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Blow-up results for a class of p-Laplacian heat equations with logarithmic nonlinearity

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Abstract. In this paper, we consider an initial-boundary value problem for the following mixed pseudo-parabolic $p(.)$-Laplacian type equation with logarithmic nonlinearity on a bounded and regular domain. By adapting the first-order differential inequality method, we establish a blow-up criterion for the solutions and obtain an upper bound for the blow-up time. Besides, we show that blow-up may be prevented under appropriate smallness conditions on the initial datum, in which case we also establish decay estimates in the $H_0^1(\Omega)$-norm as $t \to +\infty$. We also give a two-dimensional numerical example to confirm the validity of our theoretical results.

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Keywords: Pseudo-parabolic equation, logarithmic nonlinearity, variable nonlinearity, blow-up time, bounds of the blow-up time.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we look at the following pseudo-parabolic $p(.)$-Laplacian equation with logarithmic nonlinearity:

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right) = |u|^{q(.)-2}u\ln(|u|), & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) \neq 0, & x \in \Omega \end{cases} \tag{1.1}$$

In (1.1), $p(.)$ and $q(.)$ are given measurable functions on $\overline{\Omega}$ satisfying

$$2 \leq p_1 \leq p(x) \leq p_2 < q_1 \leq q(x) \leq q_2 < p^*(x),$$

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where we have denoted

$$p_1 := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p_2 = \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

$$q_1 := \operatorname*{ess\,inf}_{x\in\Omega} q(x), \qquad q_2 = \operatorname*{ess\,sup}_{x\in\Omega} q(x).$$

and

$$p^*(x) := \begin{cases} \dfrac{np(x)}{n-p(x)}, & \text{if } n > p_2, \\ +\infty, & \text{if } n \leq p_2 \end{cases}. \tag{1.2}$$

Moreover, we assume that $p(.)$ and $q(.)$ satisfy the following Zhikov-Fan uniform local continuity condition: there exist two constants, $M_1 > 0$ and $M_2 > 0$ such that

$$\begin{array}{ll} |q(x)-q(y)| \leq \frac{M_1}{|\log|x-y||}, & \text{for all } (x,y)\in\Omega\times\Omega \text{ with } |x-y|<\frac{1}{2}. \\ |p(x)-p(y)| \leq \frac{M_2}{|\log|x-y||}, & \text{for all } (x,y)\in\Omega\times\Omega \text{ with } |x-y|<\frac{1}{2}. \end{array} \tag{1.3}$$

Finally, the initial datum in (1.1) is any given function $u_0 \in W_0^{1,p(.)}(\Omega)$, where $W_0^{1,p(.)}(\Omega)$ denotes the generalization to the variable exponent case of the Sobolev space $W_0^{1,p}(\Omega)$. We refer to Section 2 for more detail.

The nonlinear term $\operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right)$ in (1.1) is the so-called $p(.)$-Laplacian operator, which is sometimes symbolized by

$$\Delta_{p(.)}u = \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right)$$

and is a generalization of the well-known p-Laplacian, that corresponds to taking $p(.) \equiv p \in \mathbb{R}$ constant.

This operator can be prolonged to a monotone operator between the space $W_0^{1,p(.)}(\Omega)$ and its dual space $W^{-1,p'(.)}(\Omega)$ as follows

$$\begin{cases} -\Delta_{p(.)}u : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega) \\ \langle -\Delta_{p(.)}u, \phi\rangle_{p(.)} = \displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\phi\,\mathrm{d}x, \quad 2 \leq p_1 \leq p(x) \leq p_2 < +\infty, \end{cases}$$

where $\langle .,.\rangle_{p(.)}$ indicates the duality pairing between $W_0^{1,p(.)}(\Omega)$ and $W^{-1,p'(.)}(\Omega)$, and $p'(.)$ denotes the conjugate exponent such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Nonlinear pseudo-parabolic equations like (1.1) arise in the description of several problems in hydrodynamics, thermodynamics, nonlinear elasticity, image processing and filtration theory (see [5, 28, 30, 31]). Furthermore, we can mention interesting applications in the study of electrorheological fluids, whose viscosity depends on the electric field in the fluid itself (see [2, 18, 19, 24, 43]).

In the case in which the exponents $p(.)$ and $q(.)$ are constants, that is when $p(.) \equiv p \in \mathbb{R}$ and $q(.) \equiv q \in \mathbb{R}$, the corresponding PDE

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{q-2}u\ln(|u|), \quad (x,t)\in\Omega\times(0,+\infty) \tag{1.4}$$

has been largely studied by the mathematical community.

In [44, 47], the linear version of (1.4) (i.e. with the right-hand side equal to zero) has been considered in the case $p = 2$ in which the p-Laplacian $-\Delta_p$ becomes the standard Laplace operator $-\Delta$. In particular, results of existence, uniqueness and regularity of the solutions were obtained.

Later on, these results have been extended to the general case of (1.4). In several papers, the asymptotic behavior of weak solutions to (1.4) with initial datum $u_0 \in W_0^{1,p}(\Omega)$ has been studied. In more detail,

- The case $p = q > 2$ has been considered in [35].
- The case $1 < p = q < 2$ has been considered in [11].
- The case $2 < p < q < p(1 + \frac{2}{n})$ has been considered in [25].
- The case $1 < p \leq q < p^*$, with p^* as in (1.2), has been considered in [20].

In all the aforementioned references, the global-in-time existence or the finite time blow-up of such solutions has been characterized in terms of the values of the exponents p and q. We shall stress, however, that the methodology employed in [25, 35, 11] to prove the blow-up of weak solutions to (1.4) does not allow the authors to address the relevant issue of estimating the blow-up time and rate (we refer to [34, 36, 37, 38, 39, 45] for the analysis of these issues in the case of other interesting non-linear evolution equations). A partial answer to these relevant questions has been recently given in [16] where, under the same conditions as in [25] on the exponents $p, q \in \mathbb{R}$ (that is, $2 < p < q < p(1 + \frac{2}{n})$), explicit estimates for the blow-up time are provided.

Finally, we refer to [10, 12, 13, 15, 16, 17, 23, 27, 29, 32, 40, 49] for the study of the asymptotic behavior of weak solutions to (1.4) under high initial energy level conditions.

As for the case of variable exponents $p(.)$ and $q(.)$, to the best of our knowledge, there are no results in the existing literature in the spirit of those that we have just recalled. As a matter of fact, many of the techniques employed in the previous references to deal with (1.4) become unsuccessful to analyze (1.1) when $p(.)$ and $q(.)$ are measurable functions on $\overline{\Omega}$.

The goal of this paper is precisely to address this variable measurable coefficients framework. In particular, we shall identify sufficient conditions on $p(.)$, $q(.)$ and the initial datum u_0 for which the blow-up and the non-blow-up phenomena of solutions to (1.1) occur, also providing estimates for the blow-up time.

