




Low-regret control of a nonlinear parabolic problem with missing data

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Abstract. In this paper, we study the optimal control of a nonlinear parabolic problem with missing data. Using the concepts of no-regret control, low-regret control and adapted low-regret control, we give a characterization of the control for ill-posed problems. More precisely, we study the control of a nonlinear parabolic problem using a regularization approach that generates incomplete information. We obtain a singular optimality system characterizing the no-regret control for the nonlinear parabolic problem.

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1. Introduction

Let Ω be a bounded open subset of \mathbb{R}^n with a sufficiently smooth boundary Γ . Let $T > 0$ and let $Q = \Omega \times (0, T)$ denote the space–time cylinder, with lateral boundary $\Sigma = \Gamma \times (0, T)$.


Consider the following Cauchy problem in Q :

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z - z^3 = v & \text{in } Q, \\ z(0) = 0 & \text{in } \Omega. \end{cases} \quad (1.1)$$

Let $z \in L^2(Q)$ denote the state of the system and $v \in L^2(Q)$ the control. The state of the system is unknown on the boundary of the domain. Thus, problem (1.1) is a problem with missing data. Due to the presence of the term z^3 in (1.1), this problem is generally unstable. In the case where a boundary condition is prescribed, the solution

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generally blows up at a time $t_0 < T$ [3, 4, 13]. This problem was studied by Lions in [3], where he obtained a characterization of optimal pairs. Optimal control problems for nonlinear distributed systems constitute an open and active field of research. Most existing works focus on deriving necessary or sufficient first- or second-order optimality conditions. Owing to the lack of obvious regularity of the state solution, the issues of existence and characterization of optimal controls still require a detailed analysis. The assumptions imposed on control problems are generally chosen so that a more or less adapted standard method can be applied to establish the existence of solutions. In the present work, we apply the low-regret control method introduced by Lions in [6] to study the control of nonlinear distributed systems with missing data. We then characterize the no-regret control of the problem with missing data through a singular optimality system. The notion of no-regret control was introduced by Leonard J. Savage [12]. Lions subsequently employed this concept in the context of optimal control, motivated by numerous applications in economics and ecology [6]. Indeed, he was the first to use both the low-regret and no-regret control concepts to control distributed systems with missing data in various areas of applied mathematics [1, 5, 7]. For example, integrating a decision criterion into a closed and nonempty subspace of uncertainties makes it possible to improve existing results and to extend the concept of low-regret control to a framework involving multiple agents in economics [1]. Picart et al. apply the concepts of no-regret and low-regret control to address problems related to the control of nutrient absorption by roots in the rhizosphere (the area surrounding the roots), considering two types of soil: healthy soil and polluted soil [9]. Nakoulima et al. successfully characterize no-regret control for missing data problems in both stationary and evolutionary cases through a singular optimality system [10, 11]. In this article, we regularize problem (1.1) as a sequence of well-posed problems to study the control problem using the low-regret method, which is particularly suited to problems with missing data. We show that the no-regret control corresponds to the limit of the low-regret control as the disturbance parameter tends to zero. The rest of the article is organized as follows: in Section 2, we present the regularization and control problem. In Section 3, we characterize low-regret and no-regret control via singular optimality systems.

2. Regularization and control problem

2.1. Regularization of the ill-posed problem

In this section, we regularize problem (1.1) following the work of Lions [2, 3] on the regularization of problems with missing data. Thus, for all $\varepsilon > 0$ and for all $g \in L^2(\Sigma)$, we consider the following regularized problem:

$$\begin{cases} \frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon - z_\varepsilon^3 + \varepsilon z_\varepsilon^5 = v & \text{in } Q, \\ z_\varepsilon(0) = 0 & \text{in } \Omega, \\ z_\varepsilon = \varepsilon g & \text{on } \Sigma. \end{cases} \quad (2.1)$$

Let G be a nonempty, closed vector subspace of $L^2(\Sigma)$, equipped with the $L^2(\Sigma)$ norm. The function g ranges over the set G and may represent pollution or a disturbance.

Remark 2.1. For any fixed $\varepsilon > 0$ and $g \in L^2(\Sigma)$, the system (2.1) has a unique solution [2].

2.2. Cost function and control problem

2.2.1. Analysis of cost function and no-regret control case. Let $z_\varepsilon := z_\varepsilon(v, g)$ denote the state of system (2.1). For all $v \in L^2(Q)$ and $g \in L^2(\Sigma)$, we associate the following cost function:

$$J_\varepsilon(v, g) = \frac{1}{2} \|z_\varepsilon(v, g) - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|v\|_{L^2(Q)}^2. \tag{2.2}$$

Here, $N > 0$ and $z_d \in L^2(Q)$ denotes the desired state.

Definition 2.2. We say that $u \in L^2(Q)$ is a no-regret control for problem (1.1) if u is a solution of the following problem:

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} (J_\varepsilon(v, g) - J_\varepsilon(0, g)) \right]. \tag{2.3}$$

Remark 2.3. To make the nonlinear problem amenable to analysis, we consider the standard definition of no-regret control associated with a linearized cost functional, and then study the corresponding no-regret control problem.

The concepts of no-regret control and low-regret control, introduced by Lions in [6], rely on the decomposition of the solution of (2.1) in the form $z_\varepsilon(v, g) = z_\varepsilon(v, 0) + z_\varepsilon(0, g) - z_\varepsilon(0, 0)$ where $z_\varepsilon(v, 0)$ is the solution of (2.1) with $g = 0$ and $z_\varepsilon(0, g)$ is the solution of (2.1) with $v = 0$. This decomposition is no longer valid in our case because the mapping $(v, g) \mapsto z_\varepsilon(v, g)$ from $L^2(Q) \times L^2(\Sigma)$ to $L^2(Q)$ is nonlinear. For this nonlinear case, we exploit the regularity of the solution of (2.1) and replace the cost function defined in (2.2) by its linear form with respect to g . Thus, following [8], we consider the new cost function given by:

$$J_{1\varepsilon}(v, g) = J_\varepsilon(v, 0) + \frac{\partial J_\varepsilon}{\partial g}(v, 0)(g). \tag{2.4}$$

Hence, we consider the following new optimization problem:

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} (J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g)) \right]. \tag{2.5}$$

The solution of (2.5) if exists, is called no-regret control for the nonlinear problem. Then, we have the following proposition.

