \left| f^{(n+1)}(z) \right|^q |dz| \right)^{\frac{2}{q}}.$$

Here we utilize on $\mathbb{C}$ the results of [1] which are for general Banach space valued functions.

Mainly we give different cases of the left fractional $\mathbb{C}$-Ostrowski type inequality and we continue with the left fractional: $\mathbb{C}$-Poincaré like and Sobolev like inequalities.

We present an Opial type left $\mathbb{C}$-fractional inequality, and we finish with the Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities.

2. Background

In this section all integrals are of Bochner type.

We need

Definition 2.1. (see [4]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ ($diam$ =diameter) for every i. Now, for every $\alpha > 0$ define

$$h_\alpha^\delta(A) := \inf\left\{\sum (diam C_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \tag{2.1}$$

Then there exists $\lim_{\delta\to 0} h_\alpha^\delta(A) = \sup_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field of all Borel subsets of T. If $T = \mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure.

Definition 2.2. ([1]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number, $f : [a,b] \to X$. We assume that $f^{(n)} \in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order ν:

$$\left(D_{*a}^\nu f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \quad \forall\, x \in [a,b]. \tag{2.2}$$

If $\nu \in \mathbb{N}$, we set $D_{*a}^\nu f := f^{(\nu)}$ the ordinary X-valued derivative, defined similarly to the numerical one, and also set $D_{*a}^0 f := f$.

By [1] $\left(D_{*a}^\nu f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^\nu f \in L_1([a,b],X)$.
If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [1] $D_{*a}^\nu f \in C([a,b],X)$.
We need the left-fractional Taylor's formula:

Theorem 2.3. ([1]) *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{2.3}$$

where $x \in [a,b]$.
Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a,x]$, such that

$$h_1(F_x(B_x)) = 0,\ \forall\, x \in [a,b]. \tag{2.4}$$

We also assume that $f^{(n)} \in L_1([a,b],X)$. Then

$$f(x) = \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_{*a}^\nu f\right)(z)\,dz, \tag{2.5}$$

$\forall\, x \in [a,b]$.

Next we mention an Ostrowski type inequality at left fractional level for Banach valued functions.

Theorem 2.4. ([1]) *Let* $\nu \geq 0$, $n = \lceil \nu \rceil$. *Here all as in Theorem* 2.3. *Assume that* $f^{(i)}(a) = 0$, $i = 1, ..., n-1$, *and that* $D_{*a}^{\nu} f \in L_{\infty}([a,b], X)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_{\infty}([a,b],X)}}{\Gamma(\nu+2)} (b-a)^{\nu}. \quad (2.6)$$

We mention an Ostrowski type L_p fractional inequality:

Theorem 2.5. ([1]) *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. *Here all as in Theorem* 2.3. *Assume that* $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, *and* $D_{*a}^{\nu} f \in L_q([a,b], X)$, *where* X *is a Banach space. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}. \quad (2.7)$$

It follows

Corollary 2.6. ([1]) *(to Theorem* 2.5, *case of* $p = q = 2$*). Let* $\nu > \frac{1}{2}$, $n = \lceil \nu \rceil$. *Here all as in Theorem* 2.3. *Assume that* $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, *and* $D_{*a}^{\nu} f \in L_2([a,b], X)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_2([a,b],X)}}{\Gamma(\nu)\left(\sqrt{2\nu-1}\right)\left(\nu + \frac{1}{2}\right)} (b-a)^{\nu - \frac{1}{2}}. \quad (2.8)$$

Next comes the L_1 case of fractional Ostrowski inequality:

Theorem 2.7. ([1]) *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$, *and all as in Theorem* 2.3. *Assume that* $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, *and* $D_{*a}^{\nu} f \in L_1([a,b], X)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_1([a,b],X)}}{\Gamma(\nu+1)} (b-a)^{\nu-1}. \quad (2.9)$$

We continue with a Poincaré like fractional inequality:

Theorem 2.8. ([1]) *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. *Here all as in Theorem* 2.3. *Assume that* $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, *and* $D_{*a}^{\nu} f \in L_q([a,b], X)$, *where* X *is a Banach space. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)}. \quad (2.10)$$

Next comes a Sobolev like fractional inequality.

Theorem 2.9. ([1]) *All as in the last Theorem* 2.8. *Let* $r > 0$. *Then*

$$\|f\|_{L_r([a,b],X)} \leq \frac{(b-a)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu - \frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)}. \quad (2.11)$$

We mention the following Opial type fractional inequality:

Theorem 2.10. ([1]) *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $\nu > \frac{1}{q}$, $n := \lceil \nu \rceil$. *Let* $[a, b] \subset \mathbb{R}$, X *a Banach space, and* $f \in C^{n-1}([a, b], X)$. *Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a, x], \text{ where } x \in [a, b]. \tag{2.12}$$

Assume that $f^{(n)}$ *exists outside a* μ*-null Borel set* $B_x \subseteq [a, x]$, *such that*

$$h_1(F_x(B_x)) = 0, \quad \forall\, x \in [a, b]. \tag{2.13}$$

We also assume that $f^{(n)} \in L_\infty([a, b], X)$.
Assume also that $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$. *Then*

$$\int_a^x \|f(w)\| \, \|(D_{*a}^\nu f)(w)\| \, dw$$

$$\leq \frac{(x-a)^{\nu - 1 + \frac{2}{p}}}{2^{\frac{1}{q}} \Gamma(\nu) \left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}} \left(\int_a^x \|(D_{*a}^\nu f)(z)\|^q \, dz\right)^{\frac{2}{q}}, \tag{2.14}$$

$\forall\, x \in [a, b]$.

We finish this section with a Hilbert-Pachpatte left fractional inequality:

Theorem 2.11. ([1]) *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $n_i := \lceil \nu_i \rceil$, $i = 1, 2$. *Here* $[a_i, b_i] \subset \mathbb{R}$, $i = 1, 2$; X *is a Banach space. Let* $f_i \in C^{n_i - 1}([a_i, b_i], X)$, $i = 1, 2$. *Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{n_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{2.15}$$

$\forall\, t_i \in [a_i, x_i]$, *where* $x_i \in [a_i, b_i]$; $i = 1, 2$. *Assume that* $f_i^{(n_i)}$ *exists outside a* μ*-null Borel set* $B_{x_i} \subseteq [a_i, x_i]$, *such that*

$$h_1(F_{x_i}(B_{x_i})) = 0, \quad \forall\, x_i \in [a_i, b_i]; \; i = 1, 2. \tag{2.16}$$

We also assume that $f_i^{(n_i)} \in L_1([a_i, b_i], X)$, *and*

$$f_i^{(k_i)}(a_i) = 0, \quad k_i = 0, 1, ..., n_i - 1; \; i = 1, 2, \tag{2.17}$$

and

$$\left(D_{*a_1}^{\nu_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{*a_2}^{\nu_2} f_2\right) \in L_p([a_2, b_2], X).$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \, \|f_2(x_2)\| \, dx_1 dx_2}{\left(\frac{(x_1 - a_1)^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(x_2 - a_2)^{q(\nu_2 - 1)+1}}{q(q(\nu_2-1)+1)}\right)}$$

$$\leq \frac{(b_1 - a_1)(b_2 - a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1, b_1], X)} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2, b_2], X)}. \tag{2.18}$$

3. Main results

We need a special case of Definition 2.2 over $\mathbb{C}$.

Definition 3.1. Let $[a,b] \subset \mathbb{R}$, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number and $f \in C^n([a,b],\mathbb{C})$. We call Caputo-Complex left fractional derivative of order ν:

$$\left(D_{*a}^{\nu} f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\, x \in [a,b], \tag{3.1}$$

where the derivatives $f', \ldots f^{(n)}$ are defined as the numerical derivative.

If $\nu \in \mathbb{N}$, we set $D_{*a}^{\nu} f := f^{(\nu)}$ the ordinary $\mathbb{C}$-valued derivative and also set $D_{*a}^{0} f := f$.

Notice here (by [1]) that $D_{*a}^{\nu} f \in C([a,b],\mathbb{C})$.

We make

Remark 3.2. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ (i.e. there exists $z'(t)$ and is continuous) and from now on f is a complex function which is continuous on γ.

Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt = \int_a^b h(t)\, dt, \tag{3.2}$$

where $h(t) := f(z(t))\, z'(t)$, $t \in [a,b]$.

We notice that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\, |dz| := \int_a^b f(z(t))\, |z'(t)|\, dt$$

and the length of the curve γ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)|\, dt.$$

We mention also the triangle inequality for the complex integral, namely

$$\left| \int_{\gamma} f(z)\, dz \right| \leq \int_{\gamma} |f(z)|\, |dz| \leq \|f\|_{\gamma,\infty}\, l(\gamma), \tag{3.3}$$

where $\|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)|$.

We give the following left-fractional $\mathbb{C}$-Taylor's formula:

Theorem 3.3. *Let $h \in C^n([a,b], \mathbb{C})$, $n = \lceil \nu \rceil$, $\nu \geq 0$. Then*

$$h(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} h^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^t (t-\lambda)^{\nu-1} \left(D_{*a}^{\nu} h\right)(\lambda)\, d\lambda, \tag{3.4}$$

$\forall\ t \in [a,b]$, in particular it holds,

$$f(z(t))\, z'(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} \left(f(z(a))\, z'(a)\right)^{(i)} + \frac{1}{\Gamma(\nu)} \int_a^t (t-\lambda)^{\nu-1} \left(D_{*a}^{\nu} f(z(\cdot))\, z'(\cdot)\right)(\lambda)\, d\lambda, \tag{3.5}$$

$\forall\ t \in [a,b]$.

Proof. By Theorem 2.3. ☐

It follows a left fractional $\mathbb{C}$-Ostroswski type inequality

Theorem 3.4. *Let $n \in \mathbb{N}$ and $h \in C^n([a,b], \mathbb{C})$, where $[a,b] \subset \mathbb{R}$, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Assume that $h^{(i)}(a) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-a} \int_a^b h(t)\, dt - f(a) \right| \leq \frac{\left\| D_{*a}^{\nu} h \right\|_{\infty,[a,b]}}{\Gamma(\nu+2)} (b-a)^{\nu}, \tag{3.6}$$

in particular when $h(t) := f(z(t))\, z'(t)$ and $\left(f(z(t))\, z'(t)\right)^{(i)} |_{t=a} = 0$, $i = 1, ...n-1$, we get

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(u)\, z'(a) \right| = \left| \frac{1}{b-a} \int_a^b f(z(t))\, z'(t)\, dt - f(z(a))\, z'(a) \right|$$

$$\leq \frac{\left\| D_{*a}^{\nu} f(z(t))\, z'(t) \right\|_{\infty,[a,b]}}{\Gamma(\nu+2)} (b-a)^{\nu}. \tag{3.7}$$

Proof. By Theorem 2.4. ☐

The corresponding $\mathbb{C}$-Ostrowski type L_p inequality follows:

Theorem 3.5. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here $h \in C^n([a,b], \mathbb{C})$. Assume that $h^{(i)}(a) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-a} \int_a^b h(t)\, dt - h(a) \right| \leq \frac{\left\| D_{*a}^{\nu} h \right\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu) \left(p(\nu-1)+1\right)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}, \tag{3.8}$$

in particular when $h(t) := f(z(t))z'(t)$ *and* $(f(z(t))z'(t))^{(i)}|_{t=a} = 0$, $i = 1,...n-1$, *we get:*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| = \left|\frac{1}{b-a}\int_a^b f(z(t))\,z'(t)\,dt - f(z(a))\,z'(a)\right|$$
$$\leq \frac{\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-\frac{1}{q}}. \tag{3.9}$$

Proof. By Theorem 2.5. □

It follows

Corollary 3.6. *(to Theorem* 3.5*, case of* $p = q = 2$*). We have that*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| \leq \frac{\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_2([a,b],\mathbb{C})}}{\Gamma(\nu)\sqrt{2\nu-1}\left(\nu+\frac{1}{2}\right)}(b-a)^{\nu-\frac{1}{2}}. \tag{3.10}$$

We continue with an L_1 fractional $\mathbb{C}$-Ostrowski type inequality:

Theorem 3.7. *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$*. Assume that* $h \in C^n([a,b],\mathbb{C})$*, where*

$$h(t) := f(z(t))\,z'(t),$$

and such that $h^{(i)}(a) = 0$, $i = 1,...,n-1$*. Then*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| \leq \frac{\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_1([a,b],\mathbb{C})}}{\Gamma(\nu+1)}(b-a)^{\nu-1}. \tag{3.11}$$

Proof. By Theorem 2.7. □

It follows a Poincaré like $\mathbb{C}$-fractional inequality:

Theorem 3.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*, and* $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$*. Let* $h \in C^n([a,b],\mathbb{C})$*. Assume that* $h^{(i)}(a) = 0$, $i = 1,...,n-1$*. Then*

$$\|h\|_{L_q([a,b],\mathbb{C})} \leq \frac{(b-a)^{\nu}\|D_{*a}^{\nu}h\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}, \tag{3.12}$$

in particular when $h(t) := f(z(t))z'(t)$ *and* $(f(z(t))z'(t))^{(i)}|_{t=a} = 0$, $i = 1,...n-1$, *we get:*

$$\|f(z(t))\,z'(t)\|_{L_q([a,b],\mathbb{C})}$$
$$\leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_q([a,b],\mathbb{C})}. \tag{3.13}$$

Proof. By Theorem 2.8. □

The corresponding Sobolev like inequality follows:

Theorem 3.9. *All as in Theorem* 3.8. *Let* $r > 0$. *Then*

$$\|f(z(t))z'(t)\|_{L_r([a,b],\mathbb{C})}$$
$$\leq \frac{(b-a)^{\nu-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}\|D_{*a}^{\nu}(f(z(t))z'(t))\|_{L_q([a,b],\mathbb{C})}. \quad (3.14)$$

Proof. By Theorem 2.9. □

We continue with an Opial type $\mathbb{C}$-fractional inequality

Theorem 3.10. *Let* $p, q > 1 : \frac{1}{p}+\frac{1}{q}=1$, *and* $\nu > \frac{1}{q}$, $n := \lceil \nu \rceil$, $h \in C^n([a,b],\mathbb{C})$. *Assume* $h^{(k)}(a)=0$, $k=0,1,...,n-1$. *Then*

$$\int_a^x |h(t)|\,|(D_{*a}^{\nu}h)(t)|\,dt$$
$$\leq \frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)((p(\nu-1)+1)(p(\nu-1)+2))^{\frac{1}{p}}}\left(\int_a^x |(D_{*a}^{\nu}h)(t)|^q\,dt\right)^{\frac{2}{q}}, \quad (3.15)$$

$\forall\, x \in [a,b]$, *in particular when* $h(t) := f(z(t))z'(t)$ *and* $(f(z(t))z'(t))^{(i)}|_{t=a}=0$, $i=1,...n-1$, *we get:*

$$\int_a^x |f(z(t))|\,|(D_{*a}^{\nu}(f(z(t))z'(t)))|\,|z'(t)|\,dt$$
$$\leq \frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)((p(\nu-1)+1)(p(\nu-1)+2))^{\frac{1}{p}}}\left(\int_a^x |D_{*a}^{\nu}(f(z(t))z'(t))|^q\,dt\right)^{\frac{2}{q}}, \quad (3.16)$$

$\forall\, x \in [a,b]$.

Proof. By Theorem 2.10. □

We finish with Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities:

Theorem 3.11. *Let* $p, q > 1 : \frac{1}{p}+\frac{1}{q}=1$, *and* $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $n_i := \lceil \nu_i \rceil$, $i=1,2$. *Let* $h_i \in C^{n_i}([a_i,b_i],\mathbb{C})$, $i=1,2$. *Assume* $h_i^{(k_i)}(a_i)=0$, $k_i=0,1,...,n_i-1$; $i=1,2$. *Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|h_1(t_1)|\,|h_2(t_2)|\,dt_1dt_2}{\left(\frac{(t_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(t_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}$$
$$\leq \frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|D_{*a_1}^{\nu_1}h_1\right\|_{L_q([a_1,b_1],\mathbb{C})}\left\|D_{*a_2}^{\nu_2}h_2\right\|_{L_p([a_2,b_2],\mathbb{C})}, \quad (3.17)$$

in particular when $h_1(t_1) := f_1(z_1(t_1))z_1'(t_1)$ *and* $h_2(t_2) := f_2(z_2(t_2))z_2'(t_2)$, *with* $h_i^{(k_i)}(a_i)=0$, $k_i=0,1,...n_i-1$; $i=1,2$, *we get:*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(z_1(t_1))z_1'(t_1)|\,|f_2(z_2(t_2))z_2'(t_2)|\,dt_1dt_2}{\left(\frac{(t_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(t_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq \frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\cdot$$
$$\left\|D_{*a_1}^{\nu_1}(f_1(z_1(t_1))z_1'(t_1))\right\|_{L_q([a_1,b_1],\mathbb{C})}\left\|D_{*a_2}^{\nu_2}(f_2(z_2(t_2))z_2'(t_2))\right\|_{L_p([a_2,b_2],\mathbb{C})}. \quad (3.18)$$

Proof. By Theorem 2.11. □

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