In this regard, we shall mention that this kind of results are already available for the hyperbolic version of (1.1). The interested reader may refer for instance to [1, 6, 8, 9, 14, 33, 41, 42, 46]. In particular, in [6, 8], the authors have discussed the Dirichlet problem for the following equation

$$u_{tt} = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) + \alpha\Delta u_t + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t),$$
$$(x,t) \in \Omega \times (0,+\infty)$$

with negative initial energy and, under suitable conditions on the functions a, b, f, p, σ, they have used Galerkin and energy methods to establish local existence, global existence and blow-up of the solutions.

The present paper is organized as follows: in Section 2, we will introduce some preliminary concepts and notations that will be of use in our further analysis. Secondly, we will study the finite-time blow-up of solutions in Section 3. The global existence of weak solutions to (1.1) is addressed in Section 4, where we also establish decay estimates in the $H_0^1(\Omega)$-norm as $t \to +\infty$. Finally, in Section 5, this decay result is illustrated by a two-dimensional numerical example.

2. Preliminaries

In this section, we present some preliminary concepts and notations that we shall employ in our further analysis. Let us start by introducing the variable-order Lebesgue space $L^{p(.)}(\Omega)$, which is defined for all $p : \Omega \to [1, +\infty]$ a measurable function as

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x < +\infty \right\}.$$

We then know that $L^{p(.)}(\Omega)$ is a Banach space, equipped with the Luxemburg-type norm

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0, \quad \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

Next, we define the variable-order Sobolev space $W^{1,p(.)}(\Omega)$ as

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) \ : \ \nabla u \in L^{p(.)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)}^2 + \|\nabla u\|_{p(.)}^2. \tag{2.1}$$

Moreover, in what follows we will need the following embedding result from [18, 21].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. The following hold.*

1. *If $p \in C(\overline{\Omega})$ and $q : \Omega \to [1, +\infty)$ is a measurable function such that*
$$\operatorname*{ess\,inf}_{x \in \Omega} \big(p^*(x) - q(x) \big) > 0,$$
with p^* defined as in (1.2), then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ with continuous and compact embedding.
2. *If p satisfy (1.3), then $\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)}$ for all $u \in W_0^{1,p(.)}(\Omega)$. In particular, $\|u\|_{1,p(.)} = \|\nabla u\|_{p(.)}$ defines a norm on $W_0^{1,p(.)}(\Omega)$ which is equivalent to (2.1).*

Let us now introduce our notion of weak solution to (1.1), and give a first result of local-in-time existence and uniqueness.

Definition 2.2. *Let $u_0 \in W_0^{1,p(.)}(\Omega)$ and $T > 0$. A function*

$$u \in L^\infty \left(0, T; W_0^{1,p(.)}(\Omega) \right) \text{ with } u_t \in L^2 \left(0, T; H_0^1(\Omega) \right),$$

is called a weak solution to (1.1) *if the identity*

$$\langle u_t, v\rangle + \langle \nabla u_t, \nabla v\rangle + \left\langle |\nabla u|^{p(.)-2}\nabla u, \nabla v\right\rangle = \int_\Omega |u|^{q(.)-2}u\ln(|u|)v\,\mathrm{d}x$$

holds for any $v \in W_0^{1,p(.)}(\Omega)$*, and for a.e.* $t \in [0,T]$*.*

Theorem 2.3. *For all* $u_0 \in W_0^{1,p(.)}(\Omega)$*, there exists* $T_0 > 0$ *such that the problem* (1.1) *admits a unique weak solution* u *on* $[0,T_0]$*.*

Proof. The result can be easily obtained by using the Faedo-Galerkin approach and combining the ideas from [3, 4, 19, 22, 48] with the ones from [7] (see also [11]). We leave the details to the reader. □

Let now $\sigma > 0$ be a positive constant satisfying

$$0 < \sigma < \begin{cases} \operatorname{ess\,inf}_{x\in\Omega}\left(p^*(x) - q(x)\right), & \text{if } p(.) < n, \\ +\infty, & \text{if } p(.) \geq n. \end{cases} \tag{2.2}$$

Then, from Lemma 2.1 we have that $W_0^{1,q(.)+\sigma}(\Omega) \hookrightarrow L^{q(.)+\sigma}(\Omega)$, and there exists a positive constant $B_\sigma > 0$ such that

$$\|u\|_{q(.)+\sigma} \leq B_\sigma \|\nabla u\|_{q(.)+\sigma}. \tag{2.3}$$

Finally, let us indicate with α_1, B_1 and E_1 the following positive constants:

$$\alpha_1 = \left(\frac{e\sigma q_1}{q_1+\sigma}B_1^{-(q_1+\sigma)}\right)^{\frac{p_2}{q_1-p_2+\sigma}} \tag{2.4a}$$

$$B_1 > \max\left(1, B_\sigma\right) \tag{2.4b}$$

$$E_1 = \left(\frac{1}{p_2} - \frac{1}{q_1+\sigma}\right)\alpha_1. \tag{2.4c}$$

3. Finite-time blow-up of solutions

In this section, we obtain a blow-up criterion for the solutions to (1.1), also providing an upper bound for the blow-up time. To this end, let us introduce the energy associated with the solution of our problem (1.1), which is defined as follows:

$$E(t) := \int_\Omega \frac{1}{p(.)}|\nabla u|^{p(.)}\,\mathrm{d}x - \int_\Omega \frac{1}{q(.)}|u|^{q(.)}\ln(|u|)\,\mathrm{d}x + \int_\Omega \frac{1}{q^2(.)}|u|^{q(.)}\,\mathrm{d}x \tag{3.1}$$

We have the following result providing upper and lower bounds for $E(t)$.

Lemma 3.1. *Let* $u_0 \in W^{1,p(.)}(\Omega)$ *and* $E(t)$ *be the energy associated with the corresponding solution of* (1.1)*, as defined in* (3.1)*. Let* $g : [0,+\infty) \to \mathbb{R}$ *be given by*

$$g(\xi) := \frac{1}{p_2}\min\left(\xi^{\frac{p_1}{p_2}}, \xi\right) - \frac{1}{e\sigma q_1}\max\left(B_1^{q_2+\sigma}\xi^{\frac{q_2+\sigma}{p_2}}, B_1^{q_1+\sigma}\xi^{\frac{q_1+\sigma}{p_2}}\right), \tag{3.2}$$

where σ and B_1 are the constants defined in (2.2) *and* (2.4b)*, respectively. Let $\alpha : [0,+\infty) \to [0,+\infty)$ be defined as*

$$\alpha(t) = \|\nabla u(t)\|_{p(.)}^{p_2}. \tag{3.3}$$

Then, we have

$$g(\alpha) \leq E(t) \leq E(0). \tag{3.4}$$

Proof. First of all, a direct computation yields that

$$\frac{dE(t)}{dt} = -\int_\Omega u_t^2 \,\mathrm{d}x - \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x \leq 0. \tag{3.5}$$

Hence, E is decreasing with respect to t, and the inequality $E(t) \leq E(0)$ immediately holds for all $t \geq 0$. As for the lower estimate in (3.4), we have that

$$\int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \,\mathrm{d}x \leq \int_\Omega \frac{1}{e\sigma q(.)} |u|^{q(.)+\sigma} \,\mathrm{d}x \leq \frac{1}{e\sigma q_1} \int_\Omega |u|^{q(.)+\sigma} \,\mathrm{d}x, \tag{3.6}$$

where we used that

$$\ln(\xi) \leq \frac{1}{e\sigma}\xi^\sigma, \quad \text{for all } \xi, \sigma > 0. \tag{3.7}$$

