Stud. Univ. Babeş-Bolyai Math. 71(2026), No. 2, 271–291
DOI: 10.24193/subbmath.2026.2.08

Existence, attractivity and controllability results for integro-differential equations with state-dependent delay

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Abstract. The objective of our research is to investigate the existence, attractivity and controllability of solutions for integro-differential equations with state-dependent delays. We employ a fixed point theorem to establish the existence of these solutions, while also utilizing the concept of measures of noncompactness. In the last section, we give an example to show that the assumed conditions can be verified and to illustrate our results.

Mathematics Subject Classification (2010): 34G20, 34K20, 34K30.

Keywords: integro-differential equation, mild solution, infinite delay, state-dependent delay, fixed point, resolvent operator, measure of noncompactness.

1. Introduction

Many branches of sciences such as physics, fluid dynamic, biology and chemistry are described and characterized by integro-differential equations. The study of integro-differential systems has captured the interest of many researchers. Grimmer was an early pioneer in this field, making significant contributions to the understanding of these complex systems through the use of resolvent operators. His work established the existence of integro-differential systems and provided critical insights into their behavior and dynamics, as documented in several of his papers [20, 21]. Resolvent operators are mathematical tools that help analyze the properties of integro-differential systems by expressing their solutions in terms of initial conditions and input functions. These operators are particularly useful for examining the stability and controllability of integro-differential systems. The existence, uniqueness, stability and controllability

Received 02 July 2025; Accepted 23 April 2026.

for various integro-differential problem with finite, infinite and state-dependent delay have been studied by many researchers, see [4, 12, 15, 16, 18].

In 1930, Kuratowski [25] introduced the notion of a measure of noncompactness, which has been proven invaluable in functional analysis, particularly in metric fixed point theory and the theory of operator equations in Banach spaces. Moreover, this concept plays a crucial role in exploring the existence of solutions across various types of equations, including ordinary and partial differential equations, as well as integral and integro-differential equations. More detailed information on this approach can be found in the works of Akhmerov et al. [2], Benchohra et al. [9, 10, 13, 14, 15], Banaś [4, 5, 6, 7].

Controllability theory is essential for understanding the behavior and dynamics of abstract control systems. Its primary goal is to identify a suitable control function that can guide the system's state to a desired final state. Exact controllability refers to the ability to steer the system precisely to the target state, while approximate controllability involves guiding the system to a neighborhood arbitrarily close to the final state. Due to the inherent uncertainty or imprecision in real-world systems, approximate controllability is often more practical and desirable. Over the years, numerous researchers have examined the approximate or exact controllability of control systems, resulting in a several published studies [8, 11, 19, 26, 27, 29, 30].

This study is dedicated to demonstrating the existence, attractivity and controllability of the solutions for an integro-differential equation. Our analysis will be conducted within the Banach space of real functions that are defined, continuous, and bounded over the real axis $\mathbb{R}$. Specifically, we will address the following problem:

$$
\begin{aligned}
&z'(\delta) = Az(\delta) + \int_0^\delta B(\delta - s)z(s)ds + \Psi\big(\delta, z_{\rho(\delta,z_\delta)},\\
&\int_0^\delta \zeta(\delta, s, z_{\rho(s,z_s)})\, ds\big), \delta \in \Re := [0, +\infty),
\end{aligned} \tag{1.1}
$$

$$
z(\delta) = \varpi(\delta), \ \delta \in (-\infty, 0],
$$

where $(\Im, |\cdot|)$ is a real Banach space, $\Psi : \Re\times\mathfrak{Q}\times\Im \to \Im$ is given function, $\zeta : \Gamma\times\mathfrak{Q} \to \Im$ is a continuous function with $\Gamma = \{(\delta, s) \in \Re \times \Re, \delta \leq s\}$, A is the infinitesimal generator of a C_0-semigroup $(\Upsilon(\delta))_{\delta\geq 0}$ on $\Im$, and $B(\delta)$ is linear closed operator on $\Im$ with $D(A) \subset D(B)$, $\varpi \in \mathfrak{Q}$. $\rho : \Re \times \mathfrak{Q} \to \mathbb{R}$, $\mathfrak{Q}$ is the phase space which will be defined later. For any function z defined on $\mathbb{R}$ and any $\delta \in \Re$, we denote by z_δ the element of $\mathfrak{Q}$ defined by

