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Second-order noncanonical difference equations with a sublinear neutral term: improved oscillation criteria

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Abstract. The topic under consideration is the study of the oscillatory behaviour of the second-order nonlinear neutral difference equation of the form

$$\Delta[r(\nu)(\Delta(\mathcal{V}(\nu)-q(\nu)\mathcal{V}^{\gamma}(\delta(\nu))))^{\alpha}]+\mathcal{P}(\nu)\mathcal{V}^{\beta}(\eta(\nu))=0,\ \nu\geq\nu_0$$

of noncanonical type. We give an answer to a question raised in [10, Remark 4.1]. Our findings generalise the work of Dinakar et al. [14], Seghar et al. [20], and Thandapani and Selvarangam [21]. Two examples are given to demonstrate the analytical results. Finally, we state two open problems for future research.

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1. Introduction

In this paper, we consider the second-order noncanonical difference equation with a sublinear neutral term given by

$$\Delta[r(\nu)(\Delta\mathcal{Z}(\nu))^{\alpha}]+\mathcal{P}(\nu)\mathcal{V}^{\beta}(\eta(\nu))=0\quad\text{for }\nu\geq\nu_0,\tag{1.1}$$

where $\nu\in\mathbb{N}(\nu_0)=\{\nu_0,\nu_0+1,\cdots\}$ and $\mathcal{Z}(\nu)=\mathcal{V}(\nu)-q(\nu)\mathcal{V}^{\gamma}(\delta(\nu))$. Everywhere, we assume that

$(\mathcal{A}_1)$ α, β and γ are ratios of odd positive integers, with $0\leq\gamma<1$;

$(\mathcal{A}_2)$ $\delta(\nu)$ and $\eta(\nu)$ are increasing sequence of positive integers satisfying $\delta(\nu)\leq\nu$ and $\eta(\nu)\leq\nu$, and $\lim_{\nu\to\infty}\delta(\nu)=\lim_{\nu\to\infty}\eta(\nu)=\infty$;

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$(\mathcal{A}_3)$ $\mathcal{P}(\nu)$, $q(\nu)$ and $r(\nu)$ are positive real sequences such that $\mathcal{P}(\nu) \not\equiv 0$, $0 \le q(\nu) \le q < 1$ and $\sum_{s=\nu_0}^{\infty} (r(s))^{-1/\alpha} < \infty$;

$(\mathcal{A}_4)$ $h(\nu) := \delta^{-1}(\eta(\nu)) \le \nu$, and $h(\nu) \to \infty$ as $\nu \to \infty$.

A real valued function $\mathcal{V}(\nu)$ defined on $\nu \in \mathbb{N}(\nu_0 - \rho)$ is called the solution of (1.1) if it satisfies (1.1) for $\nu \ge \nu_0$ with the initial conditions $\mathcal{V}(i) = \phi(i)$, where $\phi(i)$, $i = \nu_0 - \rho, \cdots, \nu_0$ are given and $\rho = \max\{\inf \delta(\nu), \inf \eta(\nu)\}$.

It is important to study (1.1) for both theoretical reasons as well as practical reasons. In fact, (1.1) and its particular cases have numerous applications in physics and engineering; see, for instance, the papers [3, 19] and the monographs [1, 2, 18]. Thus, in the past decades, a great deal of research has been done on the study of qualitative behaviours of (1.1), see [7, 8, 11, 12, 14, 15, 16, 17, 22]. However, the majority of the research has focused on the study of oscillatory behaviours of (1.1) with a linear term when $(\mathcal{A}_3)$ holds true, see [6, 10, 14, 20, 21]. In particular, Thandapani and Selvarangam [21] investigated the oscillation and asymptotic behaviour of solutions of the second-order neutral difference equation

$$\Delta\big[r(\nu)\Delta\big(\mathcal{V}(\nu) + q(\nu)\mathcal{V}(\nu - k)\big)\big] + \mathcal{P}(\nu)\mathcal{V}^{\beta}(\eta(\nu)) = 0, \tag{1.2}$$

where $0 \le q(\nu) \le q < 1$ and k is a positive integer. It is worth noting that (1.2) involves a positive neutral term with constant delay, whereas the present work considers a more general setting with a nonlinear operator, a sublinear neutral term, and variable delays, thereby extending the scope of the existing results.

Seghar et al. [20] investigated the oscillatory behaviour of the second-order neutral difference equation

$$\Delta\big[r(\nu)\Delta\big(\mathcal{V}(\nu) - q(\nu)\mathcal{V}(\nu - k)\big)\big] + \mathcal{P}(\nu)F\big(\mathcal{V}(\nu - l)\big) = 0, \tag{1.3}$$

where $0 \le q(\nu) \le q < 1$, $\frac{F(u)}{u^{\beta}} > M$ for some constant $M > 0$, and $l > k > 0$ are integers. By employing the summation averaging technique and the Riccati transformation method, the authors established sufficient conditions ensuring the oscillation and asymptotic behaviour of solutions of (1.3) in the cases $\beta = 1$, $0 < \beta < 1$, and $\beta > 1$.

Dinakar et al. [14] considered the second-order neutral difference equation

$$\Delta\big[r(\nu)\,|\Delta\mathcal{Z}(\nu)|^{\alpha-1}\Delta\mathcal{Z}(\nu)\big] + \mathcal{P}(\nu)\,|\mathcal{V}(\nu - l)|^{\beta-1}\mathcal{V}(\nu - l) = 0, \tag{1.4}$$

where $\mathcal{Z}(\nu) = \mathcal{V}(\nu) + q(\nu)\mathcal{V}(\nu - k)$, $0 \le q(\nu) \le q < 1$, and $l > k > 0$ are integers. Using the summation averaging technique, the authors obtained sufficient oscillation criteria for (1.4) in the cases $\alpha > \beta$ and $\alpha \le \beta$.

Grace and Graef [13] studied second-order delay difference equations with a superlinear neutral term and established oscillation theorems by combining Riccati-type substitutions with comparison techniques; their approach, however, was restricted to the superlinear case and to a positive neutral coefficient.