Therefore, we get from (3.1), (3.3), (3.2) and (3.6) that

$$\begin{aligned} E(t) &\geq \frac{1}{p_2} \min\left(\|\nabla u\|_{p(.)}^{p_1}, \|\nabla u\|_{p(.)}^{p_2}\right) - \frac{1}{e\sigma q_1} \max\left(\|u\|_{q(.)+\sigma}^{q_2+\sigma}, \|u\|_{q(.)+\sigma}^{q_1+\sigma}\right) \\ &\geq \frac{1}{p_2} \min\left(\|\nabla u\|_{p(.)}^{p_1}, \|\nabla u\|_{p(.)}^{p_2}\right) \\ &\quad - \frac{1}{e\sigma q_1} \max\left(\left(B_1 \|\nabla u\|_{p(.)}\right)^{q_2+\sigma}, \left(B_1 \|\nabla u\|_{p(.)}\right)^{q_1+\sigma}\right) = g(\alpha). \end{aligned} \tag{3.8}$$

This concludes the proof of (3.4). □

Next, we state some technical lemmas which will be needed in the proof of our main result.

Lemma 3.2. *Let α_1, B_1 and E_1 be given as in* (2.4a), (2.4b) *and* (2.4c)*, respectively. Define the function $h : [0,+\infty) \to \mathbb{R}$ as*

$$h(\xi) = \frac{1}{p_2}\xi - \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \xi^{\frac{q_1+\sigma}{p_2}}. \tag{3.9}$$

Then, the following assertions hold:

1. *h is increasing for $0 < \xi \leq \alpha_1$ and decreasing for $\xi \geq \alpha_1$.*
2. *$h(\alpha_1) = E_1$.*
3. *$\lim_{\xi \to +\infty} h(\xi) = -\infty$.*

Proof. First of all, from the definition (3.9) of the function h we can easily compute

$$h'(\xi) = \frac{1}{p_2} - \frac{q_1+\sigma}{p_2} \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \xi^{\frac{q_1+\sigma}{p_2}-1}.$$

Then, it is straightforward to check that $h'(\xi) > 0$ for $\xi < \alpha_1$ and $h'(\xi) < 0$ for $\xi > \alpha_1$. Besides, the fact that $h(\alpha_1) = E_1$ follows directly from (2.4c) and (3.9). Finally, since by definition of p_2, q_1 and σ we have $q_1 - p_2 + \sigma > 0$, we get

$$\lim_{\xi \to +\infty} h(\xi) = \lim_{\xi \to +\infty} \xi \left(\frac{1}{p_2} - \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \xi^{\frac{q_1-p_2+\sigma}{p_2}} \right) = -\infty.$$

This concludes our proof. □

Lemma 3.3. *Let σ, α_1, B_1 and E_1 be given as in* (2.2), (2.4a), (2.4b) *and* (2.4c), *respectively. Assume that the initial value u_0 is chosen so that*

$$0 \leq E(0) < \frac{p_2}{q_1 + p_2} \left(E_1 - \frac{\sigma \alpha_1}{p_2(q_1 + \sigma)} \right) \quad \text{and} \quad \alpha_1 < \|\nabla u_0\|_{p(.)}^{p_2} \leq B_1^{-p_2}. \tag{3.10}$$

Then there exists a positive constant $\alpha_2 > \alpha_1$ such that

$$\|\nabla u\|_{p(.)}^{p_2} \geq \alpha_2, \quad \text{for all } t \geq 0 \tag{3.11a}$$

$$\int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \mathrm{d}x \geq \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \alpha_2^{\frac{q_1+\sigma}{p_2}} \tag{3.11b}$$

$$\frac{\alpha_2}{\alpha_1} \geq \left((q_1 + \sigma) \left(\frac{1}{p_2} - \frac{E(0)}{\alpha_1} \right) \right)^{\frac{p_2}{q_1+\sigma-p_2}} > 1. \tag{3.11c}$$

Proof. Since from (3.10) we have

$$0 \leq E(0) < \frac{p_2 E_1}{q_1 + p_2} < E_1,$$

it follows from Lemma 3.2 that there exists a positive constant $\alpha_2 > \alpha_1$ such that the initial energy $E(0)$ satisfies $E(0) = h(\alpha_2)$ (see Figure 1). Moreover, since $B_1 > 1$ (see (2.4b)), it follows again from (3.10) that

$$\alpha_0 = \|\nabla u_0\|_{p(.)}^{p_2} \leq B_1^{-p_2} < 1.$$

Therefore, recalling the definitions (3.2) and (3.9) of the functions g and h, since $p_1 < p_2$, and using (3.8), we can conclude that

$$h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2).$$

Hence, it follows from Point 1 in Lemma 3.2 that $\alpha_0 \geq \alpha_2$, and (3.11a) holds for $t = 0$.

To prove (3.11a) for all $t \geq 0$, we argue by contradiction. Suppose that there exists some strictly positive time $t^* > 0$ such that $\|\nabla u(t^*)\|_{p(.)}^{p_2} < \alpha_2$. Then, by the continuity of the $L^{p(.)}$-norm, and since $\alpha_2 > \alpha_1$, we may take t^* such that

$$\alpha_2 > \|\nabla u(t^*)\|_{p(.)}^{p_2} > \alpha_1.$$

Hence, it follows from Lemma 3.2 and (3.8) that

$$E(0) = h(\alpha_2) < h\left(\|\nabla u(t^*)\|_{p(.)}^{p_2} \right) \leq E(t^*).$$

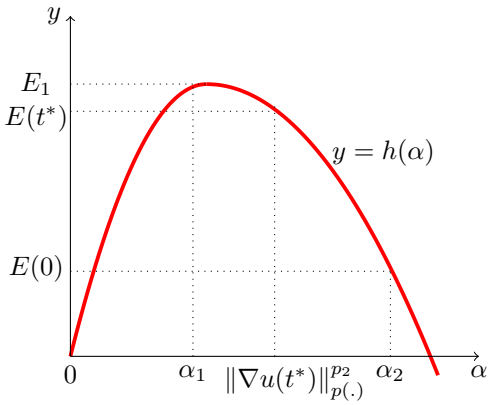

FIGURE 1. Diagram for the proof of Lemma 3.3. We plot the function h, highlighting the quantities α_1, α_2 and $\|\nabla u(t^*)\|^{p_2}_{p(.)}$ with the corresponding values of the energy (E_1, $E(0)$ and $E(t^*)$, respectively).

This contradicts the fact that the energy $E(t)$ is decreasing, as we proved in Lemma 3.1. Therefore, (3.11a) holds.

