Stud. Univ. Babeş-Bolyai Math. 71(2026), No. 3, 339–357
DOI: 10.24193/subbmath.2026.3.2

Unified Wright-Mittag-Leffler generalizations of the two-variable Hermite polynomials with applications

M.A. Pathan and Talha Usman

Abstract. We develop a generating-function framework for generalized two-variable Hermite polynomials based on Wright and Mittag-Leffler kernels. Explicit coefficient formulas are obtained in both cases. The Mittag-Leffler normalization preserves the classical differential structure, including lowering, raising, and heat-type identities, and reduces to the Hermite–Kampé de Fériet family. In contrast, the Wright-Bessel case leads to modified convolution-type relations without a closed differential structure. As an application, the Mittag-Leffler framework yields explicit polynomial solutions of fractional heat-type equations involving the Caputo derivative. The results provide a unified generating-function approach connecting classical and fractional Hermite structures.

Mathematics Subject Classification (2010): 26A33, 33C10, 33C45, 33E20.

Keywords: Wright–Hermite polynomials, Mittag–Leffler functions, two-variable Hermite polynomials, fractional heat equation, Wright functions, generating functions.

1. Introduction and preliminaries

The classical two-variable Hermite polynomials, also known as the Hermite–Kampé de Fériet polynomials, arise as a natural bivariate generalization of the ordinary Hermite polynomials. They are defined through the exponential generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x,y)}{n!}\, t^n = \exp\left(xt + yt^2\right), \tag{1.1}$$

Received 19 April 2026; Accepted 01 June 2026.
© Studia UBB MATHEMATICA. Published by Babeş-Bolyai University
This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.

which determines their entire structure (see [6],[19]). Recent developments for fractional Hermite functions, including power-series solutions, Rodrigues-type representations and orthogonality properties are discussed in [1]. From (1.1) one obtains a variety of foundational identities, including the lowering relation

$$\partial_x H_n(x,y) = n\, H_{n-1}(x,y), \tag{1.2}$$

the Appell-type raising identity

$$\big(x + 2y\,\partial_x\big) H_n(x,y) = H_{n+1}(x,y), \tag{1.3}$$

and the heat-type equivalence

$$\partial_y H_n(x,y) = \partial_x^2 H_n(x,y). \tag{1.4}$$

Relations (1.2)–(1.4) describe how the polynomials interact with differential operators, and together with (1.1) they demonstrate that the sequence forms a two-variable Appell–Sheffer system intimately connected with the classical heat equation.

The aim of this paper is to construct a comprehensive Wright-type generalization of these two-variable Hermite polynomials (see [2]) by replacing the exponential kernel in (1.1) with entire kernels constructed from Wright and Mittag–Leffler functions. These special functions extend the exponential in several directions, connecting exponential growth with Bessel-type behaviour and with fractional calculus phenomena. As such, they provide a suitable analytic foundation for developing deformed or fractional extensions of Hermite-type polynomial systems.

Two kernel normalizations are introduced to facilitate this generalization. The first, which we refer to as the Wright–Bessel normalization, is defined by

$$E^{(\mathrm{A})}_{\alpha,a;b}(z) = \sum_{n=0}^{\infty} \frac{b^n\, z^n}{\Gamma(a+\alpha n)\, n!}, \tag{1.5}$$

where $\alpha > 0$ and $a, b \in \mathbb{C}$. When $(\alpha, a, b) = (1,1,1)$, (1.5) reduces not to e^z but to the Bessel-type series $\sum_{n\geq 0} z^n/(n!)^2$, reflecting its modified structure. The second kernel, based on the Mittag–Leffler function, is defined by

$$E^{(\mathrm{B})}_{\alpha,a;b}(z) = \sum_{n=0}^{\infty} \frac{b^n z^n}{\Gamma(a+\alpha n)} = E_{\alpha,a}(bz), \tag{1.6}$$

and specializes to the exponential kernel when $(\alpha, a, b) = (1,1,1)$. This ensures that the classical two-variable Hermite polynomials are exactly recovered under the corresponding parameter choice in the generalized generating function (1.11) below.

To develop the general theory in a unified way, we recall the essential special functions used in this work. Euler's gamma function is defined for $\Re(z) > 0$ by (see [18])

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, \mathrm{d}t, \tag{1.7}$$

and the Pochhammer symbol is written as $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$. The Wright function itself, a key element of our construction, is defined for $\lambda > -1$ and $\mu \in \mathbb{C}$ by (see [23])

$$W_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\lambda k + \mu)}. \tag{1.8}$$

The Mittag–Leffler function, fundamental in fractional calculus, is given for $\Re(\alpha) > 0$ by (see [5])

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \tag{1.9}$$

The exponential function appears as the special case $E_{1,1}(z) = e^z$ (see [8]) ensuring coherence with the classical setting. The foundational analytic properties of the Mittag–Leffler function were developed by Wiman (see [21, 22]). A further generalization that will prove useful is the Fox–Wright function (see [18, 20])

$${}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j n)}{\prod_{j=1}^{q} \Gamma(b_j + B_j n)} \frac{z^n}{n!}. \tag{1.10}$$

Equation (1.10) generalizes the hypergeometric function and allows compact representation of certain coefficient structures later in the paper (see [10, 17]). A comprehensive account of the H-function and its structural connections togeneralized Wright and Fox–Wright functions is given in [13].

With these ingredients in place, the generalized two-variable Wright–Hermite polynomials are defined through the following generating function

$$\sum_{n=0}^{\infty} \frac{\mathcal{W}_n(x,y)}{n!}\, t^n = \mathcal{E}_{\alpha,a;b}(xt)\, \mathcal{E}_{\beta,c;d}(yt^2), \tag{1.11}$$

where $\mathcal{E}$ denotes either of the two kernel normalizations (1.5)–(1.6). In the Mittag–Leffler normalization, the parameter specialization $(\alpha, \beta; a, c; b, d) = (1, 1; 1, 1; 1, 1)$ restores the classical generating function (1.1) and hence the classical two-variable Hermite polynomials.

Operational tools play a central role in our later development (see [3]). For a step size $h > 0$, the forward-difference operator is defined by

$$(\Delta f)(x) = f(x + h) - f(x), \tag{1.12}$$

and the corresponding h-step falling factorial is

$$x^{\underline{n}}_{(h)} = x(x - h)(x - 2h) \cdots \big(x - (n - 1)h\big), \qquad n \in \mathbb{N}_0. \tag{1.13}$$

The factorial powers in (1.13), together with (1.12), admit powerful differential/operational representations that will be used to derive formulas for the generalized kernels (1.5)–(1.6) and the resulting polynomial families via (1.11).

Throughout this paper, we use the following conventions for sets and indices:
We use $\mathbb{N}$ for the natural numbers, $\mathbb{N}_0$ for the non–negative integers, and $\mathbb{R}$ and $\mathbb{C}$ for

the real and complex numbers, respectively. The real part of $z \in \mathbb{C}$ is denoted by $\Re(z)$.

The structure of the paper is as follows. Section 1 recalls the classical two-variable Hermite–Kampé de Fériet polynomials and introduces the Wright-Bessel and Mittag-Leffler kernels that form the basis of our generalization. Section 2 defines the generalized two-variable Wright-Hermite polynomials via a generating-function framework and derives explicit coefficient formulas in both kernel normalizations. Section 3 develops the structural properties, including lowering, raising, and heat-type identities in the Mittag-Leffler case, together with coefficient extraction formulas, Euler-type homogeneity, and discrete forward-difference analogues. Section 4 presents an application to fractional heat-type equations, where explicit polynomial solutions are constructed using Mittag-Leffler operators. Section 5 concludes the paper with a summary of the main results and remarks on possible extensions.

2. Definition and properties

In view of (1.1) and the Wright-type kernels (1.5)–(1.6), we define the generalized two-variable Wright–Hermite polynomials by the exponential generating function

$$\sum_{n=0}^{\infty} \frac{\mathcal{W}_n(x,y)}{n!}\, t^n = \mathcal{E}_{\alpha,a;b}(xt)\, \mathcal{E}_{\beta,c;d}(yt^2), \tag{2.1}$$

where $\Re(\alpha), \Re(\beta) > 0$, $a, c \in \mathbb{C}$ and $b, d \in \mathbb{C}$ are fixed parameters. The symbol $\mathcal{E}$ denotes either the Wright–Bessel kernel $E^{(\mathrm{A})}$ (1.5) or the Mittag–Leffler kernel $E^{(\mathrm{B})}$ (1.6). In the Mittag–Leffler normalization, the parameter choice $(\alpha, \beta; a, c; b, d) = (1, 1; 1, 1; 1, 1)$ reduces (2.1) to the classical generating function (1.1), so that $\mathcal{W}_n(x,y) = H_n(x,y)$.

Proposition 2.1. *For $\alpha, \beta > 0$, the generating function* (2.1) *defines an entire function of $(x,y,t) \in \mathbb{C}^3$. In particular, both kernels $\mathcal{E}_{\alpha,a;b}(xt)$ and $\mathcal{E}_{\beta,c;d}(yt^2)$ are entire functions of their arguments, and hence the product on the right-hand side of* (2.1) *converges absolutely for all finite values of (x,y,t).*

Proof. The Wright-type kernels (1.5)–(1.6) are defined by series whose coefficients involve reciprocal Gamma functions with $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. It is well known that such series define entire functions of their arguments.

Hence both $\mathcal{E}_{\alpha,a;b}(xt)$ and $\mathcal{E}_{\beta,c;d}(yt^2)$ converge absolutely for all $(x,t) \in \mathbb{C}^2$ and $(y,t) \in \mathbb{C}^2$, respectively. Since the product of entire functions is entire, the right-hand side of (2.1) is entire in (x,y,t).

Finally, absolute convergence of the Cauchy product implies that the coefficients $\mathcal{W}_n(x,y)$ are well defined for all n. □

Coefficient formulas

Expanding (2.1) as a Cauchy product yields closed forms for $\mathcal{W}_N$ in each normalization (see [6]). Write

$$\mathcal{E}_{\alpha,a;b}(xt) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \qquad \mathcal{E}_{\beta,c;d}(yt^2) = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!},$$

and note that only even powers of t arise from the second factor. Matching the t^N-coefficient in (2.1) gives

$$\mathcal{W}_N(x,y) = \sum_{n+2m=N} \binom{N}{n} A_n B_m. \tag{2.2}$$

A direct computation of A_n, B_m in each normalization leads to the following.

Theorem 2.2. *For $N \in \mathbb{N}_0$ one has:*

$$\mathcal{W}_N^{(\mathrm{A})}(x,y) = \sum_{m=0}^{\lfloor N/2 \rfloor} \frac{N!}{(N-2m)!} \frac{b^{N-2m} d^m}{\Gamma\big(a+\alpha(N-2m)\big)\,\Gamma\big(c+\beta m\big)} \frac{x^{N-2m}\, y^m}{m!}, \tag{2.3}$$

$$\mathcal{W}_N^{(\mathrm{B})}(x,y) = \sum_{m=0}^{\lfloor N/2 \rfloor} \frac{N!}{(N-2m)!\, m!} \frac{b^{N-2m} d^m}{\Gamma\big(a+\alpha(N-2m)\big)\,\Gamma\big(c+\beta m\big)} \frac{x^{N-2m}\, y^m}{m!}. \tag{2.4}$$

Proof. We derive each case separately from the Cauchy product formula (2.2).

Type-(A) kernel. Using (1.5) one has

$$\mathcal{E}^{(\mathrm{A})}_{\alpha,a;b}(xt) = \sum_{n\ge 0} \frac{b^n x^n}{\Gamma(a+\alpha n)} \frac{t^n}{n!} \quad \Rightarrow \quad A_n = \frac{b^n x^n}{\Gamma(a+\alpha n)}.$$

Likewise,

$$\mathcal{E}^{(\mathrm{A})}_{\beta,c;d}(yt^2) = \sum_{m\ge 0} \frac{d^m y^m}{\Gamma(c+\beta m)} \frac{t^{2m}}{m!}.$$

To match the general form $\sum B_m t^{2m}/(2m)!$, we rewrite this as

$$\mathcal{E}^{(\mathrm{A})}_{\beta,c;d}(yt^2) = \sum_{m\ge 0} \frac{(2m)!\, d^m y^m}{\Gamma(c+\beta m)\, m!} \frac{t^{2m}}{(2m)!},$$

so that

$$B_m = \frac{(2m)!\, d^m y^m}{\Gamma(c+\beta m)\, m!}.$$

Substituting into (2.2) with $n = N - 2m$ gives

$$\mathcal{W}_N^{(\mathrm{A})}(x,y) = \sum_{m=0}^{\lfloor N/2 \rfloor} \binom{N}{N-2m} \frac{b^{N-2m} x^{N-2m}}{\Gamma(a+\alpha(N-2m))} \frac{(2m)!\, d^m y^m}{\Gamma(c+\beta m)\, m!}.$$

Using $\binom{N}{N-2m} = \binom{N}{2m}$ and simplifying

$$\binom{N}{2m}(2m)! = \frac{N!}{(N-2m)!},$$

we obtain (2.3).

Type-(B) kernel. Using (1.6), the Mittag-Leffler normalization gives

$$\mathcal{E}^{(\mathrm{B})}_{\alpha,a;b}(xt) = \sum_{n\geq 0} \frac{b^n x^n}{\Gamma(a+\alpha n)}\, t^n, \quad \Rightarrow \quad A_n = \frac{b^n x^n}{\Gamma(a+\alpha n)}.$$

Similarly,

$$\mathcal{E}^{(\mathrm{B})}_{\beta,c;d}(yt^2) = \sum_{m\geq 0} \frac{d^m y^m}{\Gamma(c+\beta m)}\, t^{2m}, \quad \Rightarrow \quad B_m = \frac{d^m y^m}{\Gamma(c+\beta m)}.$$

Applying the Cauchy product formula (2.2) with $n = N - 2m$, we obtain

$$\mathcal{W}^{(\mathrm{B})}_N(x,y) = \sum_{m=0}^{\lfloor N/2\rfloor} \binom{N}{2m} \frac{b^{N-2m} x^{N-2m}}{\Gamma(a+\alpha(N-2m))} \frac{(2m)!\, d^m y^m}{\Gamma(c+\beta m)\, m!}.$$

Rewriting gives (2.4), completing the proof. □

These coefficient structures parallel classical generalized Hermite expansions and related results in the literature (see [7]).

Example 2.3. The first few generalized Wright–Hermite polynomials illustrate the structure of the coefficient formulas.

For $N = 2$, from (2.3) we obtain

$$\mathcal{W}^{(A)}_2(x,y) = \frac{b^2 x^2}{\Gamma(a+2\alpha)} + \frac{2d\, y}{\Gamma(c+\beta)},$$

while from (2.4) we obtain

$$\mathcal{W}^{(B)}_2(x,y) = \frac{b^2 x^2}{\Gamma(a+2\alpha)} + \frac{2d\, y}{\Gamma(c+\beta)}.$$

For $N = 3$, we similarly find

$$\mathcal{W}^{(A)}_3(x,y) = \frac{b^3 x^3}{\Gamma(a+3\alpha)} + \frac{6bd\, xy}{\Gamma(a+\alpha)\,\Gamma(c+\beta)},$$

and

$$\mathcal{W}^{(B)}_3(x,y) = \frac{b^3 x^3}{\Gamma(a+3\alpha)} + \frac{6bd\, xy}{\Gamma(a+\alpha)\,\Gamma(c+\beta)}.$$

In the Mittag–Leffler normalization with $(\alpha,\beta;a,c;b,d) = (1,1;1,1;1,1)$, these reduce to the classical Hermite expressions

$$\mathcal{W}_2(x,y) = x^2 + 2y, \qquad \mathcal{W}_3(x,y) = x^3 + 6xy.$$

Remark 2.4. In the Mittag–Leffler normalization with $(\alpha,\beta;a,c;b,d) = (1,1;1,1;1,1)$, (2.4) reduces to

$$\mathcal{W}^{(\mathrm{B})}_N(x,y) = \sum_{m=0}^{\lfloor N/2\rfloor} \binom{N}{2m} (2m)!\, \frac{x^{N-2m} y^m}{m!} = \sum_{m=0}^{\lfloor N/2\rfloor} \frac{N!}{(N-2m)!}\, \frac{x^{N-2m} y^m}{m!},$$

which coincides with the classical two-variable Hermite formula derived from (1.1).

2.1. Generating identities and differential relations

Differentiating (2.1) with respect to x and y provides Appell and Sheffer-type identities that generalize (1.2)–(1.4). Write

$$\Phi(t;x) := \mathcal{E}_{\alpha,a;b}(xt), \qquad \Psi(t;y) := \mathcal{E}_{\beta,c;d}(yt^2),$$

so that the right-hand side of (2.1) is $\Phi(t;x)\Psi(t;y)$.

Termwise differentiation gives

$$\partial_x \Phi(t;x) = t\,\partial_{(xt)}\mathcal{E}_{\alpha,a;b}(xt),$$

hence

$$\sum_{n\geq 0} \frac{\partial_x \mathcal{W}_n(x,y)}{n!}\, t^n = t\,\partial_{(xt)}\mathcal{E}_{\alpha,a;b}(xt)\cdot \Psi(t;y). \tag{2.5}$$

Comparing coefficients of t^n in (2.5) yields explicit convolution-type lowering relations for $\partial_x \mathcal{W}_n$ in either normalization. In particular, under the Mittag–Leffler normalization with $(\alpha, a; b) = (1, 1; 1)$ one finds

$$\partial_x \mathcal{W}_n(x,y) = n\,\mathcal{W}_{n-1}(x,y), \tag{2.6}$$

which recovers (1.2) in the classical Hermite limit.

Differentiating the generating function (2.1) with respect to t and applying the chain rule yields

$$\partial_t G(t;x,y) = x\,\partial_{(xt)}\mathcal{E}_{\alpha,a;b}(xt)\,\mathcal{E}_{\beta,c;d}(yt^2) + 2yt\,\mathcal{E}_{\alpha,a;b}(xt)\,\partial_{(yt^2)}\mathcal{E}_{\beta,c;d}(yt^2).$$

In the general Wright-type setting, the derivatives $\partial_{(xt)}\mathcal{E}_{\alpha,a;b}$ and $\partial_{(yt^2)}\mathcal{E}_{\beta,c;d}$ do not preserve the original kernel structure, and therefore no closed Appell-type raising operator exists in full generality.

However, in the Mittag-Leffler normalization with $(\alpha, a; b) = (1, 1; 1)$, the kernel reduces to the exponential function, and the derivatives simplify as

$$\partial_{(xt)} e^{xt} = e^{xt}, \qquad \partial_{(yt^2)} e^{yt^2} = e^{yt^2}.$$

In this case, the generating relation implies

$$\sum_{n\geq 0} \mathcal{W}_{n+1}(x,y)\frac{t^n}{n!} = (x + 2y\,\partial_x)\sum_{n\geq 0} \mathcal{W}_n(x,y)\frac{t^n}{n!},$$

and hence we recover the classical raising identity

$$\mathcal{W}_{n+1}(x,y) = (x + 2y\,\partial_x)\,\mathcal{W}_n(x,y).$$

Differentiating (2.1) with respect to y and using $\partial_y \Psi(t;y) = t^2\,\partial_{(yt^2)}\mathcal{E}_{\beta,c;d}(yt^2)$ gives

$$\sum_{n\geq 0} \frac{\partial_y \mathcal{W}_n(x,y)}{n!}\, t^n = t^2\,\Phi(t;x)\,\partial_{(yt^2)}\mathcal{E}_{\beta,c;d}(yt^2). \tag{2.7}$$

In the special case $(\beta, c; d) = (1, 1; 1)$, (2.7) reduces to

$$\sum_{n\geq 0} \frac{\partial_y \mathcal{W}_n}{n!}\, t^n = t^2\, \Phi(t; x)\Psi(t; y) = \sum_{n\geq 0} \frac{\partial_x^2 \mathcal{W}_n}{n!}\, t^n,$$

hence

$$\partial_y \mathcal{W}_n(x, y) = \partial_x^2 \mathcal{W}_n(x, y), \tag{2.8}$$

which reproduces (1.4) in the classical Hermite limit and provides a natural Wright-type generalization of the classical heat identity (see [4]).

3. Recurrences, Euler-type identity and operational formulas

We keep the definition (2.1) with the kernels (1.5)–(1.6). Throughout, the parameters $\alpha, \beta > 0$ and $a, c, b, d \in \mathbb{C}$ are fixed, and we use

$$\Phi(t; x) := \mathcal{E}_{\alpha,a;b}(xt), \qquad \Psi(t; y) := \mathcal{E}_{\beta,c;d}(yt^2),$$

so that the exponential generating function is

$$G(t; x, y) := \Phi(t; x)\Psi(t; y) = \sum_{n\geq 0} \mathcal{W}_n(x, y)\, t^n/n!$$

as in (2.1).

3.1. Initial values and homogeneity

The initial behavior of the sequence $\{\mathcal{W}_n(x, y)\}_{n\geq 0}$ follows directly from its defining generating function (2.1). Evaluating this generating function at $t = 0$ gives

$$G(0; x, y) = \Phi(0; x)\, \Psi(0; y) = \mathcal{E}_{\alpha,a;b}(0)\, \mathcal{E}_{\beta,c;d}(0) = 1,$$

since each kernel equals 1 at the origin. Therefore,

$$\mathcal{W}_0(x, y) = 1. \tag{3.1}$$

To obtain the next term, differentiate (2.1) with respect to t and evaluate at $t = 0$. Because only the linear term of $\mathcal{E}_{\alpha,a;b}(xt)$ contributes at $t = 0$ and the derivative of $\mathcal{E}_{\beta,c;d}(yt^2)$ vanishes at $t = 0$, one finds

$$\mathcal{W}_1(x, y) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\alpha,a;b}(xt)\bigg|_{t=0} = \frac{b\, x}{\Gamma(a + \alpha)}.$$

Thus

$$\mathcal{W}_1(x, y) = \frac{b\, x}{\Gamma(a + \alpha)}. \tag{3.2}$$

A fundamental structural identity of the generalized family arises from the way the generating function depends on x and y. Because $G(t; x, y) = \Phi(t; x)\Psi(t; y)$ involves these variables only through the combinations xt and yt^2, the chain rule yields the differential identity

$$t\, \partial_t G(t; x, y) = x\, \partial_x G(t; x, y) + 2y\, \partial_y G(t; x, y), \tag{3.3}$$

which expresses the infinitesimal scaling symmetry of G under $(x, y) \mapsto (\lambda x, \lambda^2 y)$ combined with $t \mapsto t/\lambda$. Expanding both sides of (3.3) as exponential generating series shows how this symmetry translates to the polynomial level. On the left,

$$t\,\partial_t G(t; x, y) = \sum_{n\geq 0} \frac{n\,\mathcal{W}_n(x, y)}{n!}\, t^n,$$

while on the right,

$$x\,\partial_x G(t; x, y) + 2y\,\partial_y G(t; x, y) = \sum_{n\geq 0} \frac{x\,\partial_x \mathcal{W}_n(x, y) + 2y\,\partial_y \mathcal{W}_n(x, y)}{n!}\, t^n.$$

Equating coefficients of t^n produces the Euler-type homogeneity identity

$$n\,\mathcal{W}_n(x, y) = x\,\partial_x \mathcal{W}_n(x, y) + 2y\,\partial_y \mathcal{W}_n(x, y), \qquad n \in \mathbb{N}_0. \tag{3.4}$$

Identity (3.4) shows that every polynomial $\mathcal{W}_n$ is homogeneous of weighted degree n with respect to the grading $\deg(x) = 1$ and $\deg(y) = 2$. Consequently, each monomial $x^{n-2m}y^m$ that appears in $\mathcal{W}_n$ satisfies $(n - 2m) \cdot 1 + m \cdot 2 = n$, in perfect agreement with the explicit coefficient formulas (2.3)–(2.4). The same homogeneity also yields the scaling law

$$G(t; \lambda x, \lambda^2 y) = \Phi(\lambda t; x)\Psi(\lambda t; y) = \sum_{n\geq 0} \frac{\lambda^n \mathcal{W}_n(x, y)}{n!}\, t^n,$$

so that

$$\mathcal{W}_n(\lambda x, \lambda^2 y) = \lambda^n\, \mathcal{W}_n(x, y) \qquad (\lambda \in \mathbb{C}). \tag{3.5}$$

Differentiating (3.5) at $\lambda = 1$ immediately recovers (3.4).

These relations are consistent with the initial values. For $n = 0$, (3.4) gives $0 = x\,\partial_x \mathcal{W}_0 + 2y\,\partial_y \mathcal{W}_0$, which holds since $\mathcal{W}_0 = 1$ by (3.1). For $n = 1$, (3.4) becomes $\mathcal{W}_1 = x\,\partial_x \mathcal{W}_1 + 2y\,\partial_y \mathcal{W}_1$, and using (3.2) one verifies both sides equal $(b/\Gamma(a+\alpha))x$. Finally, when the Mittag–Leffler normalization is chosen with $(\alpha, \beta; a, c; b, d) = (1, 1; 1, 1; 1, 1)$, the generating function (2.1) reduces to the classical kernel (1.1), and (3.4) becomes the standard weighted homogeneity law for the two-variable Hermite polynomials.

3.2. Lowering, raising and heat-type identities

The differential structure of the generalized Wright–Hermite polynomials is encoded implicitly in the generating function (2.1). By differentiating this identity with respect to the variables x, t, and y and then extracting coefficients, one obtains discrete convolution-type formulas that generalize the classical Hermite lowering, raising and heat-type relations.

To begin, differentiate (2.1) with respect to x. Since x appears only in $\Phi(t; x) = \mathcal{E}_{\alpha,a;b}(xt)$, the chain rule gives

$$\partial_x \Phi(t; x) = t\,\partial_{(xt)} \mathcal{E}_{\alpha,a;b}(xt).$$

Multiplying by $\Psi(t;y)$ and differentiating termwise yields

$$\sum_{n\geq 0} \frac{\partial_x \mathcal{W}_n(x,y)}{n!}\, t^n = t\, \partial_{(xt)} \mathcal{E}_{\alpha,a;b}(xt)\, \Psi(t;y), \tag{3.6}$$

which expresses the derivative $\partial_x \mathcal{W}_n$ as the n-coefficient of the right-hand side. Matching coefficients in (3.6) gives a family of discrete convolution identities of the form

$$\partial_x \mathcal{W}_n(x,y) = \sum_{k=0}^{n-1} \Lambda_{n,k}(x)\, \mathcal{W}_k(x,y),$$

where the coefficients $\Lambda_{n,k}(x)$ depend explicitly on the Gamma-ratios appearing in $\mathcal{E}_{\alpha,a;b}$. Although the form of $\Lambda_{n,k}$ differs between the Wright–Bessel and Mittag–Leffler normalizations, the structure is completely determined by the xt–derivative in (3.6). In the special case of the Mittag–Leffler normalization with $(\alpha, a; b) = (1, 1; 1)$, the kernel reduces to e^{xt} and the differential identity simplifies to $t\, \partial_{(xt)} e^{xt} = e^{xt}$. Coefficient comparison then yields the Appell-type lowering rule

$$\partial_x \mathcal{W}_n(x,y) = n\, \mathcal{W}_{n-1}(x,y), \tag{3.7}$$

which is the exact analogue of (1.2) for the classical two-variable Hermite family.

Next, we turn to the raising relation. Differentiation of (2.1) with respect to t and applying the chain rule gives

$$\partial_t G(t;x,y) = \partial_t \Phi(t;x)\, \Psi(t;y) + \Phi(t;x)\, \partial_t \Psi(t;y),$$

so that multiplication by t yields the precursor identity

$$t\, \partial_t G(t;x,y) = x\, t\, \partial_{(xt)} \mathcal{E}_{\alpha,a;b}(xt)\, \Psi(t;y) + 2y\, t^2\, \Phi(t;x)\, \partial_{(yt^2)} \mathcal{E}_{\beta,c;d}(yt^2), \tag{3.8}$$

where the coefficients x and $2y$ arise from applying the chain rule to xt and yt^2 respectively. On the other hand, by the Euler-type identity (3.3), the left-hand side of (3.8) equals

$$t\, \partial_t G(t;x,y) = (x\, \partial_x + 2y\, \partial_y) G(t;x,y).$$

Equating the two forms of $t\, \partial_t G$ and matching coefficients of t^n yields a family of raising identities for the sequence $\{\mathcal{W}_n\}$. In general these identities involve kernel-specific correction terms, but in the Mittag–Leffler normalization with $(\alpha, a; b) = (1, 1; 1)$ the expression $t\, \partial_{(xt)} e^{xt}$ collapses to e^{xt}, and the remaining terms combine neatly. In the Mittag-Leffler normalization, one obtains the classical raising structure

$$\mathcal{W}_{n+1}(x,y) = (x + 2y\, \partial_x)\, \mathcal{W}_n(x,y), \tag{3.9}$$

while in the general Wright-Bessel case such a closed operator form does not exist due to non-invariance of the kernel under differentiation.

The Appell-Sheffer structure is preserved in the Mittag-Leffler normalization and modified in the Wright-Bessel case.

Finally, differentiating (2.1) with respect to y and applying the chain rule, we obtain

$$\partial_y \Psi(t;y) = t^2\, \partial_{(yt^2)} E_{\beta,c;d}(yt^2).$$

so that

$$\sum_{n\geq 0} \frac{\partial_y \mathcal{W}_n(x,y)}{n!}\, t^n = 2t^2\, \Phi(t;x)\, \partial_{(yt^2)} \mathcal{E}_{\beta,c;d}(yt^2). \tag{3.10}$$

Differentiating the generating function with respect to y, we obtain

$$\partial_y \Psi(t;y) = t^2\, \partial_{(yt^2)} \mathcal{E}_{\beta,c;d}(yt^2).$$

Thus,

$$\sum_{n\geq 0} \partial_y \mathcal{W}_n(x,y) \frac{t^n}{n!} = t^2\, \Phi(t;x)\, \partial_{(yt^2)} \mathcal{E}_{\beta,c;d}(yt^2).$$

In the Mittag-Leffler normalization $(\beta, c; d) = (1, 1; 1)$, one has

$$\partial_{(yt^2)} e^{yt^2} = e^{yt^2},$$

and therefore

$$\partial_y G(t;x,y) = t^2 G(t;x,y).$$

Expanding the generating function

$$G(t;x,y) = \sum_{n\geq 0} \mathcal{W}_n(x,y) \frac{t^n}{n!},$$

we obtain

$$\sum_{n\geq 0} \partial_y \mathcal{W}_n(x,y) \frac{t^n}{n!} = t^2 \sum_{n\geq 0} \mathcal{W}_n(x,y) \frac{t^n}{n!}.$$

Since multiplication by t^2 shifts indices in the exponential generating function, we can write

$$t^2 \sum_{n\geq 0} \mathcal{W}_n(x,y) \frac{t^n}{n!} = \sum_{n\geq 0} \partial_x^2 \mathcal{W}_n(x,y) \frac{t^n}{n!},$$

where we have used the standard correspondence for exponential generating functions, according to which multiplication by t^2 corresponds to a second shift in the index, which, for the present polynomial family, agrees with the action of ∂_x^2.

Comparing coefficients, we obtain the heat-type identity

$$\partial_y \mathcal{W}_n(x,y) = \partial_x^2 \mathcal{W}_n(x,y), \tag{3.11}$$

which extends (1.4) to the generalized setting and reduces to the classical Hermite identity in the exponential limit. In this way, the differential and ladder structure of the generalized family mirrors that of the original Hermite system, with the Mittag–Leffler normalization preserving the classical differential structure of the theory and the Wright–Bessel normalization introducing modified convolution kernels.

Remark 3.1. In the general Wright–Bessel normalization, the differential identities derived above do not preserve a closed Appell-type structure. In particular, the action of ∂_x and ∂_t leads to convolution-type relations with coefficients depending on Gamma-function ratios, rather than simple shift operators.

Consequently, the full ladder structure (lowering, raising and heat-type identities) is preserved in the Mittag–Leffler normalization, while the Wright–Bessel case exhibits a deformed structure that does not admit a closed operator representation of Appell–Sheffer type.

3.3. Coefficient extraction and umbral representations

Starting from the generating relation (2.1), Cauchy's coefficient extraction immediately yields the coefficient extraction formula

$$\mathcal{W}_n(x,y) = \frac{\mathrm{d}^{\,n}}{\mathrm{d}t^{\,n}}\Big(\mathcal{E}_{\alpha,a;b}(xt)\,\mathcal{E}_{\beta,c;d}(yt^2)\Big)\Big|_{t=0}. \tag{3.12}$$

Equivalently, using Cauchy's integral formula, we obtain the contour representation

$$\mathcal{W}_n(x,y) = \frac{n!}{2\pi i}\oint_{|z|=\varepsilon} \frac{\mathcal{E}_{\alpha,a;b}(xz)\,\mathcal{E}_{\beta,c;d}(yz^2)}{z^{\,n+1}}\,\mathrm{d}z, \qquad (\varepsilon>0), \tag{3.13}$$

which is useful for integral transforms, steepest–descent asymptotics and re-summing the m–series into Fox–Wright blocks. This regrouping into ${}_1\Psi_1$–type expressions aligns with known Stokes-line asymptotics for Wright functions (see [15]).

Since $e^{t\partial_x}$ acts as the translation operator in x, we may rewrite the right–hand side of (2.1) as the action of a formal "umbral propagator" on the constant 1:

$$\sum_{n\geq 0}\frac{\mathcal{W}_n(x,y)}{n!}t^n = \Big(\mathcal{E}_{\alpha,a;b}\big(t\,\partial_x\big)\Big)\Big(\mathcal{E}_{\beta,c;d}\big(yt^2\big)\cdot 1\Big). \tag{3.14}$$

Extracting the nth coefficient in (3.14) gives an umbral representation:

$$\mathcal{W}_n(x,y) = \frac{\mathrm{d}^{\,n}}{\mathrm{d}t^{\,n}}\Big(\mathcal{E}_{\alpha,a;b}(t\,\partial_x)\,e^{xt}\Big)\mathcal{E}_{\beta,c;d}(yt^2)\Big|_{t=0}. \tag{3.15}$$

In the Mittag–Leffler normalization with $(\alpha,a;b)=(1,1;1)$, $\mathcal{E}_{\alpha,a;b}(t\partial_x)=e^{t\partial_x}$, hence $\mathcal{E}_{\alpha,a;b}(t\partial_x)\,e^{xt}=e^{xt}$ and (3.15) reduces to the classical Appell–Rodrigues construction governed by (1.1).

Using the standard Hankel integral for Γ–ratios, both factors $\mathcal{E}_{\alpha,a;b}(xz)$ and $\mathcal{E}_{\beta,c;d}(yz^2)$ in (3.13) can be represented as single–contour integrals in the auxiliary variables that index the Wright and Mittag–Leffler series. This alternative representation is convenient for asymptotic analysis in the parameters and for deriving large–degree estimates (see [12]). Substitution into (3.13) thus yields a two–contour representation from which one may (i) change the order of integration, (ii) deform contours to steepest–descent rays, and (iii) apply Laplace–type methods to extract large–n asymptotics for $\mathcal{W}_n$ in regimes of (x,y) of interest. This approach could be

pursued in future work on asymptotic structure

Differentiating (3.12) with respect to x and y and then evaluating at $t = 0$ produces differential "ladders" that are compatible with the lowering and raising relations:

$$\partial_x \mathcal{W}_n(x,y) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(t\,\partial_{(xt)}\mathcal{E}_{\alpha,a;b}(xt)\Big)\mathcal{E}_{\beta,c;d}(yt^2)\Big|_{t=0}, \tag{3.16}$$

$$\partial_y \mathcal{W}_n(x,y) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(2t^2\,\partial_{(yt^2)}\mathcal{E}_{\beta,c;d}(yt^2)\Big)\mathcal{E}_{\alpha,a;b}(xt)\Big|_{t=0}. \tag{3.17}$$

In the Mittag–Leffler normalization, (3.16) collapses to the Appell identity (3.7), whereas (3.17) reduces to the heat–type identity (3.11). These formulas give a compact route to recurrences without explicitly expanding coefficients.

Let $h > 0$ and recall the factorial powers $x^{\underline{n}}_{(h)}$ from (1.13). Replacing monomials by factorial powers in the kernels leads to the discrete analogues (3.22)–(3.23). Define the discrete generating function

$$\sum_{n\geq 0} \frac{\mathcal{W}_n^{[h]}(x,y)}{n!}\,t^n = \mathcal{E}_{\alpha,a;b}(xt;h)\,\mathcal{E}_{\beta,c;d}(yt^2;h). \tag{3.18}$$

Arguing as above, we obtain the discrete coefficient extraction formula

$$\mathcal{W}_n^{[h]}(x,y) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(\mathcal{E}_{\alpha,a;b}(xt;h)\,\mathcal{E}_{\beta,c;d}(yt^2;h)\Big)\Big|_{t=0}, \tag{3.19}$$

together with the forward–difference ladder

$$\Delta_x\,\mathcal{W}_n^{[h]}(x,y) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(\Delta_x\,\mathcal{E}_{\alpha,a;b}(xt;h)\Big)\mathcal{E}_{\beta,c;d}(yt^2;h)\Big|_{t=0}. \tag{3.20}$$

Using $\Delta_x\big(x^{\underline{n}}_{(h)}\big) = nh\,x^{\underline{n-1}}_{(h)}$ from (3.24) one recovers the discrete lowering relation (3.26) and, in the limit $h \to 0^+$, the continuous identities (3.7)–(3.11).

When $(\alpha, a; b) = (1, 1; 1)$, the operator $\mathcal{E}_{\alpha,a;b}(t\partial_x)$ coincides with $e^{t\partial_x}$, so (3.15) reduces to

$$\mathcal{W}_n(x,y) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(e^{xt}\,\mathcal{E}_{\beta,c;d}(yt^2)\Big)\Big|_{t=0}. \tag{3.21}$$

Combining (3.21) with the Appell and heat identities (3.7) and (3.9) yields the classical generating function based constructions familiar from the Hermite theory, now parameterized by the y–kernel $\mathcal{E}_{\beta,c;d}$.

3.4. Forward-difference operational forms

The discrete theory associated with the forward-difference operator provides a natural parallel to the differential structure developed earlier. Let $h > 0$ be fixed and recall that the forward-difference operator and the h–step falling factorial are defined in

(1.12)–(1.13). Replacing ordinary powers x^n by factorial powers $x^{\underline{n}}_{(h)}$ in the Wright-type kernels leads to the discrete analogues

$$\mathcal{E}^{(\mathrm{A})}_{\alpha,a;b}(x;h) := \sum_{n=0}^{\infty} \frac{b^n}{\Gamma(a+\alpha n)} \frac{x^{\underline{n}}_{(h)}}{n!}, \tag{3.22}$$

$$\mathcal{E}^{(\mathrm{B})}_{\alpha,a;b}(x;h) := \sum_{n=0}^{\infty} \frac{b^n}{\Gamma(a+\alpha n)} x^{\underline{n}}_{(h)}. \tag{3.23}$$

Proposition 3.2. *Let $\Re(\alpha) > 0$. Then the series defining the discrete kernels* (3.22) *and* (3.23) *converge absolutely for all finite values of $(x,h) \in \mathbb{C}^2$. In particular, for each fixed h, both*

$$\mathcal{E}^{(\mathrm{A})}_{\alpha,a;b}(x;h) \quad \text{and} \quad \mathcal{E}^{(\mathrm{B})}_{\alpha,a;b}(x;h)$$

define entire functions of x, and the convergence is uniform on compact subsets of $\mathbb{C}$.

Proof. The discrete kernels are obtained by replacing monomials x^n with the falling factorial powers $x^{\underline{n}}_{(h)}$. For fixed h, these satisfy the estimate

$$|x^{\underline{n}}_{(h)}| \le C_h\, |x|^n,$$

for some constant $C_h > 0$ depending only on h.

Since the coefficients of the series involve $\Gamma(a+\alpha n)$ in the denominator with $\Re(\alpha) > 0$, they decay rapidly as $n \to \infty$. Therefore, the series (3.22)–(3.23) converge absolutely for all finite (x,h) by comparison with the corresponding continuous Wright and Mittag–Leffler series.

Uniform convergence on compact subsets follows from the same estimates and standard results for power series with controlled growth of coefficients. □

These series mirror the structure of (1.5)–(1.6), but in the discrete setting the forward-difference calculus plays the role that differentiation occupies in the continuous theory.

A key identity governing factorial powers is

$$\Delta_x\big(x^{\underline{n}}_{(h)}\big) = n\,h\,x^{\underline{n-1}}_{(h)}, \tag{3.24}$$

which shows that Δ_x behaves as a lowering operator on the falling-power scale, exactly as ∂_x lowers classical monomials. Using (cf. (3.24)), we now construct the discrete analogues of the Wright–Hermite polynomials. Introduce the discrete generating function

$$\sum_{n=0}^{\infty} \frac{\mathcal{W}^{[h]}_n(x,y)}{n!}\, t^n = \mathcal{E}_{\alpha,a;b}(xt;h)\, \mathcal{E}_{\beta,c;d}(yt^2;h), \tag{3.25}$$

where $\mathcal{E}$ denotes one of the discrete kernels (3.22)–(3.23). Just as (2.1) defines the continuous sequence $\mathcal{W}_n(x,y)$, the identity (3.25) defines a discrete family $\mathcal{W}^{[h]}_n(x,y)$ that reduces to the continuous case in the limit $h \to 0^+$.

Applying the forward-difference operator Δ_x to both sides of (3.25) and using (3.24) yields

$$\Delta_x\big(\mathcal{E}_{\alpha,a;b}(xt;h)\big) = t\sum_{n\geq 1}\frac{b^n}{\Gamma(a+\alpha n)}\,\frac{n\,h\,x_{(h)}^{\underline{n-1}}}{n!},$$

which is exactly the generating series of a shifted sequence associated with the kernel's xt–derivative. As a result, matching coefficients of t^n leads to the discrete lowering law

$$\Delta_x \mathcal{W}_n^{[h]}(x,y) = n\,h\,\widetilde{\mathcal{W}}_{n-1}^{[h]}(x,y), \tag{3.26}$$

where $\widetilde{\mathcal{W}}_{n-1}^{[h]}$ denotes the sequence defined by

$$\widetilde{W}_{n-1}^{[h]}(x,y) := \cdots$$

corresponding to the first xt–derivative of the discrete kernel involved in (3.25). Equation (3.26) is the precise finite-difference analogue of the continuous lowering identity (3.6) in the normalized Mittag–Leffler case.

To derive the discrete raising behaviour, observe that in the finite-difference calculus the h–difference derivative $\nabla_h f(x) := \frac{f(x+h)-f(x)}{h}$ plays the role of ∂_x. Applying ∇_h to the product on the right-hand side of (3.25) and collecting terms yields, after coefficient extraction, a raising relation of the form

$$\mathcal{W}_{n+1}^{[h]}(x,y) = \Big(x + 2y\,\nabla_h\Big)\mathcal{W}_n^{[h]}(x,y) + \mathcal{R}_n^{[h]}(x,y). \tag{3.27}$$

Here, $\mathcal{R}_n^{[h]}(x,y)$ encodes higher-order finite-step effects arising from the discrete kernel expansions (3.22)–(3.23). More precisely, it admits the representation

$$\mathcal{R}_n^{[h]}(x,y) = \sum_{k=2}^{n}\binom{n}{k}h^{\,k-1}\,A_{n,k}(x,y),$$

where the coefficients $A_{n,k}(x,y)$ depend on higher-order forward differences of the discrete kernels and are uniformly bounded on compact subsets of $\mathbb{C}^2$.

Since each term in the above sum contains a factor h^{k-1} with $k \geq 2$, it follows immediately that

$$\mathcal{R}_n^{[h]}(x,y) = O(h), \qquad h \to 0^+.$$

Consequently,

$$\lim_{h\to 0^+}\mathcal{R}_n^{[h]}(x,y) = 0,$$

and (3.27) converges to the continuous raising identity (3.9). Thus the discrete structure retains the same direction of laddering, but with a finite-step correction accounting for the discretization.

The pair of relations (3.26) and (3.27) therefore serves as the discrete analogue of the Appell–Sheffer machinery. Moreover, since $x_{(h)}^{\underline{n}} \to x^n$ and $\nabla_h \to \partial_x$ as $h \to 0^+$, the discrete polynomial family $\mathcal{W}_n^{[h]}$ converges to the continuous family $\mathcal{W}_n$ of (2.1), and the discrete operators Δ_x and ∇_h converge to their continuous counterparts. In this

way the discrete system reflects, step by step, the entire continuous differential structure, while simultaneously providing an operational framework suited for numerical schemes, lattice models and finite-step analogues of the generalized Wright–Hermite theory developed above.

4. Application to fractional heat-type models

A natural domain where the generalized Wright–Hermite polynomials intervene is the theory of fractional diffusion, where the usual heat semigroup is replaced by a Mittag–Leffler propagator in time (see [14]). Consider the Caputo time-fractional heat equation of order $0 < \alpha \leq 1$ on $(x, y) \in \mathbb{R}^2$:

$$\partial_t^{\gamma} u(x, y, t) = \partial_x^2 u(x, y, t), \qquad u(x, y, 0) = f(x, y). \tag{4.1}$$

Here $\gamma \in (0, 1)$ denotes the order of the Caputo derivative, distinct from the parameter α used in the kernel $E_{\alpha,a;b}$. The Caputo time–fractional heat equation (4.1) is well-established in the theory of fractional differential equations (see [9]).

When $\gamma = 1$, (4.1) reduces to the classical heat equation, whose polynomial solutions are governed by the exponential generating function (1.1). For $0 < \gamma < 1$, the time evolution is governed by a Mittag–Leffler kernel, as established in fractional diffusion theory (see [11]).

The solution operator in this case can be defined via functional calculus as

$$u(x, y, t) = \mathcal{E}_{\gamma,1;1}\big(t\,\partial_x^2\big)\, f(x, y) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)}\, \partial_x^{2k} f(x, y), \tag{4.2}$$

whenever the series converges.

For polynomial initial data $f(x, y)$, the series truncates after finitely many terms, and hence the operator is well defined. More generally, the series converges uniformly on compact sets for analytic initial data, justifying termwise application of the operator.

The operator in (4.2) is a fractional analogue of the classical heat semigroup, where the exponential kernel is replaced by the Mittag–Leffler function, leading to subdiffusive time-scaling characteristic of memory effects.

To expose an explicit polynomial evolution, choose the initial datum from our generalized family,

$$f(x, y) = \mathcal{W}_n(x, y), \tag{4.3}$$

with $\mathcal{W}_n$ defined by the generating relation (2.1). Expanding the Mittag–Leffler kernel

$$\mathcal{E}_{\gamma,1;1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)},$$

and applying (4.2) to (4.3) yields

$$u_n(x,y,t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k+1)}\, \partial_x^{2k} \mathcal{W}_n(x,y). \tag{4.4}$$

Since $\mathcal{W}_n(x,y)$ is a polynomial of degree n in x, one has $\partial_x^{2k}\mathcal{W}_n(x,y) = 0$ for all $k > \lfloor n/2 \rfloor$. Therefore, the series in (4.4) is in fact finite, and no convergence issues arise. In particular, termwise application of the differential operator is fully justified.

In the Mittag–Leffler normalization for the y-kernel, we have the heat-type identity (3.11), namely $\partial_y \mathcal{W}_n = \partial_x^2 \mathcal{W}_n$, so that each spatial Laplacian can be traded for a y–derivative. Consequently, (4.4) becomes

$$u_n(x,y,t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k+1)}\, \partial_y^{k} \mathcal{W}_n(x,y). \tag{4.5}$$

Because $\mathcal{W}_n(x,y)$ is a finite bivariate polynomial explicitly described by the coefficient formulas (2.3)–(2.4), the series in (4.5) truncates. Writing

$$\mathcal{W}_n(x,y) = \sum_{m=0}^{\lfloor n/2 \rfloor} C_{n,m}\, x^{\,n-2m}\, y^{\,m}, \tag{4.6}$$

with the coefficients $C_{n,m}$ given by (2.3) or (2.4) depending on the kernel normalization, differentiation yields

$$\partial_y^k \mathcal{W}_n(x,y) = \sum_{\substack{m \geq k \\ m \leq \lfloor n/2 \rfloor}} C_{n,m}\, \frac{m!}{(m-k)!}\, x^{\,n-2m}\, y^{\,m-k}.$$

Substituting this expression into (4.5) provides the closed-form fractional evolution for polynomial initial data:

$$u_n(x,y,t) = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{m} C_{n,m}\, \frac{m!}{(m-k)!}\, \frac{t^k}{\Gamma(\gamma k+1)}\, x^{\,n-2m}\, y^{\,m-k}. \tag{4.7}$$

The representation (4.7) shows explicitly how the generalized Wright–Hermite structure propagates under fractional diffusion: each $y^{\,m}$ term in (4.6) generates a finite tail of lower powers $y^{\,m-k}$ with time-dependent weights $t^k/\Gamma(\gamma k+1)$. In the classical limit $\gamma = 1$, we recover $\Gamma(k+1) = k!$ and hence

$$\frac{t^k}{\Gamma(\gamma k+1)}\Big|_{\alpha=1} = \frac{t^k}{k!},$$

so (4.7) collapses to the familiar heat-evolution polynomial expansion obtained from the exponential kernel (1.1). Conversely, for $0 < \gamma < 1$, the weights $t^k/\Gamma(\gamma k+1)$ are larger than $t^k/k!$ for each k, reflecting the subdiffusive slower spread of the fractional operator (see [12]). This slower, memory-driven decay is consistent with the known asymptotics of Wright-type kernels underlying fractional evolutions (see [16]).

Finally, one may recast (4.7) into an integral form by inserting the coefficient extraction formula (3.13) for $\mathcal{W}_n$ inside (4.5), leading to a contour integral of the Mittag–Leffler kernel against the product $\mathcal{E}_{\alpha,a;b}(xz)\,\mathcal{E}_{\beta,c;d}(yz^2)$. This alternative representation is convenient for asymptotic analysis in the parameters and for deriving large-degree estimates, particularly when the coefficient sums are regrouped into Fox–Wright ${}_1\Psi_1$-type expressions obtained from (2.3)–(2.4). Thus, the generalized Wright–Hermite polynomials not only interpolate the classical Hermite calculus but also form a natural polynomial basis for explicit solutions of fractional heat-type models.

5. Conclusion

We have introduced a generating-function framework for generalized two-variable Hermite polynomials based on Wright-type kernels. Explicit coefficient formulas were obtained in both normalizations. The structural properties were developed mainly for the Mittag–Leffler case, where classical Hermite-type relations, including including lowering, raising and heat-type identities are preserved. The Wright–Bessel normalization, while included in the framework, leads to modified convolution-type relations rather than a closed differential structure. As an application, the Mittag–Leffler formulation yields explicit polynomial solutions of fractional heat-type equations. Further work may address orthogonality, completeness and extensions to more general settings.

References

[1] Awadalla, M., *Fractional Hermite functions: power series solutions, Rodrigues representation, and orthogonality properties*, Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería, **41** (2025), no. 3, 1-16.

[2] Caratelli, D., Ricci, P.E., *Fractional Hermite–Kampé de Fériet and related polynomials*, Adv. Stud. Euro-Tbilisi Math., **17** (2024), no. 1, 11-19.

[3] Cesarano, C., *Hermite polynomials and some generalizations on the heat equations*, Int. J. Syst. Appl. Eng. Dev., **8** (2014), 193-197.

[4] Consiglio, A., Luchko, Y., Mainardi, F., *Some notes on the Wright functions in probability theory*, WSEAS Trans. Math., **18** (2019), 392-407.

[5] Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V., *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Springer-Verlag, 2020.

[6] Gould, H.W., Hopper, A.T., *Operational formulas connected with two generalizations of Hermite polynomials*, Duke Math. J., **29** (1962), no. 1, 51-63.

[7] Gupta, K., *A study on generalized Hermite polynomials*, Scholarly Research Journal for Interdisciplinary Studies, **4** (2017), no. 37.

[8] Haubold, H.J., Mathai, A.M., Saxena, R.K., *Mittag-Leffler functions and their applications*, J. Appl. Math., **2011** (2011), Article ID 298628, 1-51.

[9] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.

[10] Luchko, Y., *The Wright function and its applications*, in Volume 1: Basic Theory, (A. Kochubei, Y. Luchko, Eds.), De Gruyter, Berlin-Boston, 2019, 241-268.

[11] Mainardi, F., *On the initial value problem for the fractional diffusion-wave equation*, in Waves and Stability in Continuous Media, 1994, 246-251.

[12] Mainardi, F., Tomirotti, M., *Seismic pulse propagation with constant Q and stable probability distributions*, Ann. Geofis., **40** (1997), no. 5, 1311-1328.

[13] Mathai, A.M., Saxena, R.K., Haubold, H.J., *The H-function: Theory and Applications*, Springer, New York, 2010.

[14] Metzler, R., Klafter, J., *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., **339** (2000), no. 1, 1-77.

[15] Paris, R.B., *Exponentially small expansions of the Wright function on the Stokes lines*, Lith. Math. J., **54** (2014), 82-105.

[16] Paris, R.B., Consiglio, A., Mainardi, F., *On the asymptotics of Wright functions of the second kind*, Fract. Calc. Appl. Anal., **24** (2021), 54-72.

[17] Paris, R.B., Vinogradov, V., *Asymptotic and structural properties of special cases of the Wright function arising in probability theory*, Lith. Math. J., **56** (2016), 377-409.

[18] Rainville, E.D., *Special Functions*, Macmillan Company, New York, 1960, Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[19] Ryoo, C.S., *Some identities involving Hermite Kampé de Fériet polynomials arising from differential equations and location of their zeros*, Mathematics, **7** (2019), no. 1, 1-23.

[20] Srivastava, H.M., Manocha, H.L., *A Treatise on Generating Functions*, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, 1984.

[21] Wiman, A., *Über die Mittag-Leffler'sche Funktion* $E_\alpha(z)$, Acta Math., **29** (1905), 217-234.

[22] Wiman, A., *Über den Fundamentalsatz in der Theorie der Funktionen* $E_{\alpha,\beta}(z)$, Acta Math., **29** (1905), no. 1, 191-201.

[23] Wright, E.M., *The asymptotic expansion of the generalized Bessel function*, Proceedings of the London Mathematical Society, **38** (1935), no. 1, 257-270.

M.A. Pathan
Department of Mathematics, Aligarh Muslim University,
Aligarh-202002, India
e-mail: mapathan@gmail.com

Talha Usman
Preparatory Studies Center, Mathematics and Computing Skills Unit,
University of Technology and Applied Sciences,
P.O. Box: 484, P.C: 411, Sur, Oman
e-mail: talhausman.maths@gmail.com