Recently, Chhatria [10] studied the second-order neutral difference equations of the form

$$\Delta[a(\nu)\Delta(\mathcal{V}(\nu) - q(\nu)\mathcal{V}(\delta(\nu))] + \mathcal{P}(\nu)G(\mathcal{V}(\eta(\nu))) = 0, \tag{1.5}$$

where $|q(\nu)| < 1$ and $G \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ such that G is either strongly sublinear or strongly superlinear. Using inequality techniques and some fixed-point theorems, they improved some known results of the literature. The following conclusions emerged from the results of [10]:

1. With $0 \leq q(\nu) < 1$, [10, Theorem 2.1] and [10, Theorem 3.1] ensure that every unbounded solution of (1.5) oscillates, which is a restrictive condition;
2. If we remove this restriction from the solution, this result guarantees that all solutions of (1.5) either oscillate or converge to zero.

Inspired by the foregoing discussion, we simplify and enhance the results discussed in the literature by finding some novel oscillation conditions for (1.1). First, we establish the primary results assuming new limsup type conditions. Second, we refine the theorem by setting forth criteria that ensure (1.1) oscillations only. To achieve the proof, we were inspired by recent articles [4, 5, 9] on the oscillation of solutions of differential equations.

Let us denote

$$\hat{\mathcal{R}}(T, M) = \sum_{s=T}^{M-1} \frac{1}{r^{\frac{1}{\alpha}}(s)} \quad \text{for } M > T \geq \nu_0.$$

Then, Eq. (1.1) is said to be of **non-canonical type** (see [23]) if

$$\hat{\mathcal{R}}(\nu_0, \nu) = \sum_{s=\nu_0}^{\nu-1} \frac{1}{r^{\frac{1}{\alpha}}(s)} < \infty \quad \text{as } \nu \to \infty.$$

2. Preliminary lemmas

Due to $(\mathcal{A}_3)$, we may let

$$\hat{\mathcal{R}}(\nu) = \sum_{s=\nu}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} \text{ for } \nu \geq \nu_0, \tag{2.1}$$

then $\hat{\mathcal{R}}(\nu) \to 0$ as $\nu \to \infty$.

Lemma 2.1. [10] *Let $\mathcal{V}(\nu) > 0$ eventually be a solution of (1.1) for $\nu \geq \nu_0$. Then $\mathcal{Z}(\nu)$ satisfies $\Delta[r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha] < 0$ along with one of the following cases:*

$$(a)\ \mathcal{Z}(\nu) > 0, r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha > 0, \quad (b)\ \mathcal{Z}(\nu) < 0, r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha < 0,$$
$$(c)\ \mathcal{Z}(\nu) > 0, r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha < 0, \quad (d)\ \mathcal{Z}(\nu) < 0, r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha > 0$$

for $\nu \geq \nu_* > \nu_0$.

Proof. Let $\mathcal{V}(\nu)$ be a non-oscillatory solution of Eq. (1.1) for $\nu \geq \nu_0$. Without loss of generality, we assume that $\mathcal{V}(\nu) > 0$, $\mathcal{V}(\delta(\nu)) > 0$, and $\mathcal{V}(\eta(\nu)) > 0$ for $\nu \geq \nu_1 > \nu_0$. Then, in view of $(\mathcal{A}_3)$, it follows from (1.1) that

$$\Delta\left[r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha\right] = -\mathcal{P}(\nu)\mathcal{V}^\beta(\eta(\nu)) < 0.$$

Hence, $r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha$ is decreasing and therefore eventually of one sign. Consequently, there exists $\nu_2 > \nu_1$ such that $r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha > 0$, or $r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha < 0$ for all $\nu \geq \nu_2$.

Since $r(\nu) > 0$, it follows that $(\Delta\mathcal{Z}(\nu))^\alpha$ is eventually of one sign; since α is a ratio of positive odd integers, it follows that $\Delta\mathcal{Z}(\nu)$ is eventually of one sign. Thus, $\mathcal{Z}(\nu)$ is monotone, and consequently $\mathcal{Z}(\nu)$ is eventually of one sign; that is, either $\mathcal{Z}(\nu) > 0$ or $\mathcal{Z}(\nu) < 0$ for all $\nu \geq \nu_*$.

Combining the eventual signs of $\mathcal{Z}(\nu)$ and $r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha$, we obtain one of the cases a–d. This completes the proof. □

Lemma 2.2. *Let $\mathcal{V}(\nu)$ be a positive solution of (1.1) such that $\mathcal{Z}(\nu) > 0$ for $\nu \geq \nu_0$. If*

$$\sum_{s=\nu^*}^{\infty} \mathcal{P}(s) = \infty, \tag{2.2}$$

then $\mathcal{Z}(\nu)$ satisfies Case-c of Lemma 2.1. Moreover, $\left(\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}\right)$ is increasing.

Proof. Suppose that $\mathcal{Z}(\nu)$ satisfies Case-a of Lemma 2.1 for $\nu \geq \nu_1 > \nu_0$. From (A3), we have $0 \leq q(\nu) \leq q < 1$, and since $\mathcal{V}^\gamma(\delta(\nu)) > 0$, it follows that

$$q(\nu)\mathcal{V}^\gamma(\delta(\nu)) > 0.$$

Hence,

$$\mathcal{Z}(\nu) = \mathcal{V}(\nu) - q(\nu)\mathcal{V}^\gamma(\delta(\nu)) \leq \mathcal{V}(\nu).$$

Therefore, Eq. (1.1) becomes

$$\Delta[r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha] + \mathcal{P}(\nu)\mathcal{Z}^\beta(\eta(\nu)) \leq 0 \text{ for } \nu \geq \nu_2. \tag{2.3}$$

Summing Eq. (2.3) from ν_2 to $\nu - 1$, we get

$$r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha - r(\nu_2)(\Delta\mathcal{Z}(\nu_2))^\alpha + \sum_{s=\nu_2}^{\nu} \mathcal{P}(s)\mathcal{Z}^\beta(\eta(s)) \leq 0.$$