Let us now prove (3.11b). In this regard, from the definition of the energy (3.1), and since $E(t) \leq E(0)$, we obtain

$$\begin{aligned}\int_\Omega \frac{1}{q(.)}\,|u|^{q(.)}\ln(|u|)\mathrm{d}x &= \int_\Omega \frac{1}{p(.)}\,|\nabla u|^{p(.)}\,\mathrm{d}x + \int_\Omega \frac{1}{q^2(.)}|u|^{q(.)}\,\mathrm{d}x - E(t)\\ &\geq \int_\Omega \frac{1}{p(.)}\,|\nabla u|^{p(.)}\,\mathrm{d}x - E(0)\\ &\geq \frac{1}{p_2}\min\left(\|\nabla u\|^{p_2}_{p(.)}, \|\nabla u\|^{p_1}_{p(.)}\right) - E(0)\\ &\geq \frac{1}{p_2}\min\left(\alpha_2^{\frac{p_1}{p_2}}, \alpha_2\right) - E(0). \end{aligned} \tag{3.12}$$

Moreover, since $\alpha_2 \leq \alpha_0 < 1$ and $p_1 \leq p_2$, we have

$$\min\left(\alpha_2^{\frac{p_1}{p_2}}, \alpha_2\right) = \alpha_2.$$

Hence, it follows from (3.9) and (3.12) that

$$\int_\Omega \frac{1}{q(.)}\,|u|^{q(.)}\ln(|u|)\mathrm{d}x \geq \frac{1}{p_2}\alpha_2 - E(0) = \frac{1}{p_2}\alpha_2 - h(\alpha_2) = \frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\alpha_2^{\frac{q_1+\sigma}{p_2}}.$$

Finally, let us prove (3.11c). To this end, since $E(0) < E_1$, recalling the definition of E_1 given in (2.4c) we can readily check that

$$\left((q_1+\sigma)\left(\frac{1}{p_2} - \frac{E(0)}{\alpha_1}\right)\right)^{\frac{p_2}{q_1-p_2+\sigma}} > \left((q_1+\sigma)\left(\frac{1}{p_2} - \frac{E_1}{\alpha_1}\right)\right)^{\frac{p_2}{q_1-p_2+\sigma}} = 1.$$

Hence, the second inequality in (3.11c) holds. As for the first inequality, we can easily compute

$$\begin{aligned} E(0) = h(\alpha_2) &= \alpha_2 \left(\frac{1}{p_2} - \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \alpha_2^{\frac{q_1+\sigma}{p_2}-1} \right) \\ &= \alpha_1 \frac{\alpha_2}{\alpha_1} \left(\frac{1}{p_2} - \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{q_1+\sigma}{p_2}-1} \alpha_1^{\frac{q_1+\sigma}{p_2}-1} \right). \end{aligned}$$

Recalling the definition of α_1 given in (2.4a), and since $\alpha_2 > \alpha_1$ we then have that

$$E(0) = \alpha_1 \frac{\alpha_2}{\alpha_1} \left(\frac{1}{p_2} - \frac{1}{q_1+\sigma} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{q_1+\sigma-p_2}{p_2}} \right) \geq \alpha_1 \left(\frac{1}{p_2} - \frac{1}{q_1+\sigma} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{q_1+\sigma-p_2}{p_2}} \right).$$

Therefore,

$$\frac{E(0)}{\alpha_1} \geq \frac{1}{p_2} - \frac{1}{q_1+\sigma} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{q_1+\sigma-p_2}{p_2}},$$

from which it follows immediately that

$$\frac{\alpha_2}{\alpha_1} \geq \left((q_1+\sigma) \left(\frac{1}{p_2} - \frac{E(0)}{\alpha_1} \right) \right)^{\frac{p_2}{q_1+\sigma-p_2}}$$

Our proof is then concluded. □

Let us define

$$H(t) := E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} - E(t), \quad \text{for } t \geq 0. \tag{3.13}$$

We have the following result.

Lemma 3.4. *Let σ, α_1, B_1 and E_1 be given as in* (2.2), (2.4a), (2.4b) *and* (2.4c), *respectively. Assume that the initial value u_0 is chosen so that*

$$0 \leq E(0) < \frac{p_2}{q_1+p_2} \left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} \right) \quad \text{and} \quad \alpha_1 < \|\nabla u_0\|_{p(.)}^{p_2} \leq B_1^{-p_2}.$$

Then, the functional $H(t)$ defined in (3.13) *satisfies the following estimates:*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \mathrm{d}x, \quad t \geq 0. \tag{3.14}$$

Proof. By Lemma 3.1, we know that $H(t)$ is non-decreasing for all $t \geq 0$. Hence

$$H(t) \geq H(0) = E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} - E(0). \tag{3.15}$$

By the definition (3.1) of $E(t)$, we have

$$\begin{aligned} &H(t) - \int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \, \mathrm{d}x \\ &= E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} - \int_\Omega \frac{1}{p(.)} |\nabla u|^{p(.)} \, dx - \int_\Omega \frac{1}{q^2(.)} |u|^{q(.)} \, \mathrm{d}x \\ &< E_1 - \int_\Omega \frac{1}{p(.)} |\nabla u|^{p(.)} \, \mathrm{d}x. \end{aligned}$$

Hence, using (2.4a), (2.4c), (3.11a) and the fact that $\alpha_2 > \alpha_1$, for all $t \geq 0$ we have

$$H(t) - \int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \mathrm{d}x < E_1 - \frac{1}{p_2} \min\left(\|\nabla u\|_{p(.)}^{p_2}, \|\nabla u\|_{p(.)}^{p_1} \right) \tag{3.16}$$
$$< \left(\frac{1}{p_2} - \frac{1}{q_1 + \sigma} \right) \alpha_1 - \frac{1}{p_2} \alpha_1 < 0.$$

Thus, (3.14) follows immediately from (3.15) and (3.16). □

We are now ready to state and prove our main result of this Section, concerning the blow-up of solutions for (1.1).

Theorem 3.5. *Let σ, α_1, B_1 and E_1 be given as in* (2.2), (2.4a), (2.4b) *and* (2.4c), *respectively. Assume that the initial value u_0 is chosen so that* (3.10) *holds. Then, the corresponding solution of problem* (1.1) *will blow-up in finite time T^*. Furthermore, we have the following upper estimate for T^*:*

$$0 < T^* < \frac{\|u_0\|_{H_0^1(\Omega)}^{p_2}}{(\mathcal{B} + p_2 - 2)\left(\frac{q_1 - p_2}{q_1 + \sigma} \alpha_1 - (\mathcal{B} + p_2) E(0) \right)},$$

where we have denoted

$$\mathcal{B} = \mathcal{B}(\alpha_1, \alpha_2, p_2, q_1, \sigma) := (q_1 - p_2) \left(1 - \left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{\sigma + q_1}{p_2}} \right) > 0, \tag{3.17}$$

with α_2 as in Lemma 3.3.

Proof. Let us define the function

$$\varphi(t) = \frac{1}{2} \int_\Omega u^2 \, \mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x = \frac{1}{2} \|u\|_{H_0^1(\Omega)}^2.$$

Then, using (1.1) and integration by parts, the derivative $\varphi'(t)$ satisfies

$$\begin{aligned} \varphi'(t) &= \int_\Omega u u_t \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla u_t \, \mathrm{d}x \\ &= \int_\Omega u \left(\Delta u_t + \mathrm{div}\left(|\nabla u|^{p(.)-2} \nabla u \right) + |u|^{q(.)-2} u \ln(|u|) \right) \mathrm{d}x - \int_\Omega u \Delta u_t \, \mathrm{d}x \\ &= - \int_\Omega |\nabla u|^{p(.)} \, \mathrm{d}x + \int_\Omega |u|^{q(.)} \ln(|u|) \, \mathrm{d}x. \end{aligned}$$

Recalling the definitions (3.1) and (3.13) of the energy $E(t)$ and the function $H(t)$, from the above identity we can estimate

$$\begin{aligned} \varphi'(t) &\geq -p_2 E(t) - p_2 \int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \mathrm{d}x + \int_\Omega |u|^{q(.)} \ln(|u|) \, \mathrm{d}x \\ &\geq - \left(p_2 E_1 - \frac{\sigma \alpha_1}{q_1 + \sigma} \right) + p_2 H(t) + \frac{q_1 - p_2}{q_1} \int_\Omega |u|^{q(.)} \ln(|u|) \, \mathrm{d}x. \end{aligned} \tag{3.18}$$

Moreover, from (2.4a), (2.4c) and (3.11b) we have

$$\begin{aligned} p_2E_1 - \frac{\sigma\alpha_1}{q_1+\sigma} &= \frac{q_1-p_2}{q_1+\sigma}\alpha_1 \\ &= \frac{q_1-p_2}{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}}\alpha_1^{-\frac{q_1-p_2+\sigma}{p_2}} = \frac{q_1-p_2}{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}}\left(\frac{e\sigma q_1}{q_1+\sigma}B_1^{-(q_1+\sigma)}\right)^{-1} \\ &= \left(\frac{e\sigma q_1}{q_1+\sigma}\right)^{-1}\frac{q_1-p_2}{q_1+\sigma}B_1^{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}} = \frac{q_1-p_2}{q_1}\left(\frac{1}{e\sigma}B_1^{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}}\right) \\ &= \frac{q_1-p_2}{q_1}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\left(\frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\alpha_2^{\frac{q_1+\sigma}{p_2}}\right) \\ &\leq \frac{q_1-p_2}{q_1}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x. \end{aligned} \tag{3.19}$$

Then, it follows (3.18) and (3.19) that

$$\varphi'(t) \geq \frac{q_1-p_2}{q_1}\left(1-\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\right)\int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x + p_2H(t). \tag{3.20}$$

Besides, using (2.2), we can readily check that

$$\zeta := \frac{q_1-p_2}{q_1}\left(1-\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\right) > 0. \tag{3.21}$$

Now, let us define the function

$$\psi(t) := -\zeta q_1E(t) + p_2H(t). \tag{3.22}$$

Then, using (3.5) and (3.13), we have

$$\begin{aligned} \psi'(t) &= -\zeta q_1E'(t) + p_2H'(t) \\ &= -(\zeta q_1+p_2)E'(t) = (\zeta q_1+p_2)\int_\Omega \left(|u_t|^2 + |\nabla u_t|^2\right)\,\mathrm{d}x > 0. \end{aligned} \tag{3.23}$$

Hence, using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} \varphi(t)\psi'(t) &= \frac{\zeta q_1+p_2}{2}\left[\int_\Omega \left(|u|^2+|\nabla u|^2\right)\mathrm{d}x\right]\left[\int_\Omega \left(|u_t|^2+|\nabla u_t|^2\right)\mathrm{d}x\right] \\ &\geq \frac{\zeta q_1+p_2}{2}\left(\int_\Omega uu_t\mathrm{d}x + \int_\Omega \nabla u\nabla u_t\mathrm{d}x\right)^2 \\ &= \frac{\zeta q_1+p_2}{2}\varphi'(t)^2. \end{aligned} \tag{3.24}$$

Since, by definition, $\zeta < 1$, we have from (3.10) that

$$E(0) < \frac{p_2}{q_1+p_2}\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right) < \frac{p_2}{\zeta q_1+p_2}\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right).$$

Hence, we obtain that

$$\begin{aligned}\psi(0) &= -\zeta q_1 E(0) + p_2 H(0) \\ &= -(\zeta q_1 + p_2)E(0) + p_2\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right) > 0\end{aligned} \tag{3.25}$$

which, thanks to (3.23), yields $\psi(t) > 0$ for all $t \geq 0$. Moreover, from (3.20) and (3.22) it is easy to check that $\varphi'(t) \geq \psi(t)$. Hence from (3.24), we get

$$\varphi(t)\psi'(t) \geq \frac{\zeta q_1 + p_2}{2}\varphi'(t)\psi(t)$$

which can be written as

$$\frac{\psi'(t)}{\psi(t)} \geq \frac{\zeta q_1 + p_2}{2}\frac{\varphi'(t)}{\varphi(t)}. \tag{3.26}$$

Integrating (3.26) from 0 to t and using (3.22), we have

$$\frac{\varphi'(t)}{\varphi(t)^{\frac{\zeta q_1+p_2}{2}}} \geq \frac{\psi(0)}{\varphi(0)^{\frac{\zeta q_1+p_2}{2}}}. \tag{3.27}$$

Integrating also (3.27) from 0 to t, we obtain

$$-\frac{2}{\zeta q_1 + p_2 - 2}\,\varphi(s)^{-\frac{\zeta q_1+p_2}{2}-1}\Big|_{s=0}^{s=t} \geq \frac{\psi(0)}{\varphi(0)^{\frac{\zeta q_1+p_2}{2}}}t,$$

that is,

$$\frac{1}{\varphi(t)^{\frac{\zeta q_1+p_2-2}{2}}} \leq \frac{1}{\varphi(0)^{\frac{\zeta q_1+p_2-2}{2}}} - \frac{\zeta q_1 + p_2 - 2}{2}\frac{\psi(0)}{\varphi(0)^{\frac{\zeta q_1+p_2}{2}}}t,$$

From the above inequality, by means of some easy algebraic manipulations, we finally get

$$\varphi(t)^{\frac{\zeta q_1+p_2-2}{2}} \geq \frac{\varphi(0)^{\frac{\zeta q_1+p_2}{2}}}{\varphi(0) - \frac{\zeta q_1+p_2-2}{2}\psi(0)t}$$

Let

$$0 < T^* = \frac{2\varphi(0)}{(\zeta q_1 + p_2 - 2)\psi(0)}. \tag{3.28}$$

Then

$$\varphi(0) - \frac{\zeta q_1 + p_2 - 2}{2}\psi(0)T^* = 0,$$

yielding the blowing-up of $\varphi(t)$ at time T^*. By definition of φ, we can then conclude that $u(x,t)$ blows-up in $H_0^1(\Omega)$-norm at $t = T^*$.

