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On a nonlocal nonlinear hybrid fractional boundary value problem

Ahmed Alsaedi, Rabab Alghamdi, Bashir Ahmad and Sotiris K. Ntouyas

Abstract. We investigate the existence criteria and Ulam–Hyers–Rassias stability for solutions of a nonlocal and nonlinear hybrid fractional boundary value problem. A variant of Krasnosel'skiĭ's fixed point theorem proposed by Burton is employed to prove the existence of at least one solution of the given problem while the uniqueness of its solutions is established via Banach's fixed point theorem. Examples illustrating the obtained results are presented.

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1. Introduction

Hybrid fractional differential equations can be interpreted as the equations containing fractional derivative of an unknown function hybrid with a nonlinearity relying on it. Also, a fractional differential equation containing a perturbed term under the fractional derivative is known as a perturbation of second type or hybrid differential equation as argued in [1]. Dhage and Lakshmikantham in [2] initiated the study of hybrid differential equations. Afterward, many investigators turned to study fractional hybrid differential equations. In [3], the authors discussed the existence of solutions for fractional hybrid differential inclusions. For some interesting results on hybrid fractional integrodifferential equations, we refer the reader to the articles [4, 5, 6]. The authors in [7] investigated a dual hybrid system of nonlinear q-fractional differential equations with nonlocal boundary conditions. In a recent work [8], a hybrid

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Erdélyi–Kober–type fractional differential equation was studied on an infinite interval. For application of fractional hybrid differential equations in physical and engineering problems related to electromagnetic waves, gravity driven flows, deflection of a curved beam, a three–layer beam, etc., we refer the reader to the articles [9, 10].

Nonlocal boundary conditions are considered to be more realistic and practical than the classical ones as such conditions model the phenomena experiencing changes at arbitrary positions or segments inside the given domain. For a detailed account of nonlocal multipoint and integral boundary value problems, we refer the reader to the texts [11, 12].

The notion of Ulam–Hyers stability, originated from a question posed by Ulam in 1940 about stability of functional equations (for details, see [13]) and refined by Hyers in [14], appears in various fields, like Black-Scholes equation [15], fractional boundary value problems [16], etc. Rassias [17] further enhanced the idea of Ulam–Hyers stability by extending its application to a wider class of functional equations. In [18], Jung studied Ulam–Hyers–Rassias stability for functional equations appearing in mathematical analysis.

Motivated by the recent interest in hybrid fractional differential equations, in this paper, we discuss the existence and Ulam–Hyers–Rassias stability for solutions of a fractional hybrid differential equation equipped with a nonlocal multipoint boundary condition given by

$$\begin{cases} {}^{c}D^{\omega}(z(t) - p(t, z(t))) = \psi(t, z(t)),\ 0 < \omega \leq 1,\ t \in [0, T], \\ z(0) - \delta z(T) = \sum_{i=1}^{n} \sigma_i z(\gamma_i),\ \delta, \sigma_i \in \mathbb{R},\ 0 < \gamma_i < T, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\omega}$ denotes the Caputo fractional derivative operator of order ω and $p, \psi : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In the rest of the paper, we present the material as follows. In Section 2, we outline some concepts of fractional calculus related to our work and establish an auxiliary result. Section 3 contains the main results for the problem (1.1). Based on a variant of Krasnosel'skiĭ's fixed point theorem proposed by Burton, we first accomplish the existence of at least one solution to the problem (1.1). As a second result, the existence of a unique solution for the given problem is established via Banach's fixed point theorem. Section 4 deals with the Ulam–Hyers–Rassias stability for the problem (1.1).

2. Preliminaries

Let us begin this section with some preliminary concepts of fractional operators [19].

Definition 2.1. *The (left) Riemann–Liouville fractional integral of order $q > 0$ for a function $x \in L^1[a, b], a, b \in \mathbb{R}$ is given by*

$$I_{a^+}^{q} x(\zeta) = \frac{1}{\Gamma(q)} \int_{a}^{\zeta} (\zeta - s)^{q-1} x(s) ds,$$

where $\Gamma(.)$ *denotes the Euler gamma function.*

Definition 2.2. [19] *We define the Riemann–Liouville fractional derivative of order* $q \in (m-1, m)$, $m \in \mathbb{N}$ *for a function* x *with* $x, x^{(m)} \in L^1[a, b]$ *as*

$$D^q_{a^+} x(\zeta) = \frac{d^m}{d\zeta^m} I^{m-q}_{a^+} x(\zeta) = \frac{1}{\Gamma(m-q)} \frac{d^m}{d\zeta^m} \int_a^\zeta (\zeta - s)^{m-1-q} x(s) ds,$$

while the Caputo fractional derivative ${}^cD^q_{a^+} x$ *of order* $q \in (m-1, m)$ *is defined by*

$${}^cD^q_{a^+} x(\zeta) = D^q_{a^+} \left(x(\zeta) - \sum_{k=0}^{m-1} \frac{x^{(k)}(a)}{k!} (\zeta - a)^k \right).$$

Remark 2.3. The (left) Caputo fractional derivative for a function $x \in AC^m[a, b]$ of order $q \in (m-1, m)$ in [19] is also given as

$${}^cD^q_{a^+} x(\zeta) = \int_a^\zeta \frac{(\zeta - s)^{m-q-1}}{\Gamma(m-q)} x^{(m)}(s) ds.$$

Remark 2.4. The Riemann–Liouville fractional integral and Caputo fractional derivative operators satisfy the relation

$$I^q_{a^+} {}^cD^q_{a^+} x(\zeta) = x(\zeta) + \sum_{j=0}^{m-1} k_j (\zeta - a)^j,$$

where $k_0, k_1, k_2, \ldots, k_{m-1}$ are real arbitrary constants.

For the sake of brevity, we write $I^q_{a^+}$ and ${}^cD^q_{a^+}$ as I^q and ${}^cD^q$, respectively when $a = 0$.

Now we present a lemma dealing with the linear version of the problem (1.1).

Lemma 2.5. *Let* $\Delta \neq 0$, $F \in C([0, T], \mathbb{R})$ *and* $G \in L^1[0, T]$. *Then, the integral solution of the hybrid fractional boundary value problem*

$$\begin{cases} {}^cD^\omega (z(t) - F(t)) = G(t), \ 0 < \omega \leq 1, \ t \in [0, T], \\ z(0) - \delta z(T) = \sum_{i=1}^n \sigma_i z(\gamma_i), \quad \delta, \sigma_i \in \mathbb{R}, \quad 0 < \gamma_i < T, \end{cases} \tag{2.1}$$

is

$$\begin{aligned} z(t) \quad = \quad & I^\omega G(t) + \frac{1}{\Delta} \Bigg(\sum_{i=1}^n \sigma_i [F(\gamma_i) + I^\omega G(\gamma_i)] - F(0) \\ & + \delta [F(T) + I^\omega G(T)] \Bigg) + F(t), \end{aligned} \tag{2.2}$$

where

$$\Delta = 1 - \delta - \sum_{i=1}^n \sigma_i. \tag{2.3}$$

Proof. Operating the Riemann–Liouville integral operator I^{ω} on the fractional differential equation in (2.1) and using Remark 2.4 with $a = 0$, we get

$$z(t) - F(t) = I^{\omega}G(t) + c_0, \tag{2.4}$$

where c_0 is an unknown arbitrary constant. Using (2.4) into the nonlocal boundary condition of (2.1), we find that

$$F(0) + c_0 - \delta[F(T) + I^{\omega}G(T) + c_0] = \sum_{i=1}^{n} \sigma_i[F(\gamma_i) + I^{\omega}G(\gamma_i) + c_0], \tag{2.5}$$

which, on solving for c_0 together with (2.3), yields

$$c_0 = \frac{1}{\Delta}\Bigg[\sum_{i=1}^{n} \sigma_i[F(\gamma_i) + I^{\omega}G(\gamma_i)] - F(0) + \delta[F(T) + I^{\omega}G(T)]\Bigg]. \tag{2.6}$$

Substituting the above expression for c_0 into (2.4) yields the solution (2.2).