Proposition 2.4. For all $(v, g) \in L^2(Q) \times L^2(\Sigma)$, we have the following equality:

$$\begin{aligned} J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) &= J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \\ &\quad - \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle, \end{aligned} \tag{2.6}$$

where J_ε is the cost function defined in (2.2).

Proof. The mapping $(v, g) \mapsto z_\varepsilon(v, g)$ is differentiable with respect to g in $L^2(\Sigma)$. Thus, the first-order (linear) expansion of z_ε at the point $(v, 0)$ along the direction of g is:

$$\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda}.$$

We have :

$$\begin{aligned} J_\varepsilon(v, \lambda g) &= J_\varepsilon(v, 0) + \frac{1}{2} \|z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)\|_{L^2(Q)}^2 \\ &\quad + \langle z_\varepsilon(v, 0) - z_d, z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0) \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{J_\varepsilon(v, \lambda g) - J_\varepsilon(v, 0)}{\lambda} &= \left\langle z_\varepsilon(v, 0) - z_d, \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda} \right\rangle \\ &\quad + \lim_{\lambda \rightarrow 0} \frac{\lambda}{2} \left\| \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda} \right\|_{L^2(Q)}^2 \\ &= \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \\ &= \frac{\partial J_\varepsilon}{\partial g}(v, 0)(g). \end{aligned} \tag{2.7}$$

By substituting (2.7) into (2.5), we obtain:

$$J_{1\varepsilon}(v, g) = J_\varepsilon(v, 0) + \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle. \tag{2.8}$$

By substituting $v = 0$ into (2.8), we obtain:

$$J_{1\varepsilon}(0, g) = J_\varepsilon(0, 0) + \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle. \tag{2.9}$$

Consequently,

$$\begin{aligned} J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) &= J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \\ &\quad - \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle. \end{aligned}$$

□

In what follows, we use the propositions below to estimate the terms

$$\left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \text{ and } \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle \text{ in (2.6).}$$

Proposition 2.5. *The partial derivative $\frac{\partial z_\varepsilon}{\partial g} := \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda}$ is the solution of the following problem:*

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g} \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g} \right) - 3z_\varepsilon^2(v, 0) \frac{\partial z_\varepsilon}{\partial g} + 5\varepsilon z_\varepsilon^4(v, 0) \frac{\partial z_\varepsilon}{\partial g} = 0 & \text{in } Q, \\ \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \Big|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \varepsilon g & \text{on } \Sigma. \end{cases} \quad (2.10)$$

Proof. Let us show that $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g)$ is a solution of (2.10).

- For the condition in Q , we have:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g} \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g} \right) - 3z_\varepsilon^2 \frac{\partial z_\varepsilon}{\partial g} + 5\varepsilon z_\varepsilon^4 \frac{\partial z_\varepsilon}{\partial g} &= \frac{\partial}{\partial g} \left(\frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon - z_\varepsilon^3 + \varepsilon z_\varepsilon^5 \right) \\ &= 0 \text{ in } Q. \end{aligned}$$

Indeed,

$$\frac{\partial z_\varepsilon}{\partial t}(v, g) - \Delta z_\varepsilon(v, g) - z_\varepsilon^3(v, g) + \varepsilon z_\varepsilon^5(v, g) = v \text{ in } Q$$

- For the initial condition in Ω , we have :

$$\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \Big|_{t=0} = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) \Big|_{t=0} - z_\varepsilon(v, 0) \Big|_{t=0}}{\lambda} = 0 \text{ because } z_\varepsilon(v, g) \Big|_{t=0} = 0 \text{ in } \Omega.$$

- For the boundary condition on Σ , we have :

$$\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\lambda \varepsilon g - 0}{\lambda} = \varepsilon g \text{ on } \Sigma.$$

□

Proposition 2.6. *The partial derivative $\frac{\partial z_\varepsilon}{\partial g} := \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(0, \lambda g) - z_\varepsilon(0, 0)}{\lambda}$ is the solution of the following problem :*

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g} \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g} \right) - 3z_\varepsilon^2(0, 0) \frac{\partial z_\varepsilon}{\partial g} + 5\varepsilon z_\varepsilon^4(0, 0) \frac{\partial z_\varepsilon}{\partial g} = 0 & \text{in } Q, \\ \frac{\partial z_\varepsilon}{\partial g} \Big|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) = \varepsilon g & \text{on } \Sigma. \end{cases} \quad (2.11)$$

Remark 2.7. The proof of Proposition 2.6 is carried out in the same way as that of Proposition 2.5.

Now we have the following lemma:

Lemma 2.8. For all $v \in L^2(Q)$ and for all $g \in L^2(\Sigma)$, we have:

$$J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}, \quad (2.12)$$

with $S(v) = \zeta_\varepsilon(v) - \zeta_\varepsilon(0)$ and $\zeta_\varepsilon(v) = \zeta_\varepsilon(x, t, v)$ the solution of the following system:

$$\begin{cases} -\frac{\partial \zeta_\varepsilon}{\partial t}(v) - \Delta \zeta_\varepsilon(v) - 3z_\varepsilon^2(v, 0)\zeta_\varepsilon(v) + 5\varepsilon z_\varepsilon^4(v, 0)\zeta_\varepsilon(v) = -z_\varepsilon(v, 0) + z_d & \text{in } Q, \\ \zeta_\varepsilon(T, v) = 0 & \text{in } \Omega, \\ \zeta_\varepsilon(v) = 0 & \text{on } \Sigma. \end{cases} \quad (2.13)$$