Finally, notice that by means of (3.17), (3.21) and (3.25), and recalling the definition of φ, we can easily obtain from (3.28)

$$T^* = \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(\mathcal{B} + p_2 - 2)\left(\frac{q_1-p_2}{q_1+\sigma}\alpha_1 - (\mathcal{B}+p_2)E(0)\right)}.$$

This concludes our proof. □

4. Non-blow-up case

In this section, we present some non blow-up conditions for the solution of (1.1). In particular, we are going to show that, when the initial datum u_0 is small enough in $H_0^1(\Omega)$, blow-up cannot manifest at any time $t \geq 0$. In addition to that, we will also provide explicit time-decay estimates for the $H_0^1(\Omega)$-norm of the solution.

In order to state our main result in this section, let us denote with λ_1 the first eigenvalue of the Dirichlet Laplacian on Ω and introduce the following constants

$$\mathrm{A} = |\Omega|^{\frac{2-p_1}{2}} \left(\frac{\lambda_1}{2(\lambda_1+1)}\right)^{\frac{p_1}{2}}, \quad \mathrm{B} = \frac{4}{e\sigma} B_\sigma \mathcal{C}_1^{q_1+\sigma}, \quad \mathrm{C} = \frac{4}{e\sigma} B_\sigma \mathcal{C}_2^{q_2+\sigma}. \tag{4.1}$$

In (4.1), $\mathcal{C}_1, \mathcal{C}_2 > 0$, are the best embedding constants of $H_0^1(\Omega)$ into $W_0^{1,q_1+\sigma}(\Omega)$ and $W_0^{1,q_2+\sigma}(\Omega)$, respectively. Moreover, σ is defined as in (2.2) and B_σ is the constant introduced in (2.3) for the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)+\sigma}(\Omega)$. We then have the following non blow-up result for the weak solutions of (1.1).

Theorem 4.1. *Let $u_0 \in W_0^{1,p(.)}(\Omega)$ satisfy*

$$\|u_0\|_{H_0^1(\Omega)} < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{1}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{1}{q_2-p_1+\sigma}}\right), \tag{4.2}$$

where the constants A, B *and* C *have been defined in* (4.1)*. Then, the corresponding weak solution u of* (1.1) *cannot blow-up in $H_0^1(\Omega)$-norm, and the following decay estimates hold*

$$\|u(t)\|_{H_0^1(\Omega)} \leq \left(\frac{4}{\mathrm{A}(p_1-2)t + 4\|u_0\|_{H_0^1(\Omega)}^{2-p_1}}\right)^{\frac{1}{p_1-2}}, \quad \text{if } p_1 > 2 \tag{4.3a}$$

$$\|u(t)\|_{H_0^1(\Omega)} \leq \left(\frac{A\|u_0\|_{H_0^1(\Omega)}^{q_1-2+\sigma}}{\mathrm{A} - B\|u_0\|_{H_0^1(\Omega)}^{q_1-2+\sigma}}\right)^{\frac{1}{q_1-2+\sigma}} e^{-\frac{A}{2}t}, \quad \text{if } p_1 = 2. \tag{4.3b}$$

Proof. Define the auxiliary function

$$\theta(t) := 2\varphi(t) = \int_\Omega u^2(t)\,\mathrm{d}x + \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x = \|u(.,t)\|_{H_0^1(\Omega)}^2 := \|u(t)\|_{H_0^1(\Omega)}^2. \tag{4.4}$$

We then have from (1.1) and integration by parts that

$$\theta'(t) = 2\int_\Omega u u_t\,\mathrm{d}x + 2\int_\Omega \nabla u \nabla u_t\,\mathrm{d}x = -2\int_\Omega |\nabla u|^{p(.)}\,\mathrm{d}x + 2\int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x. \tag{4.5}$$

For any $t > 0$, we split the domain Ω into $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 := \Big\{x \in \Omega : |\nabla u(x,t)| < 1\Big\} \quad \text{and} \quad \Omega_2 = \Big\{x \in \Omega : |\nabla u(x,t)| \geq 1\Big\}.$$

We then have

$$\int_\Omega |\nabla u|^{p(.)}\,\mathrm{d}x = \int_{\Omega_1} |\nabla u|^{p(.)}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{p(.)}\,\mathrm{d}x$$
$$\geq \int_{\Omega_1} |\nabla u|^{p_2}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x \geq \int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x. \tag{4.6a}$$

$$\int_\Omega |\nabla u|^{\rho+\sigma}\,\mathrm{d}x = \int_{\Omega_1} |\nabla u|^{\rho+\sigma}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{\rho+\sigma}\,\mathrm{d}x \leq 2\int_{\Omega_2} |\nabla u|^{\rho+\sigma}\,\mathrm{d}x. \tag{4.6b}$$

Using (2.3), (3.7), (4.6a) and (4.6b), from (4.5) we can then estimate

$$\theta'(t) \leq -2\int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x + \frac{2}{e\sigma}\int_\Omega |u|^{q(.)+\sigma}\,\mathrm{d}x \tag{4.7}$$
$$\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x + \frac{2}{e\sigma}B_\sigma \int_\Omega |\nabla u|^{q(.)+\sigma}\,\mathrm{d}x$$
$$\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x + \frac{2}{e\sigma}B_\sigma \left(\int_{\Omega_1} |\nabla u|^{q_1+\sigma}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{q_2+\sigma}\,\mathrm{d}x\right)$$
$$\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x + \frac{4}{e\sigma}B_\sigma \left(\int_{\Omega_2} |\nabla u|^{q_1+\sigma}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{q_2+\sigma},\mathrm{d}x\right)$$

By Hölder's inequality, we have

$$|\Omega|^{\frac{2-p_1}{2}} \left(\int_{\Omega_2} |\nabla u|^2\,\mathrm{d}x\right)^{\frac{p_1}{2}} \leq \int_{\Omega_2} |\nabla u|^{p_1}\,\mathrm{d}x,$$

while the Poincaré's inequality gives

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x \geq \lambda_1 \int_\Omega |u|^2\,\mathrm{d}x,$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian on Ω. Thus (4.4) satisfies the following inequality

$$\theta(t) \leq \left(\frac{1}{\lambda_1}+1\right)\int_\Omega |\nabla u|^2\,\mathrm{d}x$$
$$= \left(\frac{1}{\lambda_1}+1\right)\int_{\Omega_1} |\nabla u|^2\,\mathrm{d}x + \left(\frac{1}{\lambda_1}+1\right)\int_{\Omega_2} |\nabla u|^2\,\mathrm{d}x \leq 2\frac{\lambda_1+1}{\lambda_1}\int_{\Omega_2} |\nabla u|^2\,\mathrm{d}x.$$

Moreover, from the Sobolev embedding we have

$$\int_{\Omega_2} |\nabla u|^{q_1+\sigma}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{q_2+\sigma}\,\mathrm{d}x \leq \int_\Omega |\nabla u|^{q_1+\sigma}\,\mathrm{d}x + \int_\Omega |\nabla u|^{q_2+\sigma}\,\mathrm{d}x$$
$$\leq C_1^{q_1+\sigma}\theta(t)^{\frac{q_1+\sigma}{2}} + C_2^{q_2+\sigma}\theta(t)^{\frac{q_2+\sigma}{2}}.$$