$$
z_\delta(\chi) = z(\delta + \chi), \chi \in (-\infty, 0].
$$

Next, we study the controllability of the following problem:

$$
z'(\delta) = Az(\delta) + \int_0^\delta B(\delta - s)z(s)ds + \Psi\big(\delta, z_{\rho(\delta,z_\delta)},
$$

$$\int_0^\delta \zeta(\delta, s, z_{\rho(s,z_s)})\, ds\Big) + \mathcal{C}u(\delta), \text{a.e. } \delta \in \Re, \tag{1.2}$$

$$z(\delta) = \varpi(\delta), \;\; \delta \in (-\infty, 0],$$

where $u \in L^2(\Re, H)$ is the control function, H is the Banach space of admissible control functions and $\mathcal{C}$ is bounded linear operator from H to $\Im$.

The following is the structure of our paper. In Section 2, we begin by presenting some background and preliminary information. Section 3 delves into the study of the existence of mild solutions for the system (1.1). Building on this basis, we move to Section 4, where we establish adequate requirements for the system's attractivity. In Section 5, we delve to the study of the controllability of (1.2). Finally, we offer an example as a sample application to highlight the practical relevance of our findings.

2. Preliminaries

By $\Bbbk$ we denote the Banach space of all bounded and continuous functions from $\mathbb{R}$ into $\Im$ with

$$\|z\|_{\Bbbk} = \sup_{\delta \in \mathbb{R}} |z(\delta)|.$$

By $\widehat{\Bbbk}$ we denote the Banach space of all bounded and continuous functions from $\Re$ into $\Im$ with

$$\|z\|_{\widehat{\Bbbk}} = \sup_{\delta \in \Re} |z(\delta)|.$$

By $\widehat{B}$ we denote the Banach space $D(A)$ with

$$\|z\|_{\widehat{B}} = \|Az\| + \|z\|, z \in \widehat{B}$$

Now let us recall some information about partial integro-differential equations and resolvent operator.
We consider the following problem :

$$z'(\delta) = Az(\delta) + \int_0^\delta B(\delta - s)z(s)ds, \text{for } \delta \geq 0, \tag{2.1}$$

$$z(0) = z_0 \in \Im.$$

Definition 2.1. [20] *A resolvent operator for problem (2.1) is a bounded linear operator valued function $R(\delta) \in \mathcal{L}(\Im)$ for $\delta \geq 0$, satisfying the following properties:*

1. *$R(0) = I$(identity map of $\Im$) and $\|R(\delta)\| \leq Me^{\beta\delta}$, for some constants $M > 0$ and $\beta \in \mathbb{R}$.*
2. *For each $z \in \Im, R(\delta)z$ is strongly continuous for $\delta \geq 0$.*
3. *For any $z \in \Im, R(\cdot)z \in \mathcal{C}^1([0, +\infty), \Im) \cap \mathcal{C}([0, +\infty), \widehat{B})$ and*

$$R'(\delta)z = AR(\delta)z + \int_0^\delta B(\delta - s)R(s)zds = R(\delta)Az + \int_0^\delta R(\delta - s)B(s)zds \quad \text{for } \delta \geq 0.$$

The existence of resolvent operator has been discussed in [20, 21] under the following assumptions.

(R_1) A is the infinitesimal generator of strongly continuous semigroup $(\Upsilon(\delta))_{\delta\geq 0}$ on $\Im$.

(R_2) For all $\delta \geq 0$, $B(\delta)z$ is closed linear operator from $D(A) \to \Im$ and $B(\delta) \in \mathcal{L}(\widehat{B}, \Im)$. For any $z \in \widehat{B}$, the map $\delta \to B(\delta)$ is bounded, differentiable and the derivