

Variational analysis of Kirchhoff equations in Musielak-Orlicz-Zygmund spaces

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Abstract. We study a class of nonlocal Kirchhoff problems with nonlinearities exhibiting nonstandard growth. Using variational methods in Musielak-Orlicz-Zygmund spaces, we prove the existence of nontrivial weak solutions. The analysis uses generalized N -functions, Orlicz-Zygmund embeddings, pseudo-monotone operators, and the Palais-Smale condition, which allow handling double-phase and nonlocal Kirchhoff terms. The results extend classical variational methods to settings with borderline and logarithmic growth.

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1. Introduction

The study of nonlocal Kirchhoff problems with Sobolev–Zygmund type growth is motivated both by applications in nonlinear elasticity and composite materials, and they also raise important mathematical questions related to borderline phenomena in partial differential equations.

Kirchhoff-type nonlocal operators were first used to describe the vibration of elastic strings and membranes. The classical model of Kirchhoff [17] leads to equations of the form

$$-M\left(\int_{\Omega} |\nabla u|^p dx\right) \Delta_p u = f(x, u),$$

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where the tension depends on the average deformation. Lions extended these ideas to the PDE setting and motivated the study of nonlocal operators coupled with nonlinear source terms.

At the same time, double-phase problems were introduced to model materials with different behaviors in different regions. These problems have the form

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u),$$

where the coefficient $\mu(x)$ governs the transition between the p - and q -growth phases. Important contributions in this area were given by Colombo and Mingione [9] and Baroni–Colombo–Mingione [3].

In the last decade, researchers have studied double-phase and Kirchhoff-type operators in generalized Orlicz and Musielak–Orlicz spaces frameworks. Fiscella and Pinamonti [13] proved existence results for Kirchhoff problems in Orlicz spaces using variational methods. Diening et al. [10], and Boccardo–Murat [6] analyzed critical growth phenomena in Zygmund-Orlicz type spaces, where logarithmic perturbations naturally arise.

More recently, special attention has been given to sources of Zygmund type

$$f(x, u) \sim |u|^{q-2}u \log(e + |u|),$$

which combine superlinear behavior with logarithmic modulation. Aberqi et al. [1] studied double-phase obstacle problems with logarithmic convection, while Shen and Squassina [22] obtained existence and multiplicity results for Kirchhoff and nonlocal elliptic problems with logarithmic nonlinearities using critical point theory. Significant studies in this field include those by Aberqi et al. [1, 2, 5, 11]. These works highlight the crucial role of modular coercivity, monotonicity, and compactness recovery in borderline regimes.

Motivated by these works, we study a nonlocal double-phase Kirchhoff problem with logarithmic-type sources in Musielak–Orlicz–Zygmund spaces. Our main goal is to prove the strong convergence of Palais–Smale sequences for the associated energy functional.

This property ensures the existence of weak solutions and clarifies the interaction between the Kirchhoff term, the double-phase operator, and the logarithmic source. Our approach relies on combining modular coercivity with strict monotonicity, providing new insights into the borderline behavior of such equations.

The plan of the paper is as follows: In Section 2, we present the problem setting and assumptions. In Section 3, we recall some well-known preliminary properties and results on Orlicz-Zygmund Spaces. In Section 4, we introduce the Kirchhoff framework and the critical Point theory. Finally, in Section 5, we state and prove the main result of the paper (Theorem 5.1).

2. Problem setting and assumptions

We study a class of borderline double-phase problems with nonlocal effects and logarithmic convection in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. Our aim is to establish the existence of weak solutions under general assumptions, including

unbalanced growth and gradient-dependent nonlinearities. The model problem is to find $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$(\mathcal{P}) \begin{cases} -K\left(\int_{\Omega} \Phi(x, |\nabla u|) dx\right) \operatorname{div} a(x, \nabla u) = g(x)|u|^{q-2}u + f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

The ingredients of (2.1) are specified as follows.