Thus, (4.7) gives

$$\theta'(t) \leq -2\,|\Omega|^{\frac{2-p_1}{2}}\left(\frac{\lambda_1}{2(\lambda_1+1)}\right)^{\frac{p_1}{2}}\theta(t)^{\frac{p_1}{2}} + \frac{4}{e\sigma}B_\sigma\left(C_1^{q_1+\sigma}\theta(t)^{\frac{q_1+\sigma}{2}} + C_2^{q_2+\sigma}\theta(t)^{\frac{q_2+\sigma}{2}}\right)$$
$$= -\theta(t)^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\theta(t)^{\frac{q_1-p_1+\sigma}{2}}\right) - \theta(t)^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{C}\theta(t)^{\frac{q_2-p_1+\sigma}{2}}\right). \tag{4.8}$$

We now claim that

$$\theta(t) < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right), \quad \text{for all } t \geq 0. \tag{4.9}$$

Indeed, if we suppose (4.9) is not satisfied, by the continuity of $\theta(t)$ and (4.2), there exists $t_0 > 0$ such that

$$\theta(t_0) = \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right), \tag{4.10}$$

and

$$\theta(t) < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right) \quad \text{for } 0 \leq t < t_0. \tag{4.11}$$

Since we are assuming in (4.2) that

$$\theta(0) < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right),$$

we get from (4.10) that $\theta'(t) > 0$ for $0 < t < t_0$. On the other hand, (4.8) and (4.11) implies that $\theta'(t) < 0$ for $0 < t < t_0$. We then found a contradiction, meaning that (4.9) holds. In particular, we have that $\theta'(t) < 0$ for all $t \geq 0$. Hence, the $H_0^1(\Omega)$-norm of the solution to (1.1) decays in time and cannot blow-up.

To obtain the decay estimates (4.3a) and (4.3b), without losing generality let us assume that $\theta(t) \neq 0$ for all $t \geq 0$. We are allowed to do that because, if there exists $t_0 \geq 0$ such that $\theta(t_0) = 0$ then, since $\theta(t)$ is non-negative and decreasing for all $t \geq 0$, we would also have

$$\theta(t) = 0 \quad \text{for all } t \geq t_0$$

and (4.3a) and (4.3b) would be trivially satisfied.

If, on the other hand, $\theta(t) \neq 0$ for all $t \geq 0$, then from the differential inequality (4.8) and (4.9) we can write

$$1 \leq -\frac{\theta'(t)}{\theta(t)^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\theta(t)^{\frac{q_1-p_1+\sigma}{2}}\right)}. \tag{4.12}$$

We shall now distinguish two cases:

Case 1: $p_1 > 2$**.** Integrating (4.12) over the interval $(0, t)$, and taking into account that $\theta(t) \leq \theta(0)$ for all $t \geq 0$, we obtain

$$t \leq -\int_{\theta(0)}^{\theta(t)} \frac{d\gamma}{\gamma^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}\right)} = \int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}\right)}, \tag{4.13}$$

which is equivalent to

$$t \leq \frac{1}{\mathrm{A}}\int_{\theta(t)}^{\theta(0)} \left(\frac{1}{\gamma^{\frac{p_1}{2}}} + \frac{\mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}}{\gamma^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}\right)}\right) d\gamma. \tag{4.14}$$

Moreover, since by (4.9) we have

$$\frac{\mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}}{\mathrm{A}-\mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}} \leq 1 \quad \text{for all } \gamma \in [\theta(t), \theta(0)],$$

we obtain from (4.14) that

$$\begin{aligned} t &\leq \frac{2}{\mathrm{A}} \int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma^{\frac{p_1}{2}}} = \frac{4}{A(2-p_1)} \left(\theta(0)^{\frac{2-p_1}{2}} - \theta(t)^{\frac{2-p_1}{2}} \right) \\ &= \frac{4}{A(2-p_1)} \Big(\|u_0\|^{2-p_1} - \|u(.,t)\|^{2-p_1} \Big). \end{aligned}$$

From the above inequality, (4.3a) easily follows by means of simple algebraic manipulations.

Case 2: $p_1 = 2$**.** When $p_1 = 2$, we obtain from (4.13) that

$$t \leq \int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma \left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-2+\sigma}{2}} \right)}. \tag{4.15}$$

For simplicity of notation, we shall denote

$$\varpi := \frac{q_1 - 2 + \sigma}{2}. \tag{4.16}$$

Then if we introduce the change of variables $\xi = \gamma^{\varpi}$, we can compute

$$\begin{aligned} \int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma \left(\mathrm{A} - \mathrm{B}\gamma^{\varpi} \right)} &= \frac{1}{\varpi} \int_{\theta(t)^{\varpi}}^{\theta(0)^{\varpi}} \frac{d\xi}{\xi \left(\mathrm{A} - \mathrm{B}\xi \right)} \\ &= \frac{1}{\mathrm{A}\varpi} \ln \left(\frac{\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} \frac{\mathrm{A} - \mathrm{B}\theta(t)^{\varpi}}{\theta(t)^{\varpi}} \right). \end{aligned} \tag{4.17}$$

We then get from (4.15) and (4.17) that

$$\frac{\theta(t)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(t)^{\varpi}} \leq \frac{\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} e^{-\mathrm{A}\varpi t}.$$

Since by (4.2) we have

$$\mathrm{A} - \mathrm{B}\theta(t)^{\varpi} > 0,$$

we then obtain that

$$\theta(t)^{\varpi} \leq \left(\frac{\mathrm{A}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} - \frac{\mathrm{B}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} \theta(t)^{\varpi} \right) e^{-\mathrm{A}\varpi t} \leq \frac{\mathrm{A}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} e^{-\mathrm{A}\varpi t}.$$

Hence

$$\theta(t) \leq \left(\frac{\mathrm{A}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} \right)^{\frac{1}{\varpi}} e^{-\mathrm{A}t}.$$

Since, by definition, $\theta(t) = \|u\|^2_{H_0^1(\Omega)}$, (4.3b) follows immediately from the above estimate and (4.16). Our proof is then concluded. □

5. Numerical study

In this section, we address a 2D numerical illustration of the problem

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p(x,y)-2}\nabla u\right) = |u|^{q(x,y)-2}u\ln(|u|), & (x,y,t)\in\Omega\times(0,T) \\ u(x,y,t) = 0, & (x,y,t)\in\partial\Omega\times(0,T) \\ u(x,y,0) = u_0(x,y)\neq 0, & (x,y)\in\Omega \end{cases} \tag{5.1}$$

where $T>0$ and u represents the velocity of a fluid in the domain $\Omega=[-1,1]^2$.

Our objective is to represent numerically the decay rate obtained in Theorem 4.1 in the case $p>2$. To this end, let us start by introducing the numerical scheme that we will employ for our experiments. First of all, for $n\in\mathbb{N}$, let us consider the uniform time-mesh

$$t_n = n\delta t, \quad \delta t = \frac{T}{n},$$

and define

$$u^n = u^n(x,y) := u(x,y,t_n)$$

$$u_t^{n+1} := \frac{u^{n+1}-u^n}{\Delta t}.$$

We then approximate (5.1) through

$$\begin{cases} u_t^{n+1} - \Delta u_t^{n+1} - \operatorname{div}\left(|\nabla u^n|^{p-2}\nabla u^n\right) = |u^n|^{q-2}u^n\ln(|u^n|) & \text{in } \Omega \\ u^n = 0 & \text{on } \partial\Omega \\ u^0 = u_0, & \text{in } \Omega \end{cases} \tag{5.2}$$

For the space-discretization, we use $\mathcal{P}_1$ Lagrangian Finite Elements implemented through the software `FreeFem++` [26]. To this end, we define a triangular mesh $\mathcal{M}_h$ of mesh-size h over the domain Ω, composed by 5000 triangles and 2601 vertices. Let $\mathcal{V}_h$ be the corresponding space of piece-wise linear continuous functions. Then, approximating (5.2) amounts to solving the following fully-discrete problem: find $u_h^n\in\mathcal{V}_h$ such that

$$\int_\Omega \left[u_{h,t}^{n+1}\omega + \nabla u_{h,t}^{n+1}\cdot\nabla\omega + |\nabla u_h^n|^{p(x,y)-2}\nabla u_h^n\cdot\nabla\omega\right]\mathrm{d}x\mathrm{d}y \tag{5.3}$$
$$= \int_\Omega |u_h^n|^{q(x,y)-2}u_h^n\ln(|u_h^n|)\omega\,\mathrm{d}x\mathrm{d}y.$$

In what follows, we consider the variable exponents

$$p(x,y) = 0.2\,|\lfloor x\rfloor| + 2.5 \quad \text{and} \quad q(x,y) = 0.1\,|\lfloor x\rfloor| + 6,$$

where $\lfloor . \rfloor$ denotes the greatest integer function, so that

1. $2 < 2.5 = p_1 < p_2 = 2.7 < 6 = q_1 < q_2 = 6.1 < p^*(x,y) = +\infty$.
2. Condition (2.2) is verified for all $\sigma>0$. In what follows, we will take $\sigma = 0.1$.

Moreover, we select the initial datum

$$u_0(x,y) = 0.25e^{-x^2-y^2}.$$

whose $H_0^1(\Omega)$-norm is

$$\|u_0\|_{H_0^1(\Omega)} = \left(\|u_0\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \sim 0.477344.$$

Finally, the time horizon for our simulations will be $T = 1$.

Let us now present the results of our numerical experiments. First of all, we can see in Table 1 and Figure 2 that, with this particular choice of the initial datum and variable exponents, the condition (4.2), i.e.

$$\|u_0\|_{H_0^1(\Omega)} < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{1}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{1}{q_2-p_1+\sigma}}\right) =: \mathrm{M}$$

is satisfied.

t_n	0	0.1	0.2	0.3	0.4	0.5
M	0.671566	0.671906	0.672586	0.673228	0.673835	0.674407
t_n	0.6	0.7	0.8	0.9	1	
M	0.674946	0.675454	0.675932	0.676382	0.676804	

TABLE 1. Values of M for $t \in [0,1]$.

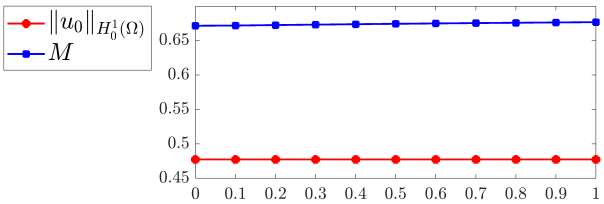

FIGURE 2. Fulfillment of the decay condition (4.2).

Hence, according to Theorem 4.1, the solution to (5.1) is expected to decay in $H_0^1(\Omega)$-norm at the rate (4.3a), i.e.

$$\|u_h^n\|_{H_0^1(\Omega)} \leq \left(\frac{4}{\mathrm{A}(p_1-2)t_n + 4\|u_0\|^{2-p_1}_{H_0^1(\Omega)}}\right)^{\frac{1}{p_1-2}} =: \delta.$$

This is precisely what we can observe in Table 2 and Figures 3, 4.

Remark 5.1. The numerical decay rate in Table 2 and Figure 3 is consistently faster than the theoretical upper bound δ. This gap is expected, as δ is derived from worst-case estimates (Holder, Poincaré, Sobolev embedding, and the elementary inequality $\ln\xi \leq \frac{1}{e\sigma}\xi^\sigma$) that are generally not sharp for a specific initial datum. The gap could

be reduced by optimizing the free parameter $\sigma > 0$ appearing in (2.2) and (4.1). Indeed, the constants A, B, C and the exponent δ depend on σ (via B_σ, $\mathcal{C}_1$, $\mathcal{C}_2$, and the condition $\operatorname{ess\,inf}_{x\in\Omega}(p^*(x) - q(x)),$). A finer, problem-dependent choice of σ-e.g., by minimizing the right-hand side of (4.3b) with respect to σ for a given u_0-would yield a tighter bound, though the present choice σ suffices to illustrate the qualitative decay behavior.

t_n	0	0.1	0.2	0.3	0.4	0.5
$\|u_h^n\|_{H_0^1(\Omega)}$	0.477344	0.43491	0.420828	0.407467	0.394777	0.382713
δ	0.477344	0.475403	0.473474	0.471557	0.469651	0.467757
t_n	0.6	0.7	0.8	0.9	1	
$\|u_h^n\|_{H_0^1(\Omega)}$	0.371233	0.360298	0.349873	0.339927	0.330429	
δ	0.465874	0.464003	0.462143	0.460294	0.458456	

TABLE 2. $H_0^1(\Omega)$-norm decay of the solution for $t \in [0, 1]$.

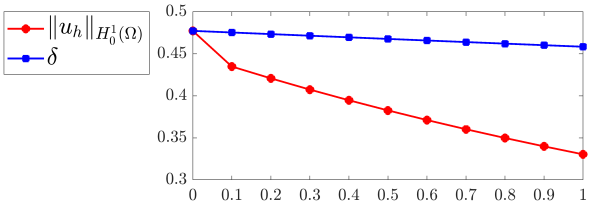

FIGURE 3. $H_0^1(\Omega)$-norm decay of the solution for $t \in [0, 1]$.

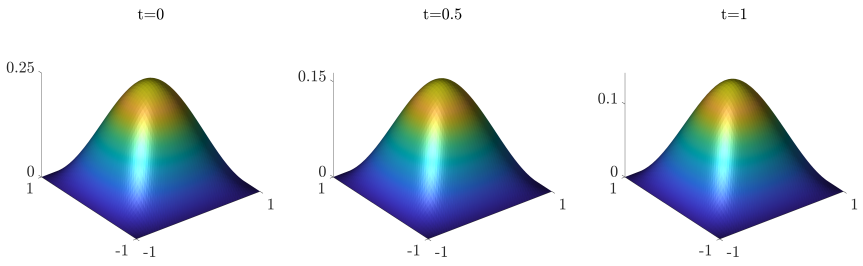

FIGURE 4. Time evolution of the solution of (5.1) for $t \in [0, 1]$.

All this confirms numerically the validity of our theoretical results presented in Section 4.

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