Since $\mathcal{Z}(\nu)$ is positive and increasing, there exist $\nu_2 > \nu_1$ and a constant $B > 0$ such that $\mathcal{Z}(\nu) \geq B$. Consequently, $\mathcal{Z}^\beta(\eta(\nu)) \geq B^\beta$ for $\nu \geq \nu_2$. Therefore, the above inequality reduces to

$$B^\beta \sum_{s=\nu_2}^{\nu-1} \mathcal{P}(s) \leq r(\nu_2)(\Delta\mathcal{Z}(\nu_2))^\alpha - r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha \leq r(\nu_2)(\Delta\mathcal{Z}(\nu_2))^\alpha,$$

a contradiction to (2.2). Therefore, $\mathcal{Z}(\nu)$ satisfies Case-c for $\nu \geq \nu_1$.
Indeed, $s > \nu \geq \nu_1$ implies that

$$r(s)(\Delta\mathcal{Z}(s))^\alpha \leq r(\nu)(\Delta\mathcal{Z}(\nu))^\alpha. \tag{2.4}$$

Summing Eq. (2.4) from ν to $s - 1$, we get

$$\mathcal{Z}(\nu) - \mathcal{Z}(s) \geq -r^{\frac{1}{\alpha}}(\nu)\Delta\mathcal{Z}(\nu) \sum_{T=\nu}^{s-1} \frac{1}{r^{\frac{1}{\alpha}}(T)}.$$

Taking $s \to \infty$, we have

$$-\mathcal{Z}(\nu) \leq r^{\frac{1}{\alpha}}(\nu)\Delta\mathcal{Z}(\nu) \sum_{T=\nu}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(T)} = r^{\frac{1}{\alpha}}(\nu)\Delta\mathcal{Z}(\nu)\hat{\mathcal{R}}(\nu).$$

Consequently,

$$\Delta\left(\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}\right)=\frac{\hat{\mathcal{R}}(\nu)\Delta\mathcal{Z}(\nu)+\mathcal{Z}(\nu)r^{-\frac{1}{\alpha}}(\nu)}{\hat{\mathcal{R}}(\nu)\hat{\mathcal{R}}(\nu+1)}\geq 0,$$

which implies that $\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}$ is increasing for $\nu\geq\nu_1$. Hence, proved. □

Lemma 2.3. *If $\mathcal{Z}(\nu)$ satisfies Case-c of Lemma 2.1 and*

$$\sum_{s=\nu_2}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\left[\sum_{T=\nu_2}^{s-1}\mathcal{P}(T)\right]^{\frac{1}{\alpha}}=\infty, \tag{2.5}$$

then $\mathcal{Z}(\nu)\to 0$ as $\nu\to\infty$. In addition to (2.5)*, assume that*

$$\sum_{s=\nu_2}^{\infty}\mathcal{P}(s)\hat{\mathcal{R}}^{\beta}(\eta(s))=\infty \tag{2.6}$$

hold, then $\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}\to\infty$ as $\nu\to\infty$.

Proof. From $(\mathcal{A}_3)$ and (2.5), it follow that (2.2) holds. As a result, Lemma 2.2 implies that $\mathcal{Z}(\nu)$ meets Case-c of Lemma 2.1. In fact, $\lim_{\nu\to\infty}\mathcal{Z}(\nu)=\mathcal{C}\geq 0$ exists. We assert that $\mathcal{C}=0$. Otherwise, we can find $\nu_2>\nu_1$ such that $\mathcal{Z}(\nu)\geq\mathcal{C}>0$ and hence $\mathcal{Z}^{\beta}(\eta(\nu))\geq\mathcal{C}^{\beta}$ for $\nu\geq\nu_2$. Upon using this relation in Eq. (2.3) and then summing from ν_2 to $\nu-1$, we get

$$\mathcal{C}^{\beta}\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\leq -r(\nu)(\Delta\mathcal{Z}(\nu))^{\alpha},$$

or,

$$\frac{\mathcal{C}^{\frac{\beta}{\alpha}}}{r^{\frac{1}{\alpha}}(\nu)}\left[\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\right]^{\frac{1}{\alpha}}\leq -\Delta\mathcal{Z}(\nu),$$

which on summation again from from ν_2 to ν gives

$$\mathcal{C}^{\frac{\beta}{\alpha}}\sum_{s=\nu_2}^{\nu}\frac{1}{r^{\frac{1}{\alpha}}(s)}\left[\sum_{T=\nu_2}^{s-1}\mathcal{P}(T)\right]^{\frac{1}{\alpha}}\leq\mathcal{Z}(\nu_2)-\mathcal{Z}(\nu+1)\leq\mathcal{Z}(\nu_2),$$

a contradiction to (2.5). Thus, $\lim_{\nu\to\infty}\mathcal{Z}(\nu)=0$.

Since $\mathcal{Z}(\nu)$ and $\hat{\mathcal{R}}(\nu)$ tends to zero as $\nu\to\infty$, then using Stolz-Cesaro theorem (see [2]), we have

$$\lim_{\nu\to\infty}\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}=\lim_{\nu\to\infty}\frac{\Delta\mathcal{Z}(\nu)}{\Delta\hat{\mathcal{R}}(\nu)}=\lim_{\nu\to\infty}\left(-r^{\frac{1}{\alpha}}(\nu)\Delta\mathcal{Z}(\nu)\right)=L,$$

where $0<L\leq\infty$. We assert that $L=\infty$. Otherwise, we can find $\nu_2>\nu_1$ such that $-r^{\frac{1}{\alpha}}(\nu)\Delta\mathcal{Z}(\nu)\leq L<\infty$ for $\nu\geq\nu_2$. Using this inequality in Eq. (2.3) and then

summing from ν_2 to $\nu-1$, we get

$$\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\mathcal{Z}^{\beta}(\eta(s))\le -r(\nu)(\Delta\mathcal{Z}(\nu))^{\alpha}\le L^{\alpha}.$$

By employing the fact that $\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}$ is positive and increasing, we obtain

$$\frac{\mathcal{Z}^{\beta}(\eta(\nu_2))}{\hat{\mathcal{R}}^{\beta}(\eta(\nu_2))}\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\hat{\mathcal{R}}^{\beta}(\eta(s))\le L^{\alpha},$$

a contradiction to (2.6). Therefore, $\lim\limits_{\nu\to\infty}\dfrac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}=\infty$. Hence, proved. □

3. Oscillation criteria

Theorem 3.1. *Let $\alpha>\beta$ and* (2.5) *hold. If*

$$\limsup_{\nu\to\infty}\hat{\mathcal{R}}^{\alpha}(\nu)\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)>0, \tag{3.1}$$

then every unbounded solutions of Eq. (1.1) *oscillates.*

Proof. Let $\mathcal{V}(\nu)$ be an unbounded non-oscillatory solution of Eq. (1.1) such that $\mathcal{V}(\nu)$, $\mathcal{V}(\delta(\nu))$, $\mathcal{V}(\eta(\nu))>0$ for $\nu\ge\nu_1>\nu_0$. Following Lemma 2.1, we raised the following cases:

Case-a. Indeed, (2.5) implies (2.2) for all $\nu\ge\nu_1$. Following the same lines as in the proof of Lemma 2.2, we conclude that this case cannot occur.

Case-b. Since $\mathcal{V}(\nu)$ is unbounded, then there exists a sequence ν_k of ν such that $\lim\limits_{k\to\infty}\nu_k=\infty$ and $\mathcal{V}(\nu_k)=\max\{\mathcal{V}(s):\nu_0\le s\le\nu_k\}$. In view of $\lim\limits_{\nu\to\infty}\delta(\nu)=\infty$, $\delta(\nu)\le\nu$ and $\delta(\nu_k)\ge\nu_0$ for all sufficiently large k, we get

$$\mathcal{V}(\delta(\nu_k))=\max\{\mathcal{V}(s):\nu_0\le s\le\delta(\nu_k)\}\le\max\{\mathcal{V}(s):\nu_0\le s\le\nu_k\}=\mathcal{V}(\nu_k).$$

Therefore, for all large k,

$$\begin{aligned}\mathcal{Z}(\nu_k)=\mathcal{V}(\nu_k)-q(\nu_k)\mathcal{V}^{\gamma}(\delta(\nu_k))&\ge\mathcal{V}(\nu_k)-q(\nu_k)\mathcal{V}^{\gamma}(\nu_k)\\&\ge\left(1-\frac{q(\nu_k)}{\mathcal{V}^{1-\gamma}(\nu_k)}\right)\mathcal{V}(\nu_k)>0\end{aligned}$$

due to $0\le q(\nu)<1$ and $0\le\gamma<1$, a contradiction to $\mathcal{Z}(\nu)<0$.

Case-c. In view of (3.1), there exist $\nu_2>\nu_1+1$ and a constant $\epsilon>0$ such that

$$\hat{\mathcal{R}}^{\alpha}(\nu)\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)>\epsilon\quad\text{for }\nu\ge\nu_2. \tag{3.2}$$

Summing Eq. (2.3) from the point ν_2 to $\nu-1$, we get

$$\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\mathcal{Z}^{\beta}(\eta(s))\le r(\nu_2)(\Delta\mathcal{Z}(\nu_2))^{\alpha}-r(\nu)(\Delta\mathcal{Z}(\nu))^{\alpha}\le -r(\nu)(\Delta\mathcal{Z}(\nu))^{\alpha}.$$

Consequently,

$$\frac{1}{r^{\frac{1}{\alpha}}(\nu)}\left[\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\mathcal{Z}^{\beta}(\eta(s))\right]^{\frac{1}{\alpha}} \leq -\Delta\mathcal{Z}(\nu).$$

Summing the inequality from ν to U and allowing $U \to \infty$, we arrive at

$$\sum_{s=\nu}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\left[\sum_{T=\nu_2}^{s-1}\mathcal{P}(T)\mathcal{Z}^{\beta}(\eta(T))\right]^{\frac{1}{\alpha}} \leq \mathcal{Z}(\nu).$$

Since η is increasing and $\mathcal{Z}$ is decreasing, we have

$$\mathcal{Z}(\eta(T)) \geq \mathcal{Z}(\eta(\nu)) \quad \text{for } \nu \geq T.$$

Therefore,

$$\hat{\mathcal{R}}(\nu)\mathcal{Z}^{\frac{\beta}{\alpha}}(\eta(\nu))\left[\sum_{T=\nu_2}^{\nu-1}\mathcal{P}(T)\right]^{\frac{1}{\alpha}} \leq \mathcal{Z}(\nu). \tag{3.3}$$

Since $\mathcal{Z}$ is positive and decreasing, it follows from Lemma 2.3 and condition (2.5) that

$$\lim_{\nu\to\infty}\mathcal{Z}(\nu) = 0.$$

Hence, for any sufficiently large constant $\ell > 0$, there exists $\nu_3 > \nu_2$ such that

$$0 < \mathcal{Z}(\nu) \leq \ell^{\frac{\alpha}{\beta-\alpha}} \quad \text{for } \nu \geq \nu_3.$$

Since $\alpha > \beta$, we have $\frac{\alpha}{\beta-\alpha} < 0$. Hence, $\ell^{\frac{\alpha}{\beta-\alpha}}$ is small for large values of ℓ. Therefore,

$$\mathcal{Z}^{\frac{\beta-\alpha}{\alpha}}(\nu) \geq \ell \quad \text{for } \nu \geq \nu_3.$$

Using this estimate in (3.3), we have

$$\ell\,\hat{\mathcal{R}}(\nu)\mathcal{Z}(\eta(\nu))\left[\sum_{T=\nu_2}^{\nu-1}\mathcal{P}(T)\right]^{\frac{1}{\alpha}} \leq \mathcal{Z}(\nu),$$

which implies that

$$\hat{\mathcal{R}}^{\alpha}(\nu)\sum_{T=\nu_2}^{\nu-1}\mathcal{P}(T) \leq \frac{1}{\ell^{\alpha}},$$

Now, selecting ℓ sufficiently large so that $\ell > \epsilon^{-\frac{1}{\alpha}}$, we obtain $\frac{1}{\ell^{\alpha}} < \epsilon$. Consequently,

$$\hat{\mathcal{R}}^{\alpha}(\nu)\sum_{T=\nu_2}^{\nu-1}\mathcal{P}(T) \leq \epsilon,$$

a contradiction to (3.2).

Case-d. Since $\mathcal{V}(\nu)$ is unbounded and $\mathcal{Z}(\nu) < 0$, the proof proceeds analogously to that of Case-b.

This completes the proof.

□

Example 3.2. Consider the Euler-type neutral difference equation

$$\Delta\left[\nu^{\alpha+1}\left(\Delta\mathcal{V}(\nu)-\frac{1}{\nu}\mathcal{V}^{\gamma}(\nu-3)\right)^{\alpha}\right]+\frac{\lambda}{\nu^{2\mu}}\,\mathcal{V}^{\beta}(\nu-2)=0,\quad \nu\geq\nu_0, \tag{3.4}$$

where α and β are ratios of odd positive integers with $\alpha>\beta$, and $0\leq\gamma<1$. Moreover,

$$\delta(\nu)=\nu-3,\quad \eta(\nu)=\nu-2,\quad \mathcal{P}(\nu)=\frac{\lambda}{\nu^{2\mu}},\quad \lambda>0,\quad \mu=\beta-\alpha,\quad q(\nu)=\frac{1}{\nu}.$$

Since $r(\nu)=\nu^{\alpha+1}$, then it follows that $r^{-\frac{1}{\alpha}}(\nu)=\nu^{-(1+\frac{1}{\alpha})}$ and hence

$$\sum_{\nu=\nu_0}^{\infty} r^{-\frac{1}{\alpha}}(\nu)<\infty.$$

Moreover,

$$\hat{\mathcal{R}}(\nu)=\sum_{s=\nu}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\sim\mathcal{C}\,\nu^{-\frac{1}{\alpha}}\Rightarrow\quad \hat{\mathcal{R}}^{\alpha}(\nu)\sim\mathcal{C}_1\,\nu^{-1}.$$

Let $k=\alpha-\beta>0$. Then $\mu=\beta-\alpha=-k$, and thus

$$\mathcal{P}(\nu)=\frac{1}{\nu^{2(\beta-\alpha)}}=\nu^{2k}.$$

Therefore,

$$\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)=\sum_{s=\nu_2}^{\nu-1}s^{2k}\sim\frac{\nu^{2k+1}}{2k+1}=\mathcal{C}_2\,\nu^{2k+1},$$

Thus,

$$\hat{\mathcal{R}}^{\alpha}(\nu)\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)\sim\mathcal{C}_1\nu^{-1}\cdot\mathcal{C}_2\nu^{2k+1}=\mathcal{C}_3\nu^{2k}.$$

Hence, from (3.1),

$$\limsup_{\nu\to\infty}\hat{\mathcal{R}}^{\alpha}(\nu)\sum_{s=\nu_2}^{\nu-1}\mathcal{P}(s)>0.$$

Further, from (2.5),

$$\begin{aligned}\sum_{s=\nu_2}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\left[\sum_{T=\nu_2}^{s-1}\mathcal{P}(T)\right]^{\frac{1}{\alpha}}&\sim\mathcal{C}_2^{\frac{1}{\alpha}}\sum_{s=\nu_2}^{\infty}s^{-(1+\frac{1}{\alpha})}s^{\frac{2k+1}{\alpha}}\\&=\mathcal{C}_2^{\frac{1}{\alpha}}\sum_{s=\nu_2}^{\infty}s^{1-\frac{2\beta}{\alpha}}.\end{aligned}$$

Since $\alpha>\beta$, we have $1-\frac{2\beta}{\alpha}>-1$, and hence the above series diverges.

Hence, all the hypotheses of Theorem 3.1 are satisfied, and therefore every unbounded solution of Eq. (3.4) is oscillatory.

In [10, Remark 4.1], it was asked "**whether it is possible to establish conditions ensuring that every solution of** (1.5) **oscillates**". The following theorem provides an affirmative answer to this question for the more general equation (1.1). To this end, we introduce the notation

$$\mathcal{Q}(\nu) = \frac{\mathcal{P}(\nu)}{(q(\delta^{-1}(\eta(\nu))))^{\frac{\beta}{\gamma}}}.$$

In the following theorem, we consider only the case where

$$q(\delta^{-1}(\eta(\nu))) > 0;$$

hence $\mathcal{Q}(\nu)$is well-defined.

Theorem 3.3. *Assume that* $\alpha > \beta$ *and that* (2.5) *and* (3.1) *hold. If*

$$\limsup_{\nu\to\infty} \sum_{s=h(\nu)}^{\nu-1} \hat{\mathcal{R}}^{\frac{\beta}{\gamma}}(h(s), h(\nu))\mathcal{Q}(s) \begin{cases} > 1 \text{ for } \beta = \alpha\gamma, \\ = \infty \text{ for } \beta < \alpha\gamma, \end{cases} \tag{3.5}$$

then Eq. (1.1) *oscillates.*

Proof. Let $\mathcal{V}(\nu)$ be a non-oscillatory solution of (1.1). Then following Lemma 2.1, we have the following four cases:

Case-a. Indeed, (2.5) implies (2.2) for $\nu \geq \nu_1$. The rest of the proof follows from Lemma 2.2.

Case-b. In this case, $\mathcal{Z}(\nu)$ is unbounded. Since $q(\nu)$ is bounded and $\mathcal{Z}(\nu)$ is unbounded, $\mathcal{V}(\nu)$ can not be bounded. Then there exists a sequence ν_k of ν such that $\lim_{k\to\infty} \nu_k = \infty$ and $\mathcal{V}(\nu_k) = \max\{\mathcal{V}(s) : \nu_0 \leq s \leq \nu_k\}$. In view of $\lim_{\nu\to\infty} \delta(\nu) = \infty$, $\delta(\nu) \leq \nu$ and $\delta(\nu_k) \geq \nu_0$ for all sufficiently large k, we get

$$\mathcal{V}(\delta(\nu_k)) = \max\{\mathcal{V}(s) : \nu_0 \leq s \leq \delta(\nu_k)\} \leq \max\{\mathcal{V}(s) : \nu_0 \leq s \leq \nu_k\} = \mathcal{V}(\nu_k).$$

Therefore, for all large k,

$$\begin{aligned} \mathcal{Z}(\nu_k) = \mathcal{V}(\nu_k) - q(\nu_k)\mathcal{V}^\gamma(\delta(\nu_k)) &\geq \mathcal{V}(\nu_k) - q(\nu_k)\mathcal{V}^\gamma(\nu_k) \\ &\geq \left(1 - \frac{q(\nu_k)}{\mathcal{V}^{1-\gamma}(\nu_k)}\right)\mathcal{V}(\nu_k) > 0, \end{aligned}$$

a contradiction to $\mathcal{Z}(\nu) < 0$.

Case-c. The proof follows the same arguments as those used in Case-c of Theorem 3.1.

Case-d. Let

$$\mathcal{U}(\nu) = -\mathcal{Z}(\nu) = -\mathcal{V}(\nu) + q(\nu)\mathcal{V}^\gamma(\delta(\nu)) \leq q(\nu)\mathcal{V}^\gamma(\delta(\nu)),$$

which implies that

$$\mathcal{V}(\delta(\nu)) \geq \left(\frac{\mathcal{U}(\nu)}{q(\nu)}\right)^{\frac{1}{\gamma}},$$

that is,

$$\mathcal{V}(\nu) \geq \left(\frac{\mathcal{U}(\delta^{-1}(\nu))}{q(\delta^{-1}(\nu))}\right)^{\frac{1}{\gamma}}.$$

Using the last inequality in (1.1), we get

$$\Delta[r(\nu)(\Delta\mathcal{U}(\nu))^{\alpha}] = \mathcal{P}(\nu)\mathcal{V}^{\beta}(\eta(\nu)) \geq \frac{\mathcal{P}(\nu)}{(q(\delta^{-1}(\eta(\nu))))^{\frac{\beta}{\gamma}}}\mathcal{U}^{\frac{\beta}{\gamma}}(\delta^{-1}(\eta(\nu))),$$

which implies that

$$\Delta[r(\nu)(\Delta\mathcal{U}(\nu))^{\alpha}] \geq \mathcal{Q}(\nu)\mathcal{U}^{\frac{\beta}{\gamma}}(h(\nu)). \tag{3.6}$$

Indeed,

$$-\Delta\mathcal{U}(s) = \frac{-r^{\frac{1}{\alpha}}(s)\Delta\mathcal{U}(s)}{r^{\frac{1}{\alpha}}(s)},$$

which for $M > T \geq \nu_1$ is equivalent to

$$\begin{aligned}\mathcal{U}(T) - \mathcal{U}(M) &\geq -r^{\frac{1}{\alpha}}(M)\Delta\mathcal{U}(M)\sum_{s=T}^{M-1}\frac{1}{r^{\frac{1}{\alpha}}(s)}\\ &= \left(-\left(r(M)(\Delta\mathcal{U}(M))^{\alpha}\right)^{\frac{1}{\alpha}}\right)\hat{\mathcal{R}}(T,M),\end{aligned}$$

that is,

$$\mathcal{U}(T) \geq \left(-\left(r(M)(\Delta\mathcal{U}(M))^{\alpha}\right)^{\frac{1}{\alpha}}\right)\hat{\mathcal{R}}(T,M).$$

For $\nu \geq s \geq \nu_2$, setting $M = h(\nu)$ and $T = h(s)$ in the preceding inequality yields

$$\mathcal{U}(h(s)) \geq \left(-\left(r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha}\right)^{\frac{1}{\alpha}}\right)\hat{\mathcal{R}}(h(s),h(\nu)).$$

Summing the inequality (3.6) from $h(\nu)$ to $\nu - 1$, we get

$$r(\nu)(\Delta\mathcal{U}(\nu))^{\alpha} - r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha} = \sum_{s=h(\nu)}^{\nu-1}\mathcal{Q}(s)\mathcal{U}^{\frac{\beta}{\gamma}}(h(s)),$$

which implies that

$$\begin{aligned}-r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha} &\geq \sum_{s=h(\nu)}^{\nu-1}\mathcal{Q}(s)\mathcal{U}^{\frac{\beta}{\gamma}}(h(s))\\ &\geq \left(-r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha}\right)^{\frac{\beta}{\alpha\gamma}}\sum_{s=h(\nu)}^{\nu-1}\hat{\mathcal{R}}^{\frac{\beta}{\gamma}}(h(s),h(\nu))\mathcal{Q}(s).\end{aligned}$$

Setting

$$\hat{\mathcal{U}}(\nu) = -r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha} \tag{3.7}$$

in the preceding inequality, we have

$$\hat{\mathcal{U}}(\nu) \geq \hat{\mathcal{U}}^{\frac{\beta}{\alpha\gamma}}(\nu)\sum_{s=h(\nu)}^{\nu-1}\hat{\mathcal{R}}^{\frac{\beta}{\gamma}}(h(s),h(\nu))\mathcal{Q}(s),$$

or,

$$\sum_{s=h(\nu)}^{\nu-1}\hat{\mathcal{R}}^{\frac{\beta}{\gamma}}(h(s),h(\nu))\mathcal{Q}(s) \leq \hat{\mathcal{U}}^{1-\frac{\beta}{\alpha\gamma}}(\nu). \tag{3.8}$$

If $\beta = \alpha\gamma$, then inequality (3.8) reduces to

$$\sum_{s=h(\nu)}^{\nu-1} \hat{\mathcal{R}}^{\frac{\beta}{\gamma}}(h(s), h(\nu))\mathcal{Q}(s) \leq 1,$$

which is in contradiction with (3.5).