- The generalized N -function is

$$\Phi(x, t) = \frac{1}{p} t^p (1 + \log(e + t)), \quad p > 1,$$

leading to the Orlicz–Zygmund space $W_0^1 L_{\Phi}(\Omega) \equiv W_0^{1,p} \log L(\Omega)$.

- The Leray-Lions type operator $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is monotone, coercive, and continuous in the gradient variable. A typical model is

$$a(x, \nabla u) = |\nabla u|^{p-2} \left(1 + \log(e + |\nabla u|) + \frac{|\nabla u|}{p(e + |\nabla u|)} \right) \nabla u.$$

- $K : [0, \infty) \rightarrow (0, \infty)$ is a Kirchhoff function (see Definition 4.1).
- There exists $\theta \in (1, \frac{p^*}{q}]$ with $p^* = \frac{Np}{N-p}$ and $q < p^*$. such that

$$tK(t) \leq \theta \widehat{K}(t) \text{ for any } t \in [0, \infty), \tag{2.2}$$

where

$$\widehat{K}(t) := \int_0^t K(\tau) d\tau, \quad t \geq 0.$$

- The source term has the form

$$h(x, u) = g(x)|u|^{q-2}u + f(x, u),$$

with $g \in L^\infty(\Omega)$, $\operatorname{ess\,inf}_{x \in \Omega} g(x) = g_0 > 0$ and f is a Carathéodory function.

Writing $F(x, u) = \int_0^u f(x, t) dt$, we assume

$$f(x, u) \leq |u|^{q-2} \log(e + |u|), \tag{2.3}$$

and

$$0 < \lambda F(x, u) \leq u f(x, u) \text{ for } |u| \text{ large}, \tag{2.4}$$

where the parameter λ satisfies $(p + 1)\theta < \lambda < q$.

Let $X = \mathbb{W}_0$ be the Orlicz-Zygmund space (see below).

Definition 2.1. A function $u \in X$ is called a weak solution if it satisfies

$$\int_{\Omega} K\left(\int_{\Omega} \Phi(x, |\nabla u|) dx\right) a(x, \nabla u) \cdot \nabla \varphi dx = \int_{\Omega} g(x)|u|^{q-2}u \varphi dx + \int_{\Omega} f(x, u) \varphi dx.$$

Under these assumptions, the associated energy functional exhibits the Mountain–Pass geometry and satisfies the Palais–Smale condition below a suitable critical level. Therefore, existence of nontrivial weak solutions to (2.1) can be established. Our framework extends the results of Tran–Nguyen [25] and El Ouardani et al. [12] by incorporating logarithmic double-phase growth within Orlicz–Zygmund spaces.

3. Orlicz-Zygmund spaces

3.1. Classical Orlicz spaces

To handle nonlinear PDEs with borderline or critical growth, one needs a refined functional framework beyond classical Sobolev spaces. Orlicz and Orlicz–Zygmund spaces naturally provide the tools to address coercivity, compactness, and critical embeddings, as shown in classical works [8, 14, 7, 15, 18, 20, 21, 23] and in recent studies [25, 12]. We now recall the basic properties of these spaces that will serve as the foundation for our analysis.

Definition 3.1. *A function $\mathcal{A} : [0, \infty) \rightarrow [0, \infty)$ is an N -function if it is convex, continuous, strictly increasing, $\mathcal{A}(0) = 0$, $\mathcal{A}(t) > 0$ for all $t > 0$, and*

$$\lim_{t \rightarrow 0} \frac{\mathcal{A}(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{\mathcal{A}(t)}{t} = \infty.$$

Definition 3.2. *The Orlicz space associated with \mathcal{A} is*

$$L^{\mathcal{A}}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \exists \lambda > 0, \varrho_{\mathcal{A}, \Omega} \left(\frac{u}{\lambda} \right) < \infty \right\},$$

where the modular functional is

$$\varrho_{\mathcal{A}, \Omega}(u) = \int_{\Omega} \mathcal{A}(|u(x)|) dx,$$

and the Luxemburg norm is defined by

$$\|u\|_{\mathcal{A}} := \inf \left\{ \lambda > 0 : \int_{\Omega} \mathcal{A} \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

When \mathcal{A} and its conjugate satisfy the Δ_2 -condition, the space $L^{\mathcal{A}}(\Omega)$ is reflexive [24, 15, 4].