Conversely, we show that $z(t)$ given by (2.2) satisfies the problem (2.1). Using (2.5) in (2.2), we get

$$z(t) - F(t) = I^{\omega}G(t) + c_0. \tag{2.7}$$

Applying the Caputo derivative operator ${}^{c}D^{\omega}$ to both sides of (2.7) and using the standard identity ${}^{c}D^{\omega}I^{\omega}G(t) = G(t)$ together with ${}^{c}D^{\omega}c_0 = 0$, we obtain

$${}^{c}D^{\omega}(z(t) - F(t)) = G(t),$$

which is the fractional differential equation in (2.1). Now, it remains to verify that $z(t)$ given by (2.2) satisfies the boundary condition in (2.1). Evaluating z at $t = 0$, $t = T$, and $t = \gamma_i$, we have

$$z(0) = F(0) + c_0,\ z(T) = I^{\omega}G(T) + F(T) + c_0,\ z(\gamma_i) = I^{\omega}G(\gamma_i) + F(\gamma_i) + c_0.$$

Then

$$\begin{aligned}
&z(0) - \delta z(T) - \sum_{i=1}^{n} \sigma_i z(\gamma_i) \\
&= \big(F(0) + c_0\big) - \delta\big(F(T) + I^{\omega}G(T) + c_0\big) - \sum_{i=1}^{n} \sigma_i\big(F(\gamma_i) + I^{\omega}G(\gamma_i) + c_0\big) \\
&= F(0) - \delta F(T) - \delta I^{\omega}G(T) - \sum_{i=1}^{n} \sigma_i F(\gamma_i) - \sum_{i=1}^{n} \sigma_i I^{\omega}G(\gamma_i) \\
&\quad + c_0\left(1 - \delta - \sum_{i=1}^{n} \sigma_i\right) \\
&= F(0) - \delta\big[F(T) + I^{\omega}G(T)\big] - \sum_{i=1}^{n} \sigma_i\big[F(\gamma_i) + I^{\omega}G(\gamma_i)\big] + c_0\,\Delta,
\end{aligned}$$

which, in view of (2.5), yields

$$z(0) - \delta z(T) - \sum_{i=1}^{n} \sigma_i z(\gamma_i) = 0.$$

Thus, the boundary condition in (2.1) is satisfied. Hence, the proof of the converse of the lemma is complete. Therefore, the lemma is completely proved. □

3. Existence and uniqueness results

Denote by $\mathcal{X} = C([0,T],\mathbb{R})$ a Banach space of all real continuous functions defined on $[0,T]$ equipped with the norm $\|z\| = \max_{t\in[0,T]} |z(t)|$. By Lemma 2.5, the problem (1.1) can be transformed into a fixed point problem as

$$z = \mathcal{Q}z,$$

where $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ is defined as

$$\begin{aligned}(\mathcal{Q}z)(t) &= I^{\omega}\psi(t,z(t)) + \frac{1}{\Delta}\bigg(\sum_{i=1}^{n}\sigma_i[p(\gamma_i,z(\gamma_i)) + I^{\omega}\psi(\gamma_i,z(\gamma_i))] - p(0,z(0)) \\ &\quad +\delta[p(T,z(T)) + I^{\omega}\psi(T,z(T))]\bigg) + p(t,z(t)),\ t\in[0,T]. \end{aligned} \tag{3.1}$$

Now, we enlist the assumptions needed in the sequel.

(A_1) $p \in C([0,T]\times\mathbb{R},\mathbb{R})$ satisfies the Lipschitz condition:

$$|p(t,z) - p(t,\widetilde{z})| \le \mathcal{K}_1|z-\widetilde{z}|,\ \mathcal{K}_1 > 0,\ \text{for all}\, z,\widetilde{z}\in\mathbb{R},\quad t\in[0,T].$$

(A_2) $\psi \in C([0,T]\times\mathbb{R},\mathbb{R})$ satisfies the Lipschitz condition:

$$|\psi(t,z) - \psi(t,\widetilde{z})| \le \mathcal{K}_2|z-\widetilde{z}|,\ \mathcal{K}_2 > 0,\ \text{for all}\, z,\widetilde{z}\in\mathbb{R},\quad t\in[0,T].$$

(A_3) There exists a nonnegative function $\Upsilon \in C([0,T],\mathbb{R})$ such that

$$|\psi(t,z(t))| \le \Upsilon(t) + b|z(t)|,\ b>0,\ t\in[0,T].$$

Remark 3.1. Using the substitutions $y = t-s$ and $x = \mu y$ in the integral

$$\int_0^t (t-s)^{\omega-1}e^{\mu s}ds,$$

we obtain the estimate

$$\begin{aligned}\int_0^t (t-s)^{\omega-1}e^{\mu s}ds &= e^{\mu t}\int_0^t y^{\omega-1}e^{-\mu y}dy \\ &= \frac{e^{\mu t}}{\mu^{\omega}}\int_0^{\mu t} x^{\omega-1}e^{-x}dx, \\ &\le \frac{e^{\mu t}}{\mu^{\omega}}\int_0^{\infty} x^{\omega-1}e^{-x}dx = \frac{e^{\mu t}}{\mu^{\omega}}\Gamma(\omega).\end{aligned}$$

3.1. Existence result by Krasnosel'skiĭ's fixed point theorem

In this subsection, we prove the existence of solutions for the problem (1.1) by using a variant of Krasnosel'skiĭ's fixed point theorem proposed by Burton [20], which is stated below.

Theorem 3.2. *Let Ω be a closed, convex and bounded subset of the Banach space $(\mathcal{X}, \|.\|)$. Suppose that $\Psi_1 : \mathcal{X} \to \mathcal{X}$ and $\Psi_2 : \Omega \to \mathcal{X}$ are two operators such that*

(a) *Ψ_1 is a contractive mapping;*
(b) *Ψ_2 is continuous and compact;*
(c) *$z = \Psi_1 z + \Psi_2 \nu$ for all $\nu \in \Omega$ implies $z \in \Omega$.*

Then, the operator equation $\Psi_1 z + \Psi_2 z = z$ has a solution in Ω.