Since $\mathcal{U}(\nu) = -\mathcal{Z}(\nu) > 0$, we have $\Delta\mathcal{U}(\nu) = -\Delta\mathcal{Z}(\nu) < 0$. Moreover, from (1.1) it follows

$$\Delta\left[r(\nu)\left(\Delta\mathcal{U}(\nu)\right)^{\alpha}\right] = \mathcal{P}(\nu)\mathcal{V}^{\beta}(\eta(\nu)) \geq 0.$$

Therefore,

$$\hat{\mathcal{U}}(\nu) = -r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha} > 0$$

and

$$\Delta\hat{\mathcal{U}}(\nu) = -\Delta\left[r(h(\nu))(\Delta\mathcal{U}(h(\nu)))^{\alpha}\right] < 0.$$

Now consider the case $\beta < \alpha\gamma$. Since $\hat{\mathcal{U}}(\nu)$ is positive and decreasing, it is bounded, and hence

$$\lim_{\nu\to\infty} \hat{\mathcal{U}}^{1-\frac{\beta}{\alpha\gamma}}(\nu) \quad \text{exists and is finite.}$$

This, together with (3.8), contradicts (3.5). This completes the proof. □

Remark 3.4. The results obtained in Theorem 3.1 extend those in [20]. Indeed, equation (1.3) is a particular case of (1.1) corresponding to $\alpha = 1$, $\gamma = 1$, $\delta(\nu) = \nu - k$, and $\eta(\nu) = \nu - l$. Therefore, Theorem 3.1 generalizes the corresponding results in [20]. Moreover, the inclusion of a nonlinear Δ-operator with $\alpha > 0$, a sublinear neutral term with $0 \leq \gamma < 1$, and variable delay arguments allows for a broader class of difference equations, thereby extending the applicability of the existing theory.

Remark 3.5. The results in Theorem 3.1 also extend those in [14]. Equation (1.4) corresponds to a special case of (1.1) with constant delays. A key difference is that (1.4) involves a positive neutral coefficient, whereas (1.1) allows a negative one, which substantially alters the oscillatory behaviour of solutions and increases the analytical complexity. Therefore, the results obtained here not only generalize but also provide a genuinely new contribution beyond the existing criteria in [14].

Theorem 3.6. *Assume that* $\alpha < \beta$, (2.5) *and* (2.6) *hold. If*

$$\limsup_{\nu\to\infty} \hat{\mathcal{R}}^{\alpha}(\nu)\hat{\mathcal{R}}^{\beta-\alpha}(\eta(\nu)) \sum_{s=\nu_2}^{\nu-1} \mathcal{P}(s) > 0, \tag{3.9}$$

then every unbounded solutions of Eq. (1.1) *oscillates.*

Proof. Let $\mathcal{V}(\nu)$ be an unbounded non-oscillatory solution of (1.1) such that $\mathcal{V}(\nu) > 0$ for $\nu \geq \nu_1 > \nu_0$. Next, we proceed as in Theorem 3.1 to obtain the following possible cases:

Case-a. The proof is same as Case-a of Theorem 3.1.

Case-c. By (3.9), there exist $\nu_2 > \nu_1 + 1$ and a constant $c > 0$ such that for infinitely many ν with $\nu \geq \nu_2$

$$\hat{\mathcal{R}}^{\alpha}(\nu)\hat{\mathcal{R}}^{\beta-\alpha}(\eta(\nu)) \sum_{s=\nu_2}^{\nu-1} \mathcal{P}(s) > c. \tag{3.10}$$

It follows from Lemma 2.3 and condition (2.6) that the function $\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)}$ is positive, increasing, and satisfies

$$\lim_{\nu\to\infty} \frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)} = \infty.$$

Therefore, for any sufficiently large constant $b > 0$, there exists $\nu_3 > \nu_2$ such that

$$\frac{\mathcal{Z}(\nu)}{\hat{\mathcal{R}}(\nu)} \geq b^{\frac{\alpha}{\beta-\alpha}} \quad \text{for } \nu \geq \nu_3.$$

Consequently,

$$\mathcal{Z}^{\frac{\beta-\alpha}{\alpha}}(\nu) \geq b\hat{\mathcal{R}}^{\frac{\beta-\alpha}{\alpha}}(\nu) \quad \text{for } \nu \geq \nu_3.$$

Since $\eta(\nu) \to \infty$, there exists $\nu_4 > \nu_3$ such that $\eta(\nu) \geq \nu_3$ for $\nu \geq \nu_4$. Hence,

$$\mathcal{Z}^{\frac{\beta-\alpha}{\alpha}}(\eta(\nu)) \geq b\hat{\mathcal{R}}^{\frac{\beta-\alpha}{\alpha}}(\eta(\nu)) \quad \text{for } \nu \geq \nu_4.$$

Using the above inequality in (3.3), we have

$$\mathcal{Z}(\nu) \geq b\hat{\mathcal{R}}(\nu)\mathcal{Z}(\eta(\nu))\hat{\mathcal{R}}^{\frac{\beta-\alpha}{\alpha}}(\eta(\nu)) \left[\sum_{T=\nu_2}^{\nu-1} \mathcal{P}(T)\right]^{\frac{1}{\alpha}}.$$

Since $\mathcal{Z}$ is decreasing and $\eta(\nu) \leq \nu$, it follows that

$$\mathcal{Z}(\nu) \leq \mathcal{Z}(\eta(\nu)).$$

Therefore,

$$\hat{\mathcal{R}}^{\alpha}(\nu)\hat{\mathcal{R}}^{\beta-\alpha}(\eta(\nu)) \left[\sum_{T=\nu_2}^{\nu-1} \mathcal{P}(T)\right] \leq \frac{1}{b^{\alpha}},$$

Now, choosing b large such that $\frac{1}{b^{\alpha}} < c$, we arrive at

$$\hat{\mathcal{R}}^{\alpha}(\nu)\hat{\mathcal{R}}^{\beta-\alpha}(\eta(\nu)) \left[\sum_{T=\nu_2}^{\nu-1} \mathcal{P}(T)\right] \leq c,$$

a contradiction to (3.10).