3.2. Orlicz-Zygmund spaces

For problems with logarithmic perturbations of polynomial growth, Orlicz-Zygmund spaces provide a natural setting. Let $p \in (1, \infty)$ and $s > 0$ and define

$$\mathcal{H}_s^p(t) := t^p \log^s(e + t), \quad t \geq 0.$$

Definition 3.3. *The Orlicz-Zygmund space associated with \mathcal{H}^p is*

$$L^{\mathcal{H}_s^p}(\Omega) := \left\{ v \in L^1(\Omega) : \int_{\Omega} |v|^p \log^s(e + |v|) dx < \infty \right\},$$

equipped with the Luxemburg norm

$$\|v\|_{L^{\mathcal{H}_s^p}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|v|^p}{\lambda^p} \log^s \left(e + \frac{|v|}{\lambda} \right) dx < 1 \right\}.$$

Remark 3.4. If $\mathcal{H}_s^p(t) = t^p$, we recover the classical $L^p(\Omega)$ space. For convenience, the modular function is defined as

$$[v]_{L^{\mathcal{H}_s^p}(\Omega)} := \left(\int_{\Omega} |v|^p \log^s(e + |v|/||v||_p) dx \right)^{1/p}.$$

Lemma 3.5 ([12]). *For $a \geq 0, b, p, s \geq 1$, the following inequalities hold:*

$$\log^s(e + ab) \leq 2^{s-1}(\log^s(e + a) + \log^s b), \tag{3.1}$$

$$\log^s b \leq (s/(ep))^s b^p. \tag{3.2}$$

Proposition 3.6 ([12]). *For all $p \in [1, \infty), s > 0$, and $q \in (p, \infty)$, the embeddings*

$$L^q(\Omega) \subset L^{\mathcal{H}_s^p}(\Omega) \subset L^p(\Omega),$$

hold. Moreover, the modular and norm satisfy

$$\|v\|_{L^{\mathcal{H}_s^p}(\Omega)} \leq [v]_{L^{\mathcal{H}_s^p}(\Omega)} \leq [2^s + (2s/(ep))^s] \|v\|_{L^{\mathcal{H}_s^p}(\Omega)}.$$

3.3. Sobolev-Orlicz-Zygmund Spaces and Compact Embeddings

The Sobolev-Orlicz-Zygmund space is defined as

$$W^{1, \mathcal{H}_s^p}(\Omega) := \left\{ u \in L^{\mathcal{H}_s^p}(\Omega) : D^\beta u \in L^{\mathcal{H}_s^p}(\Omega), |\beta| \leq 1 \right\},$$

with norm

$$\|v\|_{W^{1, \mathcal{H}_s^p}(\Omega)} := \|v\|_{L^{\mathcal{H}_s^p}(\Omega)} + \|\nabla v\|_{L^{\mathcal{H}_s^p}(\Omega)}.$$

Denote $\mathbb{W} := W^{1, \mathcal{H}_s^p}(\Omega)$, $\mathbb{W}_0 := W_0^{1, \mathcal{H}_s^p}(\Omega)$, with $\|v\|_{\mathbb{W}_0} = \|\nabla v\|_{L^{\mathcal{H}_s^p}(\Omega)}$.