In the following theorem, we split the operator $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ defined in (3.1) as $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$, where

$$(\mathcal{Q}_1 z)(t) = p(t, z(t)) + \frac{1}{\Delta}\bigg(\sum_{i=1}^{n} \sigma_i\, p(\gamma_i, z(\gamma_i)) - p(0, z(0)) + \delta\, p(T, z(T))\bigg), \tag{3.2}$$

$$(\mathcal{Q}_2 z)(t) = I^{\omega}\psi(t, z(t)) + \frac{1}{\Delta}\bigg(\sum_{i=1}^{n} \sigma_i I^{\omega}\psi(\gamma_i, z(\gamma_i)) + \delta I^{\omega}\psi(T, z(T))\bigg). \tag{3.3}$$

Theorem 3.3. *Let (A_1) and (A_3) hold with $0 < \mathcal{K}_1 < 1$ and $\Theta\mathcal{K}_1 < 1$, where*

$$\Theta = 1 + \frac{1}{|\Delta|}\left(\sum_{i=1}^{n} |\sigma_i| + 1 + |\delta|\right), \tag{3.4}$$

and Δ is defined by (2.3)*. Then, at least one solution of the problem* (1.1) *exists on $[0, T]$.*

Proof. Let us define a weighted ball B in $C([0, T], \mathbb{R})$ as

$$B = \{z \in C([0, T], \mathbb{R}) : |z(t)| \le \rho\, e^{\mu t},\ \rho \ge 0,\ \mu > 0,\ t \in [0, T]\}.$$

Notice that the set B is bounded since $\rho e^{\mu t} \le \rho e^{\mu T}$ for all $t \in [0, T]$ and $\|z\| \le \rho e^{\mu T} < \infty$. Convexity of the set B follows from the fact: for any $z_1, z_2 \in B$ and $\lambda \in [0, 1]$,

$$|\lambda z_1(t) + (1-\lambda) z_2(t)| \le \lambda |z_1(t)| + (1-\lambda)|z_2(t)| \le \lambda\rho e^{\mu t} + (1-\lambda)\rho e^{\mu t} = \rho e^{\mu t},$$

so $\lambda z_1 + (1-\lambda) z_2 \in B$. It follows from the idea of continuity of the pointwise evaluation maps that the set B is closed. Thus, the weighted ball B is a closed, convex and bounded subset of $C([0, T], \mathbb{R})$. Moreover, the exponential weight $e^{\mu t}$ is chosen deliberately to exploit the estimate in Remark 3.1 and thereby obtain a uniform bound on $\mathcal{Q}_2(B)$.

Now we verify the hypotheses of Theorem 3.2. Let us first show that $\mathcal{Q}_1$ is a contraction. For $z_1, z_2 \in \mathcal{X}$, we have

$$\begin{aligned} & |(\mathcal{Q}_1 z_1)(t) - (\mathcal{Q}_1 z_2)(t)| \\ = & \bigg| p(t, z_1(t)) - p(t, z_2(t)) + \frac{1}{\Delta}\bigg(\sum_{i=1}^{n} \sigma_i [p(\gamma_i, z_1(\gamma_i)) - p(\gamma_i, z_2(\gamma_i))] \end{aligned}$$

$$
\begin{aligned}
&-[p(0,z_1(0))-p(0,z_2(0))]+\delta[p(T,z_1(T))-p(T,z_2(T))]\Big)\Big|\\
\leq\ & |p(t,z_1(t))-p(t,z_2(t))|+\frac{1}{|\Delta|}\Big(\sum_{i=1}^{n}|\sigma_i||p(\gamma_i,z_1(\gamma_i))-p(\gamma_i,z_2(\gamma_i))|\\
&+|p(0,z_1(0))-p(0,z_2(0))|+|\delta||p(T,z_1(T))-p(T,z_2(T))|\Big)\\
\leq\ & \mathcal{K}_1|z_1-z_2|+\frac{1}{|\Delta|}\Big(\mathcal{K}_1\sum_{i=1}^{n}|\sigma_i||z_1-z_2|+\mathcal{K}_1|z_1-z_2|+\mathcal{K}_1|\delta||z_1-z_2|\Big)\\
=\ & \mathcal{K}_1\left(\frac{|\Delta|+\sum_{i=1}^{n}|\sigma_i|+1+|\delta|}{|\Delta|}\right)|z_1-z_2|=\mathcal{K}_1\Theta|z_1-z_2|,
\end{aligned}
$$

which implies that $\mathcal{Q}_1$ is a contraction as $\mathcal{K}_1\Theta<1$, where Θ is given in (3.4).

Now, for any $z\in B$, we obtain by using (A_3) that

$$
\begin{aligned}
\|(\mathcal{Q}_2z)\| &= \max_{t\in[0,T]}|(\mathcal{Q}_2z)(t)|\\
&\leq \max_{t\in[0,T]}\Big\{\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,z(s))|ds\\
&\quad+\frac{1}{|\Delta|}\Big(\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,z(s)|ds\\
&\quad+|\delta|\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,z(s))|ds\Big)\Big\}\\
&\leq \max_{t\in[0,T]}\Big\{\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}(\Upsilon(s)+b\,|z(s)|)ds\\
&\quad+\frac{1}{|\Delta|}\Big(\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}(\Upsilon(s)+b\,|z(s)|)ds\\
&\quad+|\delta|\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}(\Upsilon(s)+b\,|z(s)|)ds\Big)\Big\}\\
&\leq \|\Upsilon\|\max_{t\in[0,T]}\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}ds+b\,\rho\max_{t\in[0,T]}\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}e^{\mu s}ds\\
&\quad+\frac{1}{|\Delta|}\Big(\sum_{i=1}^{n}|\sigma_i|\|\Upsilon\|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}ds\\
&\quad+b\,\rho\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}e^{\mu s}ds\\
&\quad+|\delta|\|\Upsilon\|\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}ds+b\,\rho|\delta|\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}e^{\mu s}ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{T^{\omega}\|\Upsilon\|}{\Gamma(\omega+1)} + b\,\rho \max_{t\in[0,T]} \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} e^{\mu s} ds \\
&\quad + \frac{1}{|\Delta|}\bigg(\frac{\|\Upsilon\| \sum_{i=1}^{n} |\sigma_i|\, \gamma_i^{\omega}}{\Gamma(\omega+1)} + b\,\rho \sum_{i=1}^{n} |\sigma_i| \int_0^{\gamma_i} \frac{(\gamma_i - s)^{\omega-1}}{\Gamma(\omega)} e^{\mu s} ds \\
&\quad + \frac{|\delta| T^{\omega} \|\Upsilon\|}{\Gamma(\omega+1)} + b\,\rho |\delta| \int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)} e^{\mu s} ds \bigg).
\end{aligned}
$$

Then, by using the estimate found in Remark 3.1, the above inequality takes the form

$$
\begin{aligned}
\|(\mathcal{Q}_2 z)\| &\leq \frac{\|\Upsilon\|}{|\Delta|\Gamma(\omega+1)} \bigg(T^{\omega}[|\Delta| + |\delta|] + \sum_{i=1}^{n} |\sigma_i| \gamma_i^{\omega} \bigg) \\
&\quad + \frac{b\,\rho}{|\Delta|\mu^{\omega}} \bigg(\sum_{i=1}^{n} |\sigma_i| e^{\mu \gamma_i} + [|\Delta| + |\delta|] e^{\mu\, T} \bigg).
\end{aligned}
$$

This shows that $\mathcal{Q}_2(B)$ is uniformly bounded as $z \in B$ is an arbitrary element.

Note that continuity of $\psi(t, z(t))$ implies that of the operator $\mathcal{Q}_2$. To verify the compactness of the operator $\mathcal{Q}_2$, let us show that $\mathcal{Q}_2(B)$ forms a family of equicontinuous functions. Letting $\max\limits_{(t,z)\in[0,T]\times B} |\psi(t, z(t))| = \kappa < \infty$, $\xi_1, \xi_2 \in [0, T]$ with $\xi_1 < \xi_2$ and $z \in B$, we have

$$
\begin{aligned}
|(\mathcal{Q}_2 z)(\xi_2) - (\mathcal{Q}_2 z)(\xi_1)| &= \bigg| \int_0^{\xi_1} \bigg(\frac{(\xi_2 - s)^{\omega-1} - (\xi_1 - s)^{\omega-1}}{\Gamma(\omega)} \bigg) \psi(s, z(s)) ds \\
&\quad + \int_{\xi_1}^{\xi_2} \frac{(\xi_2 - s)^{\omega-1}}{\Gamma(\omega)} \psi(s, z(s)) ds \bigg| \\
&\leq \frac{\kappa}{\Gamma(\omega+1)} [|\xi_2^{\omega} - \xi_1^{\omega}| + 2|\xi_2 - \xi_1|^{\omega}] \to 0 \text{ as } (\xi_2 - \xi_1) \to 0,
\end{aligned}
$$

independently of $z \in B$. Thus, $\mathcal{Q}_2(B)$ is an equicontinuous set and hence it is uniformly bounded and equicontinuous. In consequence, $\mathcal{Q}_2(B)$ is relatively compact by an application of the Arzelà–Ascoli theorem [21]. Therefore, we conclude that $\mathcal{Q}_2$ is a continuous and compact operator on B.

Finally, we verify the last assumption of Theorem 3.2. Taking $z \in \mathcal{X}, y \in B$, we have

$$
\begin{aligned}
|z(t)| &\leq |(\mathcal{Q}_1 z)(t)| + |(\mathcal{Q}_2 y)(t)| \\
&\leq |p(t, z(t))| + \frac{1}{|\Delta|} \bigg| \sum_{i=1}^{n} \sigma_i p(\gamma_i, z(\gamma_i)) - p(0, z(0)) + \delta p(T, z(T)) \bigg| \\
&\quad + \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} |\psi(s, y(s))| ds + \frac{1}{|\Delta|} \bigg(\sum_{i=1}^{n} |\sigma_i| \int_0^{\gamma_i} \frac{(\gamma_i - s)^{\omega-1}}{\Gamma(\omega)} |\psi(s, y(s)| ds \\
&\quad + |\delta| \int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)} |\psi(s, y(s))| ds \bigg) \\
&\leq |p(t, z(t)) - p(t, 0)| + |p(t, 0)|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Delta|}\left|\sum_{i=1}^{n}\sigma_i p(\gamma_i,z(\gamma_i))-p(0,z(0))+\delta p(T,z(T))\right|\\
&+\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}(\Upsilon(s)+b\,|y(s)|)ds\\
&+\frac{1}{|\Delta|}\bigg(\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}(\Upsilon(s)+b\,|y(s)|)ds\\
&+|\delta|\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}(\Upsilon(s)+b\,|y(s)|)ds\bigg)\\
\leq\;&\mathcal{K}_1|z(t)|\\
&+\underbrace{\max_{t\in[0,T]}|p(t,0)|+\frac{1}{|\Delta|}\left|\sum_{i=1}^{n}\sigma_i p(\gamma_i,z(\gamma_i))-p(0,z(0))+\delta p(T,z(T))\right|}_{:=m}\\
&+\frac{\|\Upsilon\|T^{\omega}}{\Gamma(\omega+1)}+b\,\rho\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}e^{\mu s}ds+\frac{1}{|\Delta|}\bigg(\frac{\|\Upsilon\|\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}}{\Gamma(\omega+1)}\\
&+b\,\rho\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}e^{\mu s}ds+\frac{|\delta|\|\Upsilon\|T^{\omega}}{\Gamma(\omega+1)}\\
&+b|\delta|\rho\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}e^{\mu s}ds\bigg),
\end{aligned}
$$

which can alternatively be written as

$$
\begin{aligned}
|z(t)| \;\leq\;& \frac{1}{(1-\mathcal{K}_1)|\Delta|}\bigg(m|\Delta|+\frac{\|\Upsilon\|[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{\Gamma(\omega+1)}\\
&+\frac{b\,\rho[\sum_{i=1}^{n}|\sigma_i|e^{\gamma_i\mu}+|\delta|e^{T\mu}]}{\mu^{\omega}}\bigg)+\frac{b\,\rho\,e^{\mu t}}{(1-\mathcal{K}_1)\mu^{\omega}}.
\end{aligned}
$$

Now, $z\in B$ when $|z(t)|\leq\rho\,e^{\mu t}$ for any $t\in[0,T]$. Thus, we have

$$
\begin{aligned}
|z(t)| \;\leq\;& \frac{1}{(1-\mathcal{K}_1)|\Delta|}\bigg(m|\Delta|+\frac{\|\Upsilon\|[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{\Gamma(\omega+1)}\\
&+\frac{b\,\rho[\sum_{i=1}^{n}|\sigma_i|e^{\gamma_i\mu}+|\delta|e^{T\mu}]}{\mu^{\omega}}\bigg)+\frac{b\,\rho\,e^{\mu t}}{(1-\mathcal{K}_1)\mu^{\omega}}\leq\rho\,e^{\mu t}.
\end{aligned}
$$

From the above inequality, we have

$$
\rho\,e^{\mu t}\left(1-\frac{b}{(1-\mathcal{K}_1)\mu^{\omega}}\right)\;\geq\;\frac{1}{(1-\mathcal{K}_1)|\Delta|}\bigg(m|\Delta|+\frac{\|\Upsilon\|[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{\Gamma(\omega+1)}
$$

$$+\frac{b\,\rho[\sum_{i=1}^{n}|\sigma_i|e^{\gamma_i\mu}+|\delta|e^{T\mu}]}{\mu^{\omega}}\Bigg),$$

which leads to

$$\rho\,e^{\mu t} \geq \frac{\left(m|\Delta|+\frac{\|\Upsilon\|[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{\Gamma(\omega+1)}+\frac{b\,\rho[\sum_{i=1}^{n}|\sigma_i|e^{\gamma_i\mu}+|\delta|e^{T\mu}]}{\mu^{\omega}}\right)}{(1-\mathcal{K}_1)|\Delta|\left(1-\frac{b}{(1-\mathcal{K}_1)\mu^{\omega}}\right)}.$$

Let $\mu>0$ and $\rho>0$ be large enough to satisfy the inequalities

$$\mu^{\omega}>\frac{b}{1-\mathcal{K}_1}, \tag{3.5}$$

and

$$\begin{aligned}&\frac{1}{(1-\mathcal{K}_1)|\Delta|}\left(m|\Delta|+\frac{\|\Upsilon\|[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{\Gamma(\omega+1)}\right.\\&\left.+\frac{b\,\rho[\sum_{i=1}^{n}|\sigma_i|e^{\gamma_i\mu}+|\delta|e^{T\mu}]}{\mu^{\omega}}\right)\\<\;&\min_{t\in[0,T]}\rho\,e^{\mu t}=\rho.\end{aligned} \tag{3.6}$$

Note that ρ appears on both sides of (3.6) through the exponential terms. So, we provide an explicit lower bound that resolves this circularity. We rewrite (3.6) in the form

$$A+B\rho<\rho,$$

where

$$A=\frac{1}{(1-\mathcal{K}_1)|\Delta|}\left(m|\Delta|+\frac{\|\Upsilon\|\left[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}\right]}{\Gamma(\omega+1)}\right)>0,$$

$$B=\frac{b}{(1-\mathcal{K}_1)|\Delta|\,\mu^{\omega}}\left(\sum_{i=1}^{n}|\sigma_i|e^{\mu\gamma_i}+|\delta|e^{\mu T}\right)>0.$$

Since condition (3.5) guarantees $\mu^{\omega}>b/(1-\mathcal{K}_1)$, so B depends on μ but *not* on ρ. The inequality $A+B\rho<\rho$ is equivalent to $A<(1-B)\rho$. One must, therefore, first choose μ large enough so that (3.5) holds and additionally $B<1$ (which is achievable since $B\to 0$ as $\mu\to\infty$). Once such a μ is fixed, any $\rho>A/(1-B)$ satisfies (3.6). This provides the explicit lower bound on ρ and removes the circularity.

Moreover, from (3.6), we have

$$\begin{aligned}&\frac{1}{(1-\mathcal{K}_1)|\Delta|}\left(m|\Delta|+\frac{\|\Upsilon\|[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{\Gamma(\omega+1)}+\frac{b\,\rho[\sum_{i=1}^{n}|\sigma_i|e^{\gamma_i\mu}+|\delta|e^{T\mu}]}{\mu^{\omega}}\right)\\\leq\;&\rho\,e^{\mu t}.\end{aligned}$$

Therefore, we deduce from (3.5) and (3.6) that $|z(t)| \leq \rho\, e^{\mu t}$, which means that $z \in B$. Thus, all the assumptions of Theorem 3.2 are satisfied and hence its conclusion implies that the problem (1.1) has at least one solution on $[0, T]$. □

3.2. Uniqueness result via Banach's fixed point theorem

Here, we accomplish the existence of a unique solution to the problem (1.1) by means of Banach's fixed point theorem.

Theorem 3.4. *Let* (A_1) *and* (A_2) *hold. Then, there exists a unique solution to the problem* (1.1) *on* $[0, T]$ *if* $\theta_1 < 1$, *where*

$$\theta_1 = \frac{\mathcal{K}_1(|\Delta| + \sum_{i=1}^{n} |\sigma_i| + |\delta| + 1)}{|\Delta|} + \frac{\mathcal{K}_2(T^{\omega}[|\Delta| + |\delta|] + \sum_{i=1}^{n} |\sigma_i|\, \gamma_i^{\omega})}{|\Delta|\Gamma(\omega + 1)}. \tag{3.7}$$

Proof. In the first step, we define a bounded and closed ball with radius ϱ in the Banach space $\mathcal{X}$ as $B_\varrho = \{z \in \mathcal{X} : \|z\| \leq \varrho\}$ with $\varrho \geq \dfrac{\theta_2}{1 - \theta_1}$ and show that $\mathcal{Q}B_\varrho \subset B_\varrho$, where $\mathcal{Q}$ is defined in (3.1),

$$\theta_2 = \frac{M_1(|\Delta| + \sum_{i=1}^{n} |\sigma_i| + |\delta| + 1)}{|\Delta|} + \frac{M_2(T^{\omega}[|\Delta| + |\delta|] + \sum_{i=1}^{n} |\sigma_i|\, \gamma_i^{\omega})}{|\Delta|\Gamma(\omega + 1)}, \tag{3.8}$$

$M_1 = \max_{t \in [0,T]} |p(t, 0)|$ and $M_2 = \max_{t \in [0,T]} |\psi(t, 0)|$. Let $z \in B_\varrho, t \in [0, T]$. Then, by using (A_1) and (A_2), we find that

$$\begin{aligned}
|p(t, z(t))| &= |p(t, z(t)) - p(t, 0) + p(t, 0)| \leq |p(t, z(t)) - p(t, 0)| + |p(t, 0)| \\
&\leq \mathcal{K}_1\|z\| + M_1 \leq \mathcal{K}_1\varrho + M_1, \\
|\psi(t, z(t))| &= |\psi(t, z(t)) - \psi(t, 0) + \psi(t, 0)| \leq |\psi(t, z(t)) - \psi(t, 0)| + |\psi(t, 0)| \\
&\leq \mathcal{K}_2\|z\| + M_2 \leq \mathcal{K}_2\varrho + M_2.
\end{aligned}$$

Hence, for any $z \in B_\varrho, t \in [0, T]$, we find that

$$\begin{aligned}
\|(\mathcal{Q}z)\| &= \max_{t \in [0,T]} |(\mathcal{Q}z)(t)| \\
&\leq \max_{t \in [0,T]} \Bigg\{ \int_0^t \frac{(t - s)^{\omega - 1}}{\Gamma(\omega)} |\psi(s, z(s))| ds + |p(t, z(t))| \\
&\quad + \frac{1}{|\Delta|} \Bigg[\sum_{i=1}^{n} |\sigma_i| |p(\gamma_i, z(\gamma_i))| \\
&\quad + \sum_{i=1}^{n} |\sigma_i| \int_0^{\gamma_i} \frac{(\gamma_i - s)^{\omega - 1}}{\Gamma(\omega)} |\psi(s, z(s)| ds + |p(0, z(0))| \\
&\quad + |\delta| \Bigg(|p(T, z(T))| + \int_0^T \frac{(T - s)^{\omega - 1}}{\Gamma(\omega)} |\psi(s, z(s))| ds \Bigg) \Bigg] \Bigg\} \\
&\leq \frac{T^{\omega}(\mathcal{K}_2\varrho + M_2)}{\Gamma(\omega + 1)} + \mathcal{K}_1\varrho + M_1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Delta|}\Bigg[\sum_{i=1}^{n}|\sigma_i|(\mathcal{K}_1\varrho+M_1)+\frac{\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}(\mathcal{K}_2\varrho+M_2)}{\Gamma(\omega+1)}+\mathcal{K}_1\varrho+M_1\\
&+|\delta|\left(\mathcal{K}_1\varrho+M_1+\frac{T^{\omega}(\mathcal{K}_2\varrho+M_2)}{\Gamma(\omega+1)}\right)\Bigg]\\
=\;&\frac{(\mathcal{K}_1\varrho+M_1)[|\Delta|+\sum_{i=1}^{n}|\sigma_i|+|\delta|+1]}{|\Delta|}\\
&+\frac{(\mathcal{K}_2\varrho+M_2)[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{|\Delta|\Gamma(\omega+1)}\\
=\;&\varrho\left[\frac{\mathcal{K}_1[|\Delta|+\sum_{i=1}^{n}|\sigma_i|+|\delta|+1]}{|\Delta|}+\frac{\mathcal{K}_2[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{|\Delta|\Gamma(\omega+1)}\right]\\
&+\frac{M_1[|\Delta|+\sum_{i=1}^{n}|\sigma_i|+|\delta|+1]}{|\Delta|}+\frac{M_2[T^{\omega}(|\Delta|+|\delta|)+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}]}{|\Delta|\Gamma(\omega+1)}\\
=\;&\varrho\,\theta_1+\theta_2\le\varrho.
\end{aligned}
$$

Thus, $\mathcal{Q}z\in B_\varrho$. Since $z\in B_\varrho$ is an arbitrary element, we conclude that $QB_\varrho\subset B_\varrho$.