The proofs for Case-b and Case-d are same as in Theorem 3.1. Hence, proved. □

Example 3.7. Consider the difference equation

$$\Delta\left(e^{\nu}(\Delta\mathcal{V}(\nu) - q(\nu)\mathcal{V}^{\gamma}(\delta(\nu)))^{\alpha}\right) + (1+\nu)e^{3\nu}\mathcal{V}^{\beta}(\nu-5) = 0, \tag{3.11}$$

where

$$\alpha = 1, \quad \beta = 3, \quad \gamma = \frac{1}{2}, \quad \delta(\nu) = \nu - 3, \quad \eta(\nu) = \nu - 5,$$

$$\mathcal{P}(\nu) = (1+\nu)e^{3\nu}, \quad q(\nu) = \frac{1}{3\nu}, \quad r(\nu) = e^{\nu}.$$

Clearly, $\hat{\mathcal{R}}(\nu) = \sum_{s=\nu}^{\infty} e^{-s}$ is a geometric series with common ratio $0 < e^{-1} < 1$, then

$$\hat{\mathcal{R}}(\nu) = \frac{1}{1-e^{-1}}e^{-\nu} = \frac{e}{e-1}e^{-\nu} := \mathcal{C}e^{-\nu},$$

where $\mathcal{C} = \frac{e}{e-1}$ and

$$\sum_{s=\nu_2}^{\nu-1} \mathcal{P}(s) = \sum_{s=\nu_2}^{\nu-1} e^{3s} + \sum_{s=\nu_2}^{\nu-1} se^{3s} \sim \frac{e^{3\nu}}{e^3-1} + \frac{(\nu-1)e^{3\nu}}{(e^3-1)^2} \sim \mathcal{C}_1 \nu e^{3\nu},$$

where $\mathcal{C}_1 > 0$ is a constant. Hence, from (2.5),

$$\sum_{s=\nu_2}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left[\sum_{T=\nu_2}^{s-1} \mathcal{P}(T)\right]^{\frac{1}{\alpha}} \sim \mathcal{C}e^{-s} \cdot \mathcal{C}_1 s e^{3s} = \mathcal{C}_2 s e^{2s}$$

which diverges.
Using $\hat{\mathcal{R}}^{\beta}(\eta(s)) = \mathcal{C}^3 e^{-3(s-5)} = \mathcal{C}_3 e^{-3s}$, if follows from (2.6) that

$$\sum_{s=\nu_2}^{\infty} \mathcal{P}(s)\hat{\mathcal{R}}^{\beta}(\eta(s)) = \mathcal{C}_3 \sum_{s=\nu_2}^{\infty} (1+s)e^{3s}e^{-3s},$$

which diverges.
Finally, using

$$\hat{\mathcal{R}}^{\alpha}(\nu)\hat{\mathcal{R}}^{\beta-\alpha}(\eta(\nu)) \sum_{s=\nu_2}^{\nu-1} \mathcal{P}(s) \sim \left(\mathcal{C}e^{-\nu}\right) \cdot \left(\mathcal{C}^2 e^{-2(\nu-5)}\right) \cdot \left(\mathcal{C}_1 s e^{3\nu}\right) = \mathcal{C}_4 \nu\, e^{\nu},$$

if follows from (3.9) that

$$\limsup_{\nu\to\infty} \hat{\mathcal{R}}^{\alpha}(\nu)\hat{\mathcal{R}}^{\beta-\alpha}(\eta(\nu)) \sum_{s=\nu_2}^{\nu-1} \mathcal{P}(s) > 0.$$

Hence, all the hypotheses of Theorem 3.6 are satisfied, and therefore every unbounded solution of Eq. (3.11) oscillates.

4. Concluding remarks

This paper investigates the oscillatory behaviour of Eq. (1.1) under the condition $(\mathcal{A}_3)$, and establishes sufficient criteria ensuring that all solutions are oscillatory. The obtained results significantly extend and generalize earlier works by Dinakar et al. [14], Seghar et al. [20], and Thandapani and Selvarangam [21].

Future research may focus on relaxing the imposed conditions or extending the analysis to more general classes of functional difference equations, including those with variable delays or different nonlinearities.

Open Problem 1 (Negative neutral coefficient). It would be interesting to investigate the behaviour of solutions of (1.1) when the neutral coefficient $q(\nu)$ is allowed to be negative eventually. In this case, the techniques developed in the present paper may

not be directly applicable, since the positivity and monotonicity properties of the associated functional $\mathcal{Z}(\nu)$ can fail. New analytical methods are therefore required to establish oscillation or asymptotic criteria in this setting, as also suggested by Dzurina et al. [9].

Open Problem 2 (Super-linear neutral term). Another interesting problem is to extend the present results to the second-order neutral difference equation

$$\Delta[r(\nu)(\Delta\mathcal{Z}(\nu))^{\alpha}] + \mathcal{P}(\nu)\mathcal{V}^{\beta}(\eta(\nu)) = 0,\ \nu \geq \nu_0,$$

where

$$\mathcal{Z}(\nu) = \mathcal{V}(\nu) - q(\nu)\mathcal{V}^{\gamma}(\delta(\nu)), \quad \gamma > 1.$$

All other assumptions $(\mathcal{A}_2)$–$(\mathcal{A}_4)$ are retained as in the original formulation, with the only modification that $\gamma > 1$ in place of $0 \leq \gamma < 1$.

The presence of the nonlinear neutral term with $\gamma > 1$ introduces additional difficulties in estimating higher-order differences. Establishing oscillation criteria for this class of equations remains an open problem.

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