Lemma 3.7. *For all $q \in (1, p^*), p^* = \frac{np}{n-p}$, the embedding $\mathbb{W}_0 \hookrightarrow L^q(\Omega)$ is compact.*

Proof. For every Young function A and its complementary function \bar{A} satisfying the Δ_2 -condition, we introduce the following Young function

$$\bar{A}_n(t) = \int_0^t y^{\frac{n+1}{n}} \left[\bar{A}^{-1}(y^n) \right]^{-1} dy$$

where \bar{A}_n^{-1} is the right-continuous generalized inverse function of \bar{A}_n , defined by the following formula:

$$\bar{A}_n(t) = \int_0^t \frac{\bar{A}(y)}{y^{\frac{n+1}{n}}} dy$$

and

$$\bar{A}_n^{-1}(t) = \inf \{ y \geq 0 : \bar{A}_n(y) \geq t \} \quad \text{for } t \geq 0.$$

By Theorem 3 in [8], the Sobolev-Orlicz space

$$W_0^{1, \mathcal{A}}(\Omega),$$

embeds continuously into the corresponding Orlicz-Zygmund space

$$L^{\widehat{\mathcal{A}}_n}(\Omega).$$

For any N-function B satisfying the Δ_2 -condition and $B \ll \widehat{\mathcal{A}}_n$, this embedding is compact:

$$W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow L^B(\Omega).$$

Applying this to the functions

$$\mathcal{A}(t) = t^p + t^p \log(e + t) \quad \text{and} \quad B(t) = t^q \log(e + t), \quad q \in (1, p^*),$$

verifies that

$$W_0^{1,G}(\Omega) \hookrightarrow L^{\mathcal{H}_{p^*/p}^{p^*}}(\Omega),$$

where $G(t) = t^p$. The result follows directly from the general theory. □

In Orlicz-Zygmund spaces, for functions with a large norm ($\|u\|_{W_0} > 1$), there exists a constant $C > 0$ such that

$$\Psi(u) = \int_{\Omega} \frac{1}{p} |\nabla u|^p (1 + \log(e + |\nabla u_n|)) \, dx \geq C \|u\|_{W_0}^p, \tag{3.3}$$

holds.

This inequality follows from the definition of the Luxemburg norm (see [21] for the general modular-norm relationship in Musielak-Orlicz settings).

4. Kirchhoff framework and critical point theory

4.1. Kirchhoff function and operator

Kirchhoff-type models generalize classical diffusion equations by introducing a nonlocal dependence on the gradient, encoded in a Kirchhoff function.

Definition 4.1. *We assume that the Kirchhoff function $K : [0, \infty) \rightarrow (0, \infty)$ satisfies:*

- K is continuous, strictly positive, and non-decreasing;
- there exist constants $k_0, k_1 > 0$ and $\alpha \geq 0$ such that

$$k_0 \leq K(s) \leq k_1(1 + s^\alpha), \quad \forall s \geq 0. \tag{4.1}$$

A canonical example is $K(s) = k_0 + k_1 s^\alpha$, which recovers the classical Kirchhoff-type operator

$$-K\left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u.$$

In the Musielak-Orlicz setting, the associated Kirchhoff operator is defined for $u \in W_0^1 L_{\Phi}(\Omega)$ by

$$A(u) := -K\left(\int_{\Omega} \Phi(x, |\nabla u|) \, dx\right) \operatorname{div}(a(x, \nabla u)),$$

where $a(x, \xi)$ is of Leray-Lions type.

Estimates for the primitive \widehat{K}

Proposition 4.2. *Under Definition 4.1, for all $s \geq 0$:*

$$k_0 s \leq \widehat{K}(s) \leq k_1 \left(s + \frac{s^{\alpha+1}}{\alpha+1} \right), \tag{4.2}$$

$$\frac{s}{2} K\left(\frac{s}{2}\right) \leq \widehat{K}(s) \leq s K(s). \tag{4.3}$$

Corollary 4.3. *We have*

$$\lim_{s \rightarrow \infty} \widehat{K}(s) = +\infty,$$

and the polynomial growth estimate

$$k_0 s \leq \widehat{K}(s) \leq C(1 + s^{\alpha+1}), \quad \forall s \geq 0,$$

for some constant $C > 0$ depending only on k_1 and α .

Based on Definition 4.1, the following properties of the Kirchhoff function K and its primitive \widehat{K} follow immediately.