Proof. We multiply the first equation of (2.13) by $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g)$. By integrating by parts and using the boundary conditions, $\zeta_\varepsilon(T, v) = 0$ and $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \Big|_{t=0} = 0$ in Ω , we obtain that:

$$\left\langle -\frac{\partial \zeta_\varepsilon}{\partial t}(v), \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right), \zeta_\varepsilon(v) \right\rangle_{L^2(Q)}.$$

Using the conditions $\zeta_\varepsilon(v) = 0$ and $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \varepsilon g$ on Σ and by applying Green's formula, we have:

$$\begin{aligned} \left\langle -\Delta \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} &= \left\langle \frac{\partial}{\partial g} (-\Delta z_\varepsilon(v, 0))(g), \zeta_\varepsilon(v) \right\rangle_{L^2(Q)} \\ &\quad - \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

Therefore:

$$\left\langle -(z_\varepsilon(v, 0) - z_d), \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} = - \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}.$$

Indeed,

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right) - 3z_\varepsilon^2(v, 0) \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \\ + 5\varepsilon z_\varepsilon^4(v, 0) \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = 0 & \text{in } Q. \end{aligned}$$

It follows that :

$$\left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}. \quad (2.14)$$

For $v = 0$, the system (2.13) becomes:

$$\begin{cases} -\frac{\partial \zeta_\varepsilon}{\partial t}(0) - \Delta \zeta_\varepsilon(0) - 3z_\varepsilon^2(0, 0)\zeta_\varepsilon(0) + 5\varepsilon z_\varepsilon^4(0, 0)\zeta_\varepsilon(0) = -z_\varepsilon(0, 0) + z_d & \text{in } Q, \\ \zeta_\varepsilon(T, 0) = 0 & \text{in } \Omega, \\ \zeta_\varepsilon(0) = 0 & \text{on } \Sigma. \end{cases} \quad (2.15)$$

By multiplying the first equation of (2.15) by $\frac{\partial z_\varepsilon}{\partial g}(0,0)(g)$ and then applying the previous techniques, we obtain:

$$\left\langle -(z_\varepsilon(0,0) - z_d), \frac{\partial z_\varepsilon}{\partial g}(0,0) \right\rangle_{L^2(Q)} = - \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(0), \varepsilon g \right\rangle_{L^2(\Sigma)}. \quad (2.16)$$

By substituting (2.14) and (2.16) into the equation (2.6) of the Proposition 2.4, we obtain:

$$J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)},$$

with $S(v) = \zeta_\varepsilon(v) - \zeta_\varepsilon(0)$. □

Remark 2.9. Using the previous result, we obtain:

$$\sup_{g \in L^2(\Sigma)} (J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g)) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \sup_{g \in L^2(\Sigma)} \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle, \quad (2.17)$$

$$\text{with } \sup_{g \in L^2(\Sigma)} \left(\left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \right) = \begin{cases} +\infty & \text{if } \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \neq 0, \\ 0 & \text{if } \frac{\partial S}{\partial \nu}(v) = 0. \end{cases}$$

The result of $\sup_{g \in L^2(\Sigma)} \left(\left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \right)$ does not allow us to characterize the no-regret control of the following problem:

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} (J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g)) \right]. \quad (2.18)$$

To make sense of problem (2.18), we consider the set E_1 defined by:

$$E_1 = \left\{ v \in L^2(Q), \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} = 0, \forall g \in L^2(\Sigma) \right\}.$$

In what follows, we use the notion of low-regret control to seek a control in the set E_1 .

2.2.2. Low-regret control.

In this subsection, we focus on the low-regret problem, which depends on the disturbance g and its norm. This problem is obtained by relaxing problem (2.18). We then consider the following definition:

Definition 2.10. *Let $\gamma > 0$ be a fixed number representing the relaxation parameter. Then, problem (2.18) becomes:*

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} \left(J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) - \gamma \|g\|_{L^2(\Sigma)}^2 \right) \right]. \quad (2.19)$$

The solution of (2.19), if it exists, is called low-regret control.

Lemma 2.11. *We consider the function J_ε defined in (2.2). Thus, for all $v \in L^2(Q)$, the problem (2.19) becomes:*

$$\inf_{v \in L^2(Q)} \left[J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \right]. \tag{2.20}$$

Proof. Using the result of Lemma 2.8, we have:

$$\begin{aligned} \sup_{g \in L^2(\Sigma)} \left(J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) - \gamma \|g\|_{L^2(\Sigma)}^2 \right) &= J_\varepsilon(v, 0) - J_\varepsilon(0, 0) \\ &- \frac{\gamma}{2} \|g\|_{L^2(\Sigma)}^2 + \sup_{g \in L^2(\Sigma)} \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}. \end{aligned} \tag{2.21}$$

According to the Fenchel transformation, we have:

$$\begin{aligned} 2 \sup_{g \in L^2(\Sigma)} \left(\left\langle \frac{1}{2} \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} - \frac{\gamma}{2} \|g\|_{L^2(\Sigma)}^2 \right) &= \frac{\varepsilon^2}{2\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2 \\ - \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2 &= \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2. \end{aligned}$$

Thus, the problem (2.19) becomes :

$$\inf_{v \in L^2(Q)} \mathcal{J}_\varepsilon^\gamma(v) \tag{2.22}$$

with $\mathcal{J}_\varepsilon^\gamma(v) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2$. □

Now we reformulate the problem (2.19) as follows:

$$\left\{ \begin{array}{l} \text{For all } \gamma > 0, \text{ find } u_\varepsilon^\gamma \in L^2(Q) \text{ such that :} \\ \mathcal{J}_\varepsilon^\gamma(u_\varepsilon^\gamma) = \inf_{v \in L^2(Q)} \mathcal{J}_\varepsilon^\gamma(v). \end{array} \right. \tag{2.23}$$

Problem (2.23) is a low-regret problem and its solution is called low-regret control.

Remark 2.12. In contrast to the linear case, the function J_ε^γ is not convex. So we don't necessarily have uniqueness of u_ε^γ . Moreover, we are not convinced that u_ε^γ converges in E_1 . Then, we will adapt the function J_ε^γ like J.L. Lions [6] by using a regularization method for finding the low-regret control related to the problem.