Next, we take $z_1,z_2\in B_\varrho, t\in[0,T]$ and find that

$$
\begin{aligned}
&\|(\mathcal{Q}z_1)-(\mathcal{Q}z_2)\|=\max_{t\in[0,T]}|(\mathcal{Q}z_1)(t)-(\mathcal{Q}z_2)(t)|\\
\le\;&\max_{t\in[0,T]}\Bigg\{\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,z_1(s))-\psi(s,z_2(s))|ds\\
&+|p(t,z_1(t))-p(t,z_2(t))|\\
&+\frac{1}{|\Delta|}\Bigg[\sum_{i=1}^{n}|\sigma_i||p(\gamma_i,z_1(\gamma_i))-p(\gamma_i,z_2(\gamma_i))|\\
&+\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,z_1(s))-\psi(s,z_2(s))|ds\\
&+|p(0,z_1(0))-p(0,z_2(0))|+|\delta|\Bigg(|p(T,z_1(T))-p(T,z_2(T))|\\
&+\int_0^T\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,z_1(s))-\psi(s,z_2(s))|ds\Bigg)\Bigg]\Bigg\}\\
\le\;&\frac{T^{\omega}\mathcal{K}_2\|z_1-z_2\|}{\Gamma(\omega+1)}+\mathcal{K}_1\|z_1-z_2\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Delta|}\Bigg[\sum_{i=1}^{n}|\sigma_i|\mathcal{K}_1\|z_1-z_2\|+\frac{\mathcal{K}_2\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}\|z_1-z_2\|}{\Gamma(\omega+1)}\\
&+\mathcal{K}_1\|z_1-z_2\|+|\delta|\left(\mathcal{K}_1\|z_1-z_2\|+\frac{T^{\omega}\mathcal{K}_2\|z_1-z_2\|}{\Gamma(\omega+1)}\right)\Bigg]\\
=&\quad\left[\frac{\mathcal{K}_1(|\Delta|+\sum_{i=1}^{n}|\sigma_i|+|\delta|+1)}{|\Delta|}+\frac{\mathcal{K}_2(T^{\omega}[|\Delta|+|\delta|]+\sum_{i=1}^{n}|\sigma_i|\,\gamma_i^{\omega})}{|\Delta|\Gamma(\omega+1)}\right]\|z_1-z_2\|\\
=&\quad\theta_1\|z_1-z_2\|,
\end{aligned}
$$

where θ_1 defined by (3.7). Thus, $\mathcal{Q}$ is a contraction as $\theta_1<1$. Therefore, the hypothesis of Banach's fixed point theorem is satisfied and hence, $\mathcal{Q}$ has a unique fixed point which corresponds to a unique solution to the problem (1.1) on $[0,T]$. □

Remark 3.5. Assumption (A_2) states that ψ is Lipschitz in its second argument with constant $\mathcal{K}_2>0$, that is,

$$|\psi(t,z)-\psi(t,\tilde{z})|\le\mathcal{K}_2|z-\tilde{z}|\quad\text{for all } z,\tilde{z}\in\mathbb{R},\ t\in[0,T].$$

Setting $\tilde{z}=0$ and letting $M_2=\max_{t\in[0,T]}|\psi(t,0)|$, one can obtain

$$|\psi(t,z(t))|\le\mathcal{K}_2|z(t)|+M_2,$$

which is exactly (A_3) with $\Upsilon(t)\equiv M_2$ and $b=\mathcal{K}_2$. Hence (A_2) implies (A_3). However, (A_3) is only needed independently of the assumption (A_2) in the hypothesis of Theorem 3.3 (the existence result via Krasnosel'skiĭ's theorem), while Theorem 3.4 (uniqueness result) assumes the condition (A_2).

Example 3.6. Consider the following hybrid fractional boundary value problem

$$
\begin{cases}
{}^{c}D^{\frac{1}{4}}(z(t)-p(t,z(t)))=\psi(t,z(t)),\ t\in[0,1],\\
z(0)-\dfrac{1}{3}z(1)=\displaystyle\sum_{i=1}^{3}\sigma_i z(\gamma_i),\ \ 0<\gamma_i<1,
\end{cases}
\tag{3.9}
$$

where $\omega=\dfrac{1}{4},\delta=\dfrac{1}{3},\sigma_1=\dfrac{1}{4},\sigma_2=\dfrac{1}{3},\sigma_3=\dfrac{1}{2},\gamma_1=\dfrac{1}{9},\gamma_2=\dfrac{1}{8},\gamma_3=\dfrac{1}{6}$.

(a) For illustrating Theorem 3.3, we take

$$p(t,z(t))=\frac{\tan^{-1}z(t)}{35(t^2+1)}+3\cos(t^2+4),\ \ \psi(t,z(t))=\frac{6}{t^3+83}+\frac{1}{8}\sin z(t).\tag{3.10}$$

Using the given data, we find that

$$
\begin{aligned}
&\sum_{i=1}^{3}\sigma_i=\frac{13}{12},\ \ \Delta=1-\delta-\sum_{i=1}^{3}\sigma_i\approx-0.41666667,\\
&\Theta=1+\frac{1}{|\Delta|}\left(\sum_{i=1}^{n}|\sigma_i|+1+|\delta|\right)\approx 6.799999985.
\end{aligned}
$$

Also, we have $\Upsilon(t) = \dfrac{6}{t^3+83}, b = \dfrac{1}{8}$ as

$$|\psi(t,z(t))| \leq \frac{6}{t^3+83} + \frac{1}{8}|z(t)|.$$

Moreover, $p(t,z(t))$ satisfies the Lipschitz condition with $\mathcal{K}_1 = \dfrac{1}{35}$ and $\Theta\mathcal{K}_1 \simeq 0.19428571 < 1$. Therefore, the hypotheses of Theorem 3.3 holds true and its conclusion implies that the problem (3.9) with $p(t,z(t))$ and $\psi(t,z(t))$ given in (3.10) has at least one solution on $[0,1]$.

(b) For demonstrating the application of Theorem 3.4, let us consider

$$p(t,z(t)) = \frac{\sin z(t)}{5\sqrt{t^2+25}} + \sqrt[3]{t^4+32},\ \psi(t,z(t)) = \frac{\cos z(t)}{9\sqrt{t^4+81}} + \sin(t^2+6) \tag{3.11}$$

in problem (3.9). We compute the Lipschitz constants from (3.11) via mean value theorem (MVT) as follows:

$$\left|\frac{\partial p}{\partial z}\right| = \frac{|\cos z|}{5\sqrt{t^2+25}} \leq \frac{1}{25} = K_1,\ \left|\frac{\partial \psi}{\partial z}\right| = \frac{|\sin z|}{9\sqrt{t^4+81}} \leq \frac{1}{81} = K_2.$$

Now, we compute θ_1 from (3.7). Using the given values, we find that

$$|\Delta| + \sum_{i=1}^{3} |\sigma_i| + |\delta| + 1 = \frac{17}{6}.$$

$$\frac{1}{\Gamma(\omega+1)}\left(T^{\omega}(|\Delta|+|\delta|) + \sum |\sigma_i|\gamma_i^{\omega}\right) = \frac{1.41126}{0.90640} \approx 1.556995.$$

$$\begin{aligned}\theta_1 &= \frac{\mathcal{K}_1(|\Delta| + \sum_{i=1}^{n}|\sigma_i| + |\delta| + 1)}{|\Delta|} + \frac{\mathcal{K}_2(T^{\omega}[|\Delta|+|\delta|] + \sum_{i=1}^{n}|\sigma_i|\,\gamma_i^{\omega})}{|\Delta|\Gamma(\omega+1)} \\ &= \frac{12}{5}\left(\frac{2.83333}{25} + \frac{1.556995}{81}\right) \approx 0.318133 < 1.\end{aligned}$$

Since the hypothesis Theorem 3.4 is satisfied, therefore, we deduce from its conclusion that there exists a unique solution to the problem (3.9) with $p(t,z(t))$ and $\psi(t,z(t))$ given in (3.11) on $[0,1]$.

4. Ulam–Hyers–Rassias stability

Before presenting the Ulam–Hyers–Rassias stability for the problem(1.1), we formulate some fundamental definitions [18].