Scaling inequalities for $K(t\theta)$

Proposition 4.4. *For all $t, \theta \geq 0$, there exist constants $C_1, C_2 > 0$ depending only on k_0, k_1, α such that:*

$$k_0 \leq K(t\theta) \leq k_1(1 + (t\theta)^\alpha), \tag{4.4}$$

$$K(t\theta) \leq C_1(1 + t^\alpha)K(\theta), \tag{4.5}$$

$$K(t\theta) \geq K(\theta), \quad t \geq 1. \tag{4.6}$$

Scaling inequalities for $\widehat{K}(t\theta)$

Proposition 4.5. *Under Definition 4.1, for all $t, \theta \geq 0$ there exists a constant $C > 0$ depending only on k_0, k_1, α such that*

$$t \widehat{K}(\theta) \leq \widehat{K}(t\theta) \leq C t(1 + t^\alpha) \widehat{K}(\theta). \tag{4.7}$$

Proof. Set $\tau = t\sigma$ so that $d\tau = t d\sigma$. Then

$$\widehat{K}(t\theta) = \int_0^{t\theta} K(\tau) d\tau = t \int_0^\theta K(t\sigma) d\sigma.$$

Since K is non-decreasing and $K(\sigma) \geq 0$,

$$K(t\sigma) \geq K(\sigma) \implies \widehat{K}(t\theta) \geq t \widehat{K}(\theta).$$

Using Proposition 4.4, $K(t\sigma) \leq C(1 + t^\alpha)K(\sigma)$, hence

$$\widehat{K}(t\theta) = t \int_0^\theta K(t\sigma) d\sigma \leq C t(1 + t^\alpha) \widehat{K}(\theta).$$

□

4.2. Critical point theory

The Palais–Smale theorem is a foundational result in critical point theory. Establishing the existence of critical points of a functional is a key step in variational methods for PDEs, and the Palais–Smale condition ensures compactness of sequences that might otherwise diverge.

Definition 4.6. (see [16])

- The derivative $DJ(u)$ at u is the unique continuous linear operator from X to \mathbb{R} such that $J(u + h) - J(u) - DJ(u)h = o(\|h\|)$ as $\|h\| \rightarrow 0$.
- u is a critical point if $DJ(u) = 0$; otherwise it is a regular point.
- If u is a critical point, $c = J(u)$ is called a critical value.

Definition 4.7. A C^1 -functional $J : X \rightarrow \mathbb{R}$ satisfies the Palais–Smale condition (PS) if every sequence (u_n) in X with

$$J(u_n) \text{ bounded, } J'(u_n) \rightarrow 0$$

has a convergent subsequence. Any such sequence is called a Palais–Smale sequence.

Lemma 4.8 ([16]). Let $J \in C^1(X, \mathbb{R})$ satisfy the PS condition, and let c be a regular value. Then there exists a flow $\eta \in C(\mathbb{R} \times X, X)$ satisfying properties (1)–(6) of [16].

4.3. Mountain pass theorem

Let

$$\Gamma = \{\varphi \in C^1([0, 1], X) : \varphi(0) = 0, \varphi(1) = u_0\}, \quad c = \inf_{\varphi \in \Gamma} \max_{t \in [0, 1]} J(\varphi(t)).$$

Theorem 4.9 (Mountain pass theorem, [16]). Let $J \in C^1(X, \mathbb{R})$ satisfy the PS condition. Assume:

1. $J(0) = 0$.
2. There exist $R, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\|_X = R$.
3. There exists u with $\|u\|_X > R$ such that $J(u) \leq 0$.

Then c is a critical value of J , and $c \geq \delta$.

5. Main result

Equipped with the functional framework and preliminary results presented above, we now prove the existence of weak solutions to problem (2.1) by employing the variational approach based on the Palais–Smale condition.

Theorem 5.1. Under the above assumptions, there exists at least one nontrivial weak solution $u \in W_0^1 L_\Phi(\Omega)$ of the Kirchhoff problem.

Proof. The proof of Theorem 5.1 is divided into several steps.