2.3. Adapted low-regret control.

Since the function J_ε^γ is not convex, we define a new function to study the control problem. We are therefore interested in finding a solution to the following minimization problem:

$$\inf_{v \in L^2(Q)} \mathcal{J}_{\varepsilon a}^\gamma(v) \tag{2.24}$$

with $\mathcal{J}_{\varepsilon a}^\gamma(v) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{1}{2} \|v - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2$, where $\tilde{u} \in L^2(Q)$ is the no-regret control. The control solution u_ε^γ of problem (2.24) is called the adapted low-regret control.

2.3.1. Existence of adapted low-regret control. In this section, we establish the existence of the adapted low-regret control, inspired by the work of T. Thomas et al. in [14, 15]. We consider the following proposition:

Proposition 2.13. *There exists at least one solution $u_\varepsilon^\gamma \in L^2(Q)$, called the adapted low-regret control, for the optimal control problem (2.24).*

Proof. According to the definition of $\mathcal{J}_{\varepsilon a}^\gamma$, for all $v \in L^2(Q)$, we have: $\mathcal{J}_{\varepsilon a}^\gamma(v) \geq -J_\varepsilon(0, 0) \Leftrightarrow -J_\varepsilon(0, 0) \leq J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{1}{2} \|v - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2$.

Let us consider the nonempty set $E_2 = \{v \in L^2(Q), \mathcal{J}_{\varepsilon a}^\gamma(v) \geq -J_\varepsilon(0, 0)\}$. Indeed, by taking $v = 0$, we have: $\mathcal{J}_{\varepsilon a}^\gamma(0) = \frac{1}{2} \|\tilde{u}\|_{L^2(Q)}^2 \geq -J_\varepsilon(0, 0)$. Thus, there exists a real constant C , which depends on ε and γ , such that $C(\varepsilon, \gamma) = \inf_{v \in L^2(Q)} \mathcal{J}_{\varepsilon a}^\gamma(v)$. Indeed, we consider a minimizing sequence $(v_n)_{n \in \mathbb{N}} = (v_n(\varepsilon, \gamma))_{n \in \mathbb{N}}$ that converges to d_ε^γ . Then, $d_\varepsilon^\gamma = \lim_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon a}^\gamma(v_n)$.

$$\begin{aligned} -J_\varepsilon(0, 0) &\leq \frac{1}{2} \|z_\varepsilon(v_n, 0) - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|v_n\|_{L^2(Q)}^2 - \frac{1}{2} \|z_\varepsilon(0, 0) - z_d\|_{L^2(Q)}^2 \\ &\quad + \frac{1}{2} \|v_n - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v_n) \right\|_{L^2(\Sigma)}^2 \\ &\leq d_\varepsilon^\gamma + 1. \end{aligned}$$

We deduce the following inequalities, which show that the sequences are bounded.

$$\begin{cases} \|v_n\|_{L^2(Q)} \leq C_1(\varepsilon, \gamma), \\ \|z_\varepsilon(v_n, 0) - z_d\|_{L^2(Q)} \leq C_2(\varepsilon, \gamma), \\ \frac{\varepsilon}{\sqrt{\gamma}} \left\| \frac{\partial S}{\partial \nu}(v_n) \right\|_{L^2(\Sigma)} \leq C_3(\varepsilon, \gamma). \end{cases}$$

Here, $C_i(\varepsilon, \gamma)$, for $i \in 1, 2, 3$, are constants independent of n . Therefore, we can extract a subsequence, still denoted by $(v_n)_{n \in \mathbb{N}}$, such that $v_n(\varepsilon, \gamma) \rightharpoonup u_\varepsilon^\gamma$ in $L^2(Q)$. \square

In order to characterize low-regret control, we consider the following propositions:

Proposition 2.14. For all $w \in L^2(Q)$, the partial derivative $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda}$ be the solution of the following problem:

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v} \right) - \Delta \left(\frac{\partial z_\varepsilon^\gamma}{\partial v} \right) - 3z_\varepsilon^2 \frac{\partial z_\varepsilon^\gamma}{\partial v} + 5\varepsilon z_\varepsilon^4 \frac{\partial z_\varepsilon^\gamma}{\partial v} = w & \text{in } Q, \\ \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \Big|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) = 0 & \text{on } \Sigma. \end{cases} \quad (2.25)$$

Proof. Let us show that $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w)$ is a solution of (2.25). $\forall w \in L^2(Q)$ and $\forall \lambda \in \mathbb{R}$, $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda}$

- For the condition in Q , on the one hand, we have: $\frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta(z_\varepsilon^\gamma) - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 =$
 $\frac{\partial}{\partial v} \left(\frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta(z_\varepsilon^\gamma) - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 \right) (v, 0)(w) = \lim_{\lambda \rightarrow 0} \frac{v + \lambda w - v}{\lambda} = w$ in Q .

On the other hand, we have:

$$\begin{aligned} & \frac{\partial}{\partial v} \left(\frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta(z_\varepsilon^\gamma) - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 \right) (v, 0)(w) = \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \right) \\ & - \Delta \left(\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \right) - 3(z_\varepsilon^\gamma)^2(v, 0) \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) + 5\varepsilon(z_\varepsilon^\gamma)^4(v, 0) \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w). \end{aligned}$$

- For the initial condition in Ω .
 $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \Big|_{t=0} = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0)|_{t=0} - z_\varepsilon(v, 0)|_{t=0}}{\lambda} = 0$ in Ω .
- For the condition on Σ .
 $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda} = 0$ on Σ .

□

Proposition 2.15. For all $w \in L^2(Q)$, the partial derivative $\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(v + \lambda w) - \zeta_\varepsilon(v)}{\lambda}$ is the solution of the following problem:

$$\begin{cases} -\frac{\partial}{\partial t} \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v} \right) - \Delta \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v} \right) - 3z_\varepsilon^2 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v} + 5\varepsilon z_\varepsilon^4 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v} = -\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) & \text{in } Q, \\ \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \Big|_{t=T} = 0 & \text{in } \Omega, \\ \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) = 0 & \text{on } \Sigma. \end{cases} \quad (2.26)$$

Proof. Let us show that $\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w)$ is a solution of problem (2.26).