Definition 4.1. *For $\widetilde{\delta} > 0$, consider the inequality*

$$|{}^cD^{\omega}(\widehat{z}(t) - p(t,\widehat{z}(t))) - \psi(t,\widehat{z}(t))| \leq \widetilde{\delta},\ 0 < \omega \leq 1 \tag{4.1}$$

with the boundary condition

$$\widehat{z}(0) - \delta\widehat{z}(T) = \sum_{i=1}^{n} \sigma_i \widehat{z}(\gamma_i),\ \delta, \sigma_i \in \mathbb{R},\ 0 < \gamma_i < T. \tag{4.2}$$

Then, the problem (1.1) *is called Ulam–Hyers stable if we can find* $\varpi > 0$, *such that, for each solution* $\widehat{z} \in C([0,T],\mathbb{R})$ *of* (4.1)-(4.2), *there exists a unique solution* $z \in C([0,T],\mathbb{R})$ *of* (1.1) *satisfying*

$$\|\widehat{z} - z\| \leq \varpi\widetilde{\delta},\ t \in [0,T].$$

Definition 4.2. *If there exists a function* $\Psi : [0,\infty) \to [0,\infty)$ *with* $\Psi(0) = 0$, *such that, for each solution* $\widehat{z} \in C([0,T],\mathbb{R})$ *of* (4.1)-(4.2), *there exists a unique solution* $z \in C([0,T],\mathbb{R})$ *of* (1.1) *satisfying*

$$\|\widehat{z} - z\| \leq \Psi(\widetilde{\delta}),\ t \in [0,T],\ \widetilde{\delta} > 0,$$

then, the problem (1.1) *is generalized Ulam–Hyers stable.*

Remark 4.3. The function $\widehat{z} \in C([0,T],\mathbb{R})$ satisfies the problem (4.1)-(4.2) if and only if there exists a function $\varepsilon \in C([0,T],\mathbb{R})$ such that, $|\varepsilon(t)| \leq \widetilde{\delta},\ \widetilde{\delta} > 0$ and

$${}^{c}D^{\omega}(\widehat{z}(t) - p(t,\widehat{z}(t))) = \psi(t,\widehat{z}(t)) + \varepsilon(t), t \in [0,T]$$

with the boundary condition (4.2).

Theorem 4.4. *Let the hypotheses of Theorem 3.4 hold. Then, the problem* (1.1) *is Ulam–Hyers stable and hence, generalized Ulam–Hyers stable in* $C([0,T],\mathbb{R})$.

Proof. According to Theorem 3.4, there exists a unique solution z to the problem (1.1), where $z = \mathcal{Q}z$. Let $\widehat{z}$ be a solution to the fractional hybrid differential inequality (4.1) with the boundary condition (4.2). Then, by Remark 4.3, there is a function $\varepsilon \in C([0,T])$, with $|\varepsilon(t)| \leq \widetilde{\delta},\ \widetilde{\delta} > 0$, $t \in [0,T]$, such that

$${}^{c}D^{\omega}(\widehat{z}(t) - p(t,\widehat{z}(t))) = \psi(t,\widehat{z}(t)) + \varepsilon(t), t \in [0,T], \tag{4.3}$$

equipped with the boundary condition (4.2). By Lemma 2.5, the solution of (4.3) with the boundary condition (4.2) can be expressed as

$$\begin{aligned}
\widehat{z}(t) \quad = \quad & \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}(\psi(s,\widehat{z}(s)) + \varepsilon(s))ds + p(t,\widehat{z}(t)) \\
& + \frac{1}{\Delta}\Bigg[\sum_{i=1}^{n}\sigma_i p(\gamma_i,\widehat{z}(\gamma_i)) + \sum_{i=1}^{n}\sigma_i \int_0^{\gamma_i} \frac{(\gamma_i - s)^{\omega-1}}{\Gamma(\omega)}(\psi(s,\widehat{z}(s)) + \varepsilon(s))ds \\
& - p(0,\widehat{z}(0)) + \delta\Bigg(p(T,\widehat{z}(T)) + \int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)}(\psi(s,\widehat{z}(s)) + \varepsilon(s))ds\Bigg)\Bigg].
\end{aligned}$$

Since $|\varepsilon| \leq \widetilde{\delta}$, we have

$$\begin{aligned}
& \|\widehat{z} - (\mathcal{Q}\widehat{z})\| = \max_{t\in[0,T]} |\widehat{z}(t) - (\mathcal{Q}\widehat{z})(t)| \\
= \quad & \max_{t\in[0,T]} \Bigg\{\Bigg| \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}(\psi(s,\widehat{z}(s)) + \varepsilon(s))ds + p(t,\widehat{z}(t))
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Delta}\bigg[\sum_{i=1}^{n}\sigma_i p(\gamma_i,\widehat{z}(\gamma_i))+\sum_{i=1}^{n}\sigma_i\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}(\psi(s,\widehat{z}(s))+\varepsilon(s))ds-p(0,\widehat{z}(0))\\
&+\delta\bigg(p(T,\widehat{z}(T))+\int_0^{T}\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}(\psi(s,\widehat{z}(s))+\varepsilon(s))ds\bigg)\bigg]\\
&-\int_0^{t}\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\psi(s,\widehat{z}(s))ds-p(t,\widehat{z}(t))\\
&-\frac{1}{\Delta}\bigg[\sum_{i=1}^{n}\sigma_i p(\gamma_i,\widehat{z}(\gamma_i))+\sum_{i=1}^{n}\sigma_i\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}\psi(s,\widehat{z}(s))ds-p(0,\widehat{z}(0))\\
&+\delta\bigg(p(T,\widehat{z}(T))+\int_0^{T}\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}\psi(s,\widehat{z}(s))ds\bigg)\bigg]\bigg|\bigg\}\\
\leq\ &\max_{t\in[0,T]}\bigg\{\int_0^{t}\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}|\varepsilon(s)|ds\\
&+\frac{1}{|\Delta|}\bigg[\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}|\varepsilon(s)|ds+|\delta|\int_0^{T}\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}|\varepsilon(s)|ds\bigg]\bigg\}\\
\leq\ &\widetilde{\delta}\bigg(\frac{T^{\omega}[|\Delta|+|\delta|]+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}}{|\Delta|\Gamma(\omega+1)}\bigg)=\widetilde{\delta}\,\upsilon,
\end{aligned}
\tag{4.4}
$$

where

$$
\upsilon=\frac{T^{\omega}[|\Delta|+|\delta|]+\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}}{|\Delta|\Gamma(\omega+1)}. \tag{4.5}
$$