Solutions of problem (2.1) correspond to the critical points of the Euler–Lagrange energy functional

$$\mathcal{J}(u) = \widehat{K} \left(\int_{\Omega} \Phi(x, |\nabla u|) dx \right) - \int_{\Omega} \frac{g(x)}{q} |u|^q dx - \int_{\Omega} F(x, u) dx.$$

Step 1: Verification of the Palais–Smale condition

We do not provide full proofs of compactness, coercivity, or weak lower semicontinuity. We only verify that the essential structural properties are satisfied, which allows the use of the Palais-Smale framework in this restricted setting. The key points are that the functional J is of class C^1 on X and that the nonnegativity of Φ gives $J(0) = 0$. Moreover, according to (2.2) and for $t \in [0, 1]$, we have, $\widehat{K}(t) \geq \widehat{K}(1)t^\theta$, we obtain

$$\begin{aligned}
 J(u) &= \widehat{K}(\Phi(x, |\nabla u|)) - \int_{\Omega} \frac{g(x)}{q} |u|^q dx - \int_{\Omega} F(x, u) dx \\
 &\geq \widehat{K}(1) (\Phi(x, |\nabla u|))^\theta - \frac{1}{q} \|g\|_{\infty} \|u\|_q^q \\
 &\geq \frac{\widehat{K}(1)}{p^\theta} \|u\|_{\mathbb{W}_0}^{p\theta} - \frac{1}{q} \|g\|_{\infty} \|u\|_q^q \\
 &\geq \frac{\widehat{K}(1)}{p^\theta} \|u\|_{\mathbb{W}_0}^{p\theta} - \frac{1}{q} \|g\|_{\infty} [u]_{L^{\mathcal{H}_g^q}(\Omega)}^q \\
 &\geq \frac{\widehat{K}(1)}{p^\theta} \|u\|_{\mathbb{W}_0}^{p\theta} - \frac{1}{q} \|g\|_{\infty} C \|u\|_{\mathbb{W}_0}^q \quad (\text{from injection Lemma 3.7}) \\
 &\geq \left(\frac{\widehat{K}(1)}{p^\theta} - \left[\frac{1}{q} \|g\|_{\infty} C \right] \|u\|_{\mathbb{W}_0}^{q-p\theta} \right) \|u\|_{\mathbb{W}_0}^{p\theta}.
 \end{aligned} \tag{5.1}$$

Therefore, for any $u \in \mathbb{W}_0$ with $\|u\|_{\mathbb{W}_0} = R \in \left(0, \min\left\{1, \left[\frac{\widehat{K}(1)}{p^\theta \left(\frac{1}{q} \|g\|_{\infty} C\right)}\right]^{\frac{1}{q-p\theta}}\right\}\right)$.

We conclude that $J(u) \geq \delta$ where $\delta = \left(\frac{\widehat{K}(1)}{p^\theta} - \left[\frac{1}{q} \|g\|_{\infty} C\right] R^{q-p\theta}\right) R^{p\theta}$.

We choose, for $u_0 \in \mathbb{W}_0$, $\|u_0\|_{\mathbb{W}_0} > 1$ and t sufficiently large, we obtain:

$$\begin{aligned}
 J(tu_0) &= \widehat{K}(\Phi(x, |\nabla tu_0|)) - \int_{\Omega} \frac{g(x)}{q} |tu_0|^q dx - \int_{\Omega} F(x, u_0) dx \\
 &\leq C \widehat{K}(1) t^{p\theta} \|u_0\|_{\mathbb{W}_0}^{p\theta} - \frac{g_0}{q} t^q \|u_0\|_q^q. \\
 &\leq t^{p\theta} \left(C \widehat{K}(1) \|u_0\|_{\mathbb{W}_0}^{p\theta} - \frac{g_0}{q} t^{q-p\theta} \|u_0\|_q^q \right).
 \end{aligned} \tag{5.2}$$

Since $1 \leq p\theta < q$, it is easy to see that $J(tu_0) \rightarrow -\infty$ when $t \rightarrow +\infty$.