- For the condition in Q , we have :

$$\begin{aligned}
 & -\frac{\partial}{\partial t} \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \right) - \Delta \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \right) - 3z_\varepsilon^2 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) + 5\varepsilon z_\varepsilon^4 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \\
 = & \lim_{\lambda \rightarrow 0} \left(\frac{\frac{\partial \zeta_\varepsilon}{\partial t}(v + \lambda w) - \Delta \zeta_\varepsilon(v + \lambda w) - 3z_\varepsilon^2 \zeta_\varepsilon(v + \lambda w) + 5\varepsilon z_\varepsilon^4 \zeta_\varepsilon(v + \lambda w)}{\lambda} \right) \\
 & - \lim_{\lambda \rightarrow 0} \left(\frac{\frac{\partial \zeta_\varepsilon}{\partial t}(v, 0) - \Delta \zeta_\varepsilon(v, 0) - 3z_\varepsilon^2 \zeta_\varepsilon(v, 0) + 5\varepsilon z_\varepsilon^4 \zeta_\varepsilon(v, 0)}{\lambda} \right) \\
 = & \lim_{\lambda \rightarrow 0} \frac{-(z_\varepsilon(v + \lambda w, 0) - z_d) + z_\varepsilon(v, 0) - z_d}{\lambda} \text{ according to (2.13)} \\
 = & -\lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda} \\
 = & -\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \text{ in } Q,
 \end{aligned}$$

- For the condition in Ω , we have:

$$\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \Big|_{t=T} = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(v + \lambda w)|_{t=T} - \zeta_\varepsilon(v)|_{t=T}}{\lambda} = 0 \text{ because } \zeta_\varepsilon(v)|_{t=T} = 0 \text{ in } \Omega.$$

- For the condition on Σ , we have:

$$\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(v + \lambda w) - \zeta_\varepsilon(v)}{\lambda} = 0 \text{ because } \zeta_\varepsilon(v) = 0 \text{ on } \Sigma.$$

□

In the following, we provide the characterization of the adapted low-regret control.

2.3.2. Characterization of adapted low-regret control.

This characterization is given by the following proposition:

Proposition 2.16. *The adapted low-regret control u_ε^γ solution of (2.24) is characterized by $\{\zeta_\varepsilon^\gamma, z_\varepsilon^\gamma, \rho_\varepsilon^\gamma, p_\varepsilon^\gamma\}$ the unique solution of:*

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} -\frac{\partial \zeta_\varepsilon^\gamma}{\partial t} - \Delta \zeta_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \zeta_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 \zeta_\varepsilon^\gamma = -(z_\varepsilon^\gamma - z_d) \\ \zeta_\varepsilon^\gamma(T, u_\varepsilon^\gamma) = 0 \\ \zeta_\varepsilon^\gamma = 0 \end{array} \right. & \begin{array}{l} \text{in } Q, \\ \text{in } \Omega, \\ \text{on } \Sigma. \end{array} \end{array} \right. \quad (2.27)$$

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta z_\varepsilon^\gamma - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 = u_\varepsilon^\gamma \\ z_\varepsilon^\gamma(0) = 0 \\ z_\varepsilon^\gamma = 0 \end{array} \right. & \begin{array}{l} \text{in } Q, \\ \text{in } \Omega, \\ \text{on } \Sigma. \end{array} \end{array} \right. \quad (2.28)$$

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \frac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta \rho_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma = 0 \\ \rho_\varepsilon^\gamma(0) = 0 \\ \rho_\varepsilon^\gamma = -\frac{\varepsilon^2}{2\gamma} \frac{\partial S_\varepsilon^\gamma}{\partial \nu} \end{array} \right. & \begin{array}{l} \text{in } Q, \\ \text{in } \Omega, \\ \text{on } \Sigma. \end{array} \end{array} \right. \quad (2.29)$$

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} -\frac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 p_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 p_\varepsilon^\gamma = z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma \\ p_\varepsilon^\gamma(T, u_\varepsilon^\gamma) = 0 \\ p_\varepsilon^\gamma = 0 \end{array} \right. & \begin{array}{l} \text{in } Q, \\ \text{in } \Omega, \\ \text{on } \Sigma. \end{array} \end{array} \right. \quad (2.30)$$

$$\left\{ \begin{array}{l} p_\varepsilon^\gamma + (N + 1) u_\varepsilon^\gamma = \tilde{u} \quad \text{in } L^2(Q), \text{ where } z_\varepsilon^\gamma := z_\varepsilon(u_\varepsilon^\gamma, 0). \end{array} \right. \quad (2.31)$$

Proof. The Euler-Lagrange condition on $J_{\varepsilon a}^\gamma$, for all $w \in L^2(Q)$, is given by:

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma + \lambda w) - \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma)}{\lambda} &= \left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d \right\rangle_{L^2(Q)} \\ &+ \langle Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} + \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial \nu} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma) \right\rangle_{L^2(\Sigma)} \geq 0. \end{aligned} \quad (2.32)$$

Indeed, for all $\lambda \in \mathbb{R}$ and for all $w \in L^2(Q)$, we have:

$$\begin{aligned} &\frac{\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma + \lambda w) - \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma)}{\lambda} = \frac{\lambda}{2} \left\| \frac{z_\varepsilon(u_\varepsilon^\gamma + \lambda w, 0) - z_\varepsilon(u_\varepsilon^\gamma, 0)}{\lambda} \right\|_{L^2(Q)}^2 \\ &+ \frac{\lambda}{2} (N + 1) \|w\|_{L^2(Q)}^2 + \left\langle \frac{z_\varepsilon(u_\varepsilon^\gamma + \lambda w, 0) - z_\varepsilon(u_\varepsilon^\gamma, 0)}{\lambda}, z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d \right\rangle_{L^2(Q)} \\ &+ N \langle u_\varepsilon^\gamma, w \rangle_{L^2(Q)} + \langle u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} + \frac{\varepsilon^2 \lambda}{4\gamma} \left\| \frac{\frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma + \lambda w) - \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma)}{\lambda} \right\|_{L^2(\Sigma)}^2 \\ &+ \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma + \lambda w) - \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma)}{\lambda}, \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma) \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

By passing to the limit, we obtain:

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma + \lambda w) - \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma)}{\lambda} &= \left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d \right\rangle_{L^2(Q)} \\ &+ \langle Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} + \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial v} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S}{\partial v}(u_\varepsilon^\gamma) \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

Indeed, we have:

$$\begin{cases} \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(u_\varepsilon^\gamma + \lambda w, 0) - z_\varepsilon(u_\varepsilon^\gamma, 0)}{\lambda} = \frac{\partial z_\varepsilon^\gamma}{\partial v}(u_\varepsilon^\gamma, 0)(w), \\ \lim_{\lambda \rightarrow 0} \frac{S(u_\varepsilon^\gamma + \lambda w) - S(u_\varepsilon^\gamma)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(u_\varepsilon^\gamma + \lambda w) - \zeta_\varepsilon(u_\varepsilon^\gamma)}{\lambda} = \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(u_\varepsilon^\gamma)(w). \end{cases}$$

In addition, the terms $\left\| \frac{\partial z_\varepsilon^\gamma}{\partial v} \right\|_{L^2(Q)}^2$ and $\left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial v} \right\|_{L^2(\Sigma)}^2$ are bounded and tend to zero when λ tends to zero.

By multiplying the first equation of (2.29) by $\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w)$ we have:

$$\begin{aligned} \left\langle \frac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta \rho_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle_{L^2(Q)} &= 0 \Leftrightarrow \left\langle \frac{\partial \rho_\varepsilon^\gamma}{\partial t}, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle \\ &+ \left\langle -\Delta \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle + \left\langle -3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle + \left\langle 5(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle = 0 \end{aligned}$$

By integrating twice by parts, we obtain:

$$\left\langle \frac{\partial \rho_\varepsilon^\gamma}{\partial t}, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle_{L^2(Q)} = \left\langle -\frac{\partial}{\partial t} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \rho_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

By applying Green's formula, we obtain :

$$\begin{aligned} \left\langle -\Delta \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle_{L^2(Q)} &= \left\langle -\Delta \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \rho_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &- \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial v} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S_\varepsilon^\gamma}{\partial v} \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

It follows that :

$$\frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial v} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S_\varepsilon^\gamma}{\partial v} \right\rangle_{L^2(\Sigma)} = \left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), -\rho_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

The Euler-Lagrange condition becomes : $\forall w \in L^2(Q)$,

$$\left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0), z_\varepsilon(u_\varepsilon^\gamma, 0)(w) - z_d - \rho_\varepsilon^\gamma \right\rangle_{L^2(Q)} + \langle Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} \geq 0. \quad (2.33)$$

By multiplying the first equation of (2.30) by $\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w)$, we have:

$$\begin{aligned} \left\langle -\frac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 p_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 p_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} \\ = \left\langle z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle. \end{aligned}$$

By integrating twice by parts, we obtain:

$$\left\langle -\frac{\partial p_\varepsilon^\gamma}{\partial t}, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

By applying Green's formula, we obtain:

$$\left\langle -\Delta p_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} = \left\langle -\Delta \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

It follows that:

$$\begin{aligned} \left\langle z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} &= \left\langle \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &+ \left\langle -\Delta \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &+ \left\langle -3(z_\varepsilon^\gamma)^2 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) + 5\varepsilon(z_\varepsilon^\gamma)^4 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), p_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &= \langle p_\varepsilon^\gamma, w \rangle_{L^2(Q)}. \end{aligned}$$

Indeed, we have:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right) - 3(z_\varepsilon^\gamma)^2 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \\ + 5\varepsilon(z_\varepsilon^\gamma)^4 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) = w \text{ in } Q. \end{aligned}$$

Thus, the condition (2.33) becomes :

$$\langle p_\varepsilon^\gamma + Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} \geq 0, \forall w \in L^2(Q). \quad (2.34)$$

$\forall w \in L^2(Q)$, $-w \in L^2(Q)$ because $L^2(Q)$ is a vector space. Considering $w' = -w$, (2.34) becomes:

$$\langle p_\varepsilon^\gamma + Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w' \rangle_{L^2(Q)} \leq 0, \forall w' = -w \in L^2(Q).$$

Therefore, the adapted low-regret control u_ε^γ is characterized by the following equation:

$$p_\varepsilon^\gamma + (N+1)u_\varepsilon^\gamma = \tilde{u} \text{ in } L^2(Q).$$

□

3. Singular optimality system (SOS)

3.1. Characterization of low-regret control

In this section, we present the singular optimality system characterizing the low-regret control of system (2.19). To this end, we first establish the inequalities given by the following lemma:

Lemma 3.1. *There exists some positive constants C_i for $i \in \{1, 2, 3\}$ such that:*

$$\begin{cases} \|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C_1; \\ \|z_\varepsilon^\gamma\|_{L^2(Q)} \leq C_2; \\ \frac{1}{2} \frac{\varepsilon}{\sqrt{\gamma}} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C_3. \end{cases} \quad (3.1)$$

Proof. According to Proposition 2.13, u_ε^γ is the solution of (2.24). Thus, we have :

$$\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma) \leq \mathcal{J}_{\varepsilon a}^\gamma(v), \quad \forall v \in L^2(Q).$$

By taking $v = 0$, we obtain:

$$\begin{aligned} \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma) &= \frac{1}{2} \|z_\varepsilon^\gamma - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|u_\varepsilon^\gamma\|_{L^2(Q)}^2 + \frac{1}{2} \|u_\varepsilon^\gamma - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \\ &\leq \frac{1}{2} \left(\|z_d\|_{L^2(Q)}^2 + \|\tilde{u}\|_{L^2(Q)}^2 \right) = C. \end{aligned}$$

We deduce the following inequalities:

$$\begin{cases} \|z_\varepsilon^\gamma - z_d\|_{L^2(Q)} \leq C_1 \\ \|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C_2 \\ \frac{\varepsilon}{2\sqrt{\gamma}} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C_3. \end{cases}$$

By the reverse triangle inequality, we obtain:

$$\begin{aligned} \left| \|z_\varepsilon^\gamma\|_{L^2(Q)} - \|z_d\|_{L^2(Q)} \right| &\leq \|z_\varepsilon^\gamma - z_d\|_{L^2(Q)} \leq C_1 \\ \implies \|z_\varepsilon^\gamma\|_{L^2(Q)} &\leq C_1 + \|z_d\|_{L^2(Q)} = C'. \end{aligned}$$

Hence the inequalities of the Lemma 3.1. □

Theorem 3.2. *The low-regret control u^γ of the problem (2.19) is characterized by $\{\zeta^\gamma, z^\gamma, \rho^\gamma, p^\gamma\}$ the unique solution of :*

$$\begin{cases} \left\{ \begin{array}{ll} -\frac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma - 3(z^\gamma)^2 \zeta^\gamma = -(z^\gamma - z_d) & \text{in } Q, \\ \zeta^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ \zeta^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (3.2) \\ \\ \left\{ \begin{array}{ll} \frac{\partial z^\gamma}{\partial t} - \Delta z^\gamma - (z^\gamma)^3 = u^\gamma & \text{in } Q, \\ z^\gamma(0) = 0 & \text{in } \Omega, \\ z^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (3.3) \\ \\ (SOS) \left\{ \begin{array}{ll} \frac{\partial \rho^\gamma}{\partial t} - \Delta \rho^\gamma - 3(z^\gamma)^2 \rho^\gamma = 0 & \text{in } Q, \\ \rho^\gamma(0) = 0 & \text{in } \Omega, \\ \rho^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (3.4) \\ \\ \left\{ \begin{array}{ll} -\frac{\partial p^\gamma}{\partial t} - \Delta p^\gamma - 3(z^\gamma)^2 p^\gamma = z^\gamma - z_d - \rho^\gamma & \text{in } Q, \\ p^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ p^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (3.5) \\ \\ \left. \begin{array}{l} p^\gamma + (N + 1) u^\gamma = \tilde{u} \quad \text{in } L^2(Q), \text{ where } z^\gamma := z(u^\gamma, 0). \end{array} \right. & (3.6) \end{cases}$$

Proof. Consider z_ε^γ the solution of the problem (2.28). According to Lemma 3.1, we have : $\|z_\varepsilon^\gamma\|_{L^2(Q)} \leq C$. Therefore, we can extract a subsequence $(z_\varepsilon^\gamma)_\varepsilon$ which weakly converges to z^γ in $L^2(Q)$ when $\varepsilon \rightarrow 0$. In addition, $\|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C$. Thus, $u_\varepsilon^\gamma \rightharpoonup u^\gamma$ in $L^2(Q)$. Therefore, from (2.28), we have:

$$\left\| \frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta z_\varepsilon^\gamma - (z_\varepsilon^\gamma)^3 - u_\varepsilon^\gamma \right\|_{L^2(Q)} = \|\varepsilon(z_\varepsilon^\gamma)^5\|_{L^2(Q)} \leq \varepsilon C \xrightarrow{\varepsilon \rightarrow 0} 0$$

when $\varepsilon \rightarrow 0$, the problem (2.28) becomes:

$$\begin{cases} \frac{\partial z^\gamma}{\partial t} - \Delta z^\gamma - (z^\gamma)^3 = u^\gamma & \text{in } Q, \\ z^\gamma(0) = 0 & \text{in } \Omega, \\ z^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

We consider $\zeta_\varepsilon^\gamma \in L^2(Q)$ as the solution to the problem (2.27). Since $\zeta_\varepsilon^\gamma \in L^2(Q)$, the sequence $(\zeta_\varepsilon^\gamma)$ is bounded in $L^2(Q)$. There exists a positive constant M_1 such that $\|\zeta_\varepsilon^\gamma\|_{L^2(Q)} \leq M_1$. Thus, from (2.27) we have:

$$\left\| -\frac{\partial \zeta_\varepsilon^\gamma}{\partial t} - \Delta \zeta_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \zeta_\varepsilon^\gamma + (z_\varepsilon^\gamma - z_d) \right\|_{L^2(Q)} = \|5\varepsilon(z_\varepsilon^\gamma)^4 \zeta_\varepsilon^\gamma\|_{L^2(Q)} \leq 5\varepsilon C M_1 \xrightarrow{\varepsilon \rightarrow 0} 0$$

Therefore, it exists a subsequence $(\zeta_\varepsilon^\gamma)_\varepsilon$ such that $\zeta_\varepsilon^\gamma \rightharpoonup \zeta^\gamma$ in $L^2(Q)$.

When $\varepsilon \rightarrow 0$, the problem (2.27) becomes:

$$\begin{cases} -\frac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma - 3(z^\gamma)^2 \zeta^\gamma = -(z^\gamma - z_d) & \text{in } Q, \\ \zeta^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ \zeta^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

We consider $\rho_\varepsilon^\gamma \in L^2(Q)$ the solution of the problem (2.29). According to the Lemma 3.1, we deduce the following limit:

$\frac{\varepsilon}{\sqrt{\gamma}} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \rightharpoonup \lambda^\gamma$ converges weakly in $L^2(\Sigma)$. So, when $\varepsilon \rightarrow 0$, $\frac{\varepsilon^2}{\gamma} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \rightarrow 0$, $\rho_\varepsilon^\gamma \in L^2(Q)$. Therefore, the sequence $(\rho_\varepsilon^\gamma)$ is bounded in $L^2(Q)$. Then, there exists a positive constant M_2 such that $\|\rho_\varepsilon^\gamma\|_{L^2(Q)} \leq M_2$. Thus, from (2.29) we have:

$$\left\| \frac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta \rho_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma \right\|_{L^2(Q)} = \|5\varepsilon(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma\|_{L^2(Q)} \leq 5\varepsilon C M_2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore, it exists a subsequence $(\rho_\varepsilon^\gamma)_\varepsilon$ such that $\rho_\varepsilon^\gamma \rightharpoonup \rho^\gamma$ in $L^2(Q)$. When $\varepsilon \rightarrow 0$, the problem (2.29) becomes:

$$\begin{cases} \frac{\partial \rho^\gamma}{\partial t} - \Delta \rho^\gamma - 3(z^\gamma)^2 \rho^\gamma = 0 & \text{in } Q, \\ \rho^\gamma(0) = 0 & \text{in } \Omega, \\ \rho^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

We consider p_ε^γ the solution of the problem (2.30). According to the equation (2.31), (u_ε^γ) is bounded in $L^2(Q)$ and $\tilde{u} \in L^2(Q)$. Thus, there exists a positive constant M_3 such that $\|p_\varepsilon^\gamma\|_{L^2(Q)} = \|-u_\varepsilon^\gamma + \tilde{u} - N u_\varepsilon^\gamma\| \leq M_3$. Therefore, (p_ε^γ) is bounded in $L^2(Q)$ and we have:

$$\left\| -\frac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 p_\varepsilon^\gamma - (z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma) \right\|_{L^2(Q)} = \|5\varepsilon(z_\varepsilon^\gamma)^4 p_\varepsilon^\gamma\|_{L^2(Q)} \leq 5\varepsilon C M_3 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore, it exists a subsequence $(p_\varepsilon^\gamma)_\varepsilon$ such that $p_\varepsilon^\gamma \rightharpoonup p^\gamma$ in $L^2(Q)$.

When $\varepsilon \rightarrow 0$, the system (2.30) becomes:

$$\begin{cases} -\frac{\partial p^\gamma}{\partial t} - \Delta p^\gamma - 3(z^\gamma)^2 p^\gamma = z^\gamma - z_d - \rho^\gamma & \text{in } Q, \\ p^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ p^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

According to the equation (2.31) and the inequalities of the Lemma 3.1, we have:

$$u_\varepsilon^\gamma \rightharpoonup u^\gamma \text{ in } L^2(Q). \tag{3.7}$$

In addition, $p_\varepsilon^\gamma \rightharpoonup p^\gamma$ in $L^2(Q)$. We can deduce that:

$$p_\varepsilon^\gamma \rightharpoonup p^\gamma = -(N + 1) u^\gamma + \tilde{u} \text{ in } L^2(Q), \text{ where } z^\gamma := z(u^\gamma, 0). \tag{3.8}$$

□

3.2. Characterization of no-regret control

We now deduce, in this section, the singular optimality system of the no-regret control for problem (2.18) by passing to the limit as $\gamma \rightarrow 0$. The limit problem is not well posed in the sense of Hadamard. This singular optimality system is presented in the following theorem:

Theorem 3.3. *The no-regret control \tilde{u} of the problem (2.18) is characterized by $\{\zeta, z, \rho, p\}$ a solution of the optimality system:*

$$\begin{cases}
 \left\{ \begin{array}{ll}
 -\frac{\partial \zeta}{\partial t} - \Delta \zeta - 3z^2 \zeta = -(z - z_d) & \text{in } Q, \\
 \zeta(T, \tilde{u}) = 0 & \text{in } \Omega, \\
 \zeta = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.9) \\
 \left\{ \begin{array}{ll}
 \frac{\partial z}{\partial t} - \Delta z - z^3 = \tilde{u} & \text{in } Q, \\
 z(0) = 0 & \text{in } \Omega, \\
 z = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.10) \\
 (SOS) \left\{ \begin{array}{ll}
 \frac{\partial \rho}{\partial t} - \Delta \rho - 3z^2 \rho = 0 & \text{in } Q, \\
 \rho(0) = 0 & \text{in } \Omega, \\
 \rho = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.11) \\
 \left\{ \begin{array}{ll}
 -\frac{\partial p}{\partial t} - \Delta p - 3z^2 p = z - z_d - \rho & \text{in } Q, \\
 p(T, \tilde{u}) = 0 & \text{in } \Omega, \\
 p = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.12) \\
 p + N\tilde{u} = 0 & \text{in } L^2(Q), \text{ where } z := z(\tilde{u}, 0). & (3.13)
 \end{cases}$$

Proof. According to the results of Theorem 3.2 and by passing to the limit as $\gamma \rightarrow 0$, we obtain:

$$\begin{cases}
 \left\{ \begin{array}{ll}
 \zeta^\gamma \rightharpoonup \zeta = 0, \\
 z^\gamma \rightharpoonup z = 0, \\
 \rho^\gamma \rightharpoonup \rho = 0, \\
 p^\gamma \rightharpoonup p = 0.
 \end{array} \right. & \text{on } \Sigma \\
 \left\{ \begin{array}{ll}
 \zeta^\gamma \rightharpoonup \zeta = 0, \\
 z^\gamma(0) \rightharpoonup z(0) = 0, \\
 \rho^\gamma \rightharpoonup \rho = 0, \\
 p^\gamma(T, u^\gamma) \rightharpoonup p(T, \tilde{u}) = 0.
 \end{array} \right. & \text{in } \Omega
 \end{cases}$$

According to equation (3.8), we deduce that:

$$u^\gamma \rightharpoonup \tilde{u} \text{ in } L^2(Q).$$

It follows that there exists a no-regret control \tilde{u} , characterized by $\{\zeta, z, \rho, p\}$ solution of (2.18). □

4. Conclusion

In this article, we analyzed an ill-posed parabolic problem with missing data by combining regularization and low-regret control. Thanks to the regularization method, we


were able to recover the missing boundary data on Σ and transform the initial problem into a family of well-posed problems. To study the control problem, we applied the low-regret control method, which is better suited to problems involving missing data. We were able to characterize the no-regret control of system (1.1) thanks to the convergence of low-regret control, then by taking the limit as the perturbation parameter tends towards zero. This work can be approached numerically.

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