It follows by the assumptions (A_1) and (A_2) that

$$
\begin{aligned}
&\|(\mathcal{Q}\widehat{z})-(\mathcal{Q}z)\|=\max_{t\in[0,T]}|(\mathcal{Q}\widehat{z})(t)-(\mathcal{Q}z)(t)|\\
\leq\ &\max_{t\in[0,T]}\bigg\{\int_0^{t}\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,\widehat{z}(s))-\psi(s,z(s))|ds+|p(t,\widehat{z}(t))-p(t,z(t))|\\
&+\frac{1}{|\Delta|}\bigg[\sum_{i=1}^{n}|\sigma_i||p(\gamma_i,\widehat{z}(\gamma_i))-p(\gamma_i,z(\gamma_i))|\\
&+\sum_{i=1}^{n}|\sigma_i|\int_0^{\gamma_i}\frac{(\gamma_i-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,\widehat{z}(s))-\psi(s,z(s))|ds\\
&+|p(0,\widehat{z}(0))-p(0,z(0))|\\
&+|\delta|\bigg(|p(T,\widehat{z}(T))-p(T,z(T))|+\int_0^{T}\frac{(T-s)^{\omega-1}}{\Gamma(\omega)}|\psi(s,\widehat{z}(s))-\psi(s,z(s))|ds\bigg)\bigg]\bigg\}\\
\leq\ &\frac{T^{\omega}\mathcal{K}_2\|\widehat{z}-z\|}{\Gamma(\omega+1)}+\mathcal{K}_1\|\widehat{z}-z\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Delta|}\Bigg[\mathcal{K}_1\sum_{i=1}^{n}|\sigma_i|\|\widehat{z}-z\|+\frac{\mathcal{K}_2\sum_{i=1}^{n}|\sigma_i|\gamma_i^{\omega}\|\widehat{z}-z\|}{\Gamma(\omega+1)}+\mathcal{K}_1\|\widehat{z}-z\|\\
&+|\delta|\left(\mathcal{K}_1\|\widehat{z}-z\|+\frac{T^{\omega}\mathcal{K}_2\|\widehat{z}-z\|}{\Gamma(\omega+1)}\right)\Bigg]\\
=&\left[\frac{\mathcal{K}_1(|\Delta|+\sum_{i=1}^{n}|\sigma_i|+|\delta|+1)}{|\Delta|}+\frac{\mathcal{K}_2(T^{\omega}[|\Delta|+|\delta|]+\sum_{i=1}^{n}|\sigma_i|\,\gamma_i^{\omega})}{|\Delta|\Gamma(\omega+1)}\right]\|\widehat{z}-z\|\\
=&\ \theta_1\|\widehat{z}-z\|, \qquad (4.6)
\end{aligned}
$$

where θ_1 given by (3.7). Then, from (4.4) and (4.6), we obtain

$$
\begin{aligned}
\|\widehat{z}-z\| &= \|\widehat{z}-(\mathcal{Q}z)\|\\
&\le \|\widehat{z}-(\mathcal{Q}\widehat{z})\|+\|(\mathcal{Q}\widehat{z})-(\mathcal{Q}z)\|\\
&\le \widetilde{\delta}\upsilon+\theta_1\|\widehat{z}-z\|,
\end{aligned}
$$

which can be rewritten as

$$\|\widehat{z}-z\|\le\frac{\widetilde{\delta}\upsilon}{1-\theta_1}, \qquad (4.7)$$

where υ and θ_1 are given by (4.5) and (3.7), respectively. Letting $\varpi=\dfrac{\upsilon}{1-\theta_1}$ in (4.7), we get

$$\|\widehat{z}-z\|\le\varpi\widetilde{\delta}.$$

Therefore, the problem (1.1) is Ulam–Hyers stable. Moreover, it is generalized Ulam–Hyers stable in $C([0,T],\mathbb{R})$ if we take $\Psi(\widetilde{\delta})=\dfrac{\widetilde{\delta}\upsilon}{1-\theta_1}=\varpi\widetilde{\delta}$, with $\Psi(0)=0$, that is,

$$\|\widehat{z}-z\|\le\Psi(\widetilde{\delta}).$$

□

Definition 4.5. *Let $\Phi\in C([0,T],[0,\infty))$ be a given function. The problem* (1.1) *is called* Ulam–Hyers–Rassias stable *with respect to Φ if there exists a constant $c_\Phi>0$ such that, for each $\tilde{\delta}>0$ and each $\hat{z}\in C([0,T],\mathbb{R})$ satisfying*

$$|{}^cD^{\omega}(\hat{z}(t)-p(t,\hat{z}(t)))-\psi(t,\hat{z}(t))|\le\tilde{\delta}\,\Phi(t),\quad t\in[0,T],$$

together with the boundary condition (4.2), there exists a unique solution $z\in C([0,T],\mathbb{R})$ of (1.1) *satisfying*

$$\|\hat{z}-z\|\le c_\Phi\,\tilde{\delta}\,\|\Phi\|.$$

Theorem 4.6. *Let the hypotheses of Theorem 3.4 hold, and let $\Phi\in C([0,T],[0,\infty))$ be a nondecreasing function. Then, the problem* (1.1) *is Ulam–Hyers–Rassias stable with respect to Φ.*

Proof. Following the argument of Theorem 4.4 with $|\varepsilon(t)| \le \tilde{\delta}\,\Phi(t)$ in place of $|\varepsilon(t)| \le \tilde{\delta}$, the estimate (4.4) becomes

$$\|\hat{z} - \mathcal{Q}\hat{z}\| \le \tilde{\delta}\left(\frac{T^{\omega}(|\Delta| + |\delta|) + \sum_{i=1}^{n} |\sigma_i| \gamma_i^{\omega}}{|\Delta|\,\Gamma(\omega+1)} \right) \|\Phi\| = \tilde{\delta}\,\upsilon\,\|\Phi\|,$$

where υ is given by (4.5) and we used the monotonicity of Φ to bound $\Phi(s) \le \|\Phi\|$ under each integral. The remainder of the proof is identical to that of Theorem 4.4, yielding

$$\|\hat{z} - z\| \le \frac{\upsilon}{1-\theta_1}\,\tilde{\delta}\,\|\Phi\| =: c_{\Phi}\,\tilde{\delta}\,\|\Phi\|.$$

Hence, the problem is Ulam–Hyers–Rassias stable with $c_{\Phi} = \upsilon/(1-\theta_1)$. □

Example 4.7. The problem (3.9) with p and ψ given in (3.11) is Ulam–Hyers stable since $\theta_1 \simeq 0.318133 < 1, \upsilon \simeq 3.73876374$ and $\varpi \simeq 5.48332522$. Also, it is generalized Ulam–Hyers stable with $\Psi(\widetilde{\delta}) \simeq 5.48332522\,\widetilde{\delta}$. It is also Ulam–Hyers–Rassias stable with $\Phi(t) = e^t$.

5. Conclusions

In this work, we have investigated a nonlinear hybrid fractional differential equation subject to nonlocal multipoint boundary conditions. By transforming the given problem into an equivalent integral equation, we established the existence of at least one solution through a variant of Krasnosel'skiĭ's fixed point theorem due to Burton. Moreover, sufficient conditions ensuring the uniqueness of the solution to the problem at hand were obtained via Banach's contraction principle. In addition to the solvability analysis, we examined the Ulam–Hyers–Rassias stability of the given problem, thereby demonstrating the robustness of its solutions with respect to small perturbations. Several examples were provided to illustrate the applicability of the obtained theoretical results.

The analysis presented here contributes to the growing literature on hybrid fractional differential equations, particularly those involving nonlocal multipoint boundary conditions, which are known to better model various real-world phenomena than classical boundary conditions. The adopted approach is flexible and can be extended to a wider class of problems.

Possible directions for future research include the study of similar hybrid fractional differential equations with other boundary conditions, such as integral, multi-strip, periodic, or anti-periodic boundary conditions. Furthermore, the results may be generalized to systems of hybrid fractional differential equations involving other types of fractional derivative operators like Hilfer, Hadamard, or Erdélyi–Kober derivatives. The present study can also be extended to infinite or semi-infinite intervals. Other possible extensions include inclusions, higher-order fractional operators, or equations with delays and stochastic perturbations. Thus, there are several promising topics for further investigation.

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Ahmed Alsaedi
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: aalsaedi@hotmail.com

Rabab Alghamdi
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia;
College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU),
Riyadh, Saudi Arabia
e-mail: ralalghamdi@gmail.com

Bashir Ahmad
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com; bahmad@kau.edu.sa

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina,
45110, Ioannina, Greece
e-mail: sntouyas@uoi.gr