Hence, we find t sufficiently large and $\epsilon > 0$ such that

$$J(tu_0) \leq -\epsilon < 0.$$

Step 2: Coercivity and monotonicity of the operator

Before establishing compactness results for Palais–Smale sequences, it is essential to analyze the structural properties of the underlying nonlinear operator,

$$a(x, \xi) = g(|\xi|) \xi, \quad g(r) = r^{p-2} \left(1 + \log(e+r) + \frac{r}{p(e+r)} \right),$$

belongs to the class of radial, Leray–Lions type operators with logarithmic perturbations, a setting naturally encountered in borderline double-phase problems. The following proposition establishes precise coercivity and monotonicity inequalities.

Proposition 5.2. *There exist constants $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$ and $C_3 = 1$, such that for all $\xi, \eta \in \mathbb{R}^N$,*

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq C_1 \psi(|\xi - \eta|), \quad a(x, \xi) \cdot \xi \geq C_2 \phi(|\xi|) - C_3,$$

with

$$\psi(r) = r^p(1 + \log(e + r)) + r^p \log(e + r), \quad \phi(r) = r^p(1 + \log(e + r)).$$

Proof. Define

$$h(r) := g(r)r = r^{p-1} \left(1 + \log(e + r) + \frac{r}{p(e + r)} \right), \quad r \geq 0.$$

Then

$$h'(r) = (p - 1)r^{p-2} \left(1 + \log(e + r) + \frac{r}{p(e + r)} \right) + r^{p-1} \left(\frac{1}{e + r} + \frac{e}{p(e + r)^2} \right),$$

hence

$$h'(r) \geq \frac{1}{2}r^{p-2}(1 + \log(e + r)), \quad \forall r \geq 0.$$

Scalar monotonicity. For $\xi, \eta \in \mathbb{R}$, by the mean value theorem,

$$(g(|\xi|)\xi - g(|\eta|)\eta)(\xi - \eta) = (h(\xi) - h(\eta))(\xi - \eta) = h'(\theta)(\xi - \eta)^2,$$

with θ between $|\xi|$ and $|\eta|$.

Hence

$$(g(|\xi|)\xi - g(|\eta|)\eta)(\xi - \eta) \geq \frac{1}{2}|\xi - \eta|^p(1 + \log(e + |\xi - \eta|)).$$

Vectorial monotonicity. For $\xi, \eta \in \mathbb{R}^N$, the standard inequality for radial operators (see [19, 26]) implies

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq C_1 \psi(|\xi - \eta|).$$

Coercivity.

$$a(x, \xi) \cdot \xi = g(|\xi|)|\xi|^2 = h(|\xi|) \geq \frac{1}{2}\phi(|\xi|) - 1, \tag{5.3}$$

so that coercivity holds with $C_2 = \frac{1}{2}$ and $C_3 = 1$. □

Remark 5.3. The constants $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$, and $C_3 = 1$ are uniform, ensuring coercivity and strict monotonicity independently of ξ, η . For more details, see Lindqvist [19] and Ziemer [26].

The coercivity and monotonicity properties established above provide the structural backbone of our operator. They ensure that the variational setting is well-posed and allow us to invoke Minty’s lemma, a key tool to upgrade weak convergence into strong convergence.

Lemma 5.4 (Minty Lemma). *Let $A : \mathbb{W}_0 \rightarrow \mathbb{W}_0^*$ be strictly monotone and weakly continuous. If $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ strongly in \mathbb{W}_0 and $A(u_n) \rightarrow A(u)$ in \mathbb{W}_0^* .*

For a proof, see Minty [20] or Brézis [7].

5.1. Step 3: Strong convergence of Palais–Smale sequences

Proposition 5.5. *Let $\{u_n\}$ be a bounded Palais–Smale sequence for J . Then, up to a subsequence,*

$$u_n \rightarrow u \quad \text{strongly in } \mathbb{W}_0.$$

Proof. Assume, by contradiction, that $\|u_n\|_{\mathbb{W}_0} \rightarrow \infty$.

Let $\Psi(u_n) = \int_{\Omega} \Phi(x, |\nabla u_n|) dx$, then

$$\begin{aligned} J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle &= \left[\widehat{K}(\Psi(u_n)) - \frac{1}{\lambda} K(\Psi(u_n)) \int_{\Omega} a(x, \nabla u_n) \cdot \nabla u_n dx \right] \\ &\quad + \left[\frac{1}{\lambda} \int_{\Omega} g(x) |u_n|^q dx - \int_{\Omega} \frac{g(x)}{q} |u_n|^q dx \right] \\ &\quad + \left[\frac{1}{\lambda} \int_{\Omega} f(x, u_n) u_n dx - \int_{\Omega} F(x, u_n) dx \right]. \end{aligned}$$

□

Using $a(x, \nabla u_n) \cdot \nabla u_n \leq (p+1)\Phi(x, |\nabla u_n|)$ since $\frac{|\nabla u_n|}{e+|\nabla u_n|} \leq 1$ and (2.4) we have

$$\begin{aligned} J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle &\geq \widehat{K}(\Psi(u_n)) - \frac{p+1}{\lambda} K(\Psi(u_n)) \Psi(u_n) + \left(\frac{q-\lambda}{\lambda q} \right) \int_{\Omega} g(x) |u_n|^q dx \\ &\geq \left(\frac{1}{\theta} - \frac{p+1}{\lambda} \right) K(\Psi(u_n)) \Psi(u_n) + \left(\frac{q-\lambda}{\lambda q} \right) \int_{\Omega} g(x) |u_n|^q dx. \end{aligned}$$

Since $(p+1)\theta < \lambda < q$ and $g \geq 0$, we have

$$J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle \geq \left(\frac{1}{\theta} - \frac{p+1}{\lambda} \right) K(\Psi(u_n)) \Psi(u_n),$$

and by (4.1) and (3.3), we deduce

$$J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle \geq \left(\frac{1}{\theta} - \frac{p+1}{\lambda} \right) k_0 C \|u_n\|_{\mathbb{W}_0}^p.$$

The right-hand side diverges to $+\infty$, contradicting boundedness of $J(u_n)$. Thus, $\{u_n\}$ is bounded in \mathbb{W}_0 . On the other-hand, by reflexivity, there exists $u \in \mathbb{W}_0$ such that (up to subsequence)

$$u_n \rightharpoonup u \quad \text{in } \mathbb{W}_0, \quad u_n \rightarrow u \quad \text{a.e. in } \Omega.$$

Now, Take $\varphi_n = u_n - u$ in $J'(u_n)$, then

$$\langle J'(u_n), \varphi_n \rangle = K(\Psi(u_n)) \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) dx - T_g - T_f,$$

where

$$T_g = \int_{\Omega} g(x) |u_n|^{q-2} u_n (u_n - u) dx, \quad T_f = \int_{\Omega} f(x, u_n) (u_n - u) dx.$$

By compactness $\mathbb{W}_0 \hookrightarrow L^q$, we have $T_g \rightarrow 0$. By dominated convergence (subcritical growth of f), $T_f \rightarrow 0$. Hence,

$$K(\Psi(u_n)) \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0.$$

That means that $\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0$. Since u_n is bounded and K is continuous and $K \geq k_0 > 0$. Thus,

$$I_n = \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0. \quad (5.4)$$

Using strict monotonicity (Proposition 5.2), we deduce

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \geq C \int_{\Omega} \Phi(x, |\nabla u_n - \nabla u|) \, dx,$$

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx = I_n - \int_{\Omega} a(x, \nabla u) \cdot (\nabla u_n - \nabla u) \, dx.$$

Since $u_n \rightharpoonup u$ in \mathbb{W}_0 , the term $\int_{\Omega} a(x, \nabla u) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0$ as $n \rightarrow \infty$ by the definition of weak convergence.

Combined with (5.4), we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx = 0.$$


Hence, the Palais–Smale sequence $\{u_n\}$ is bounded and converges strongly to a critical point u of J . This establishes the existence of a weak solution in the Musielak–Orlicz–Zygmund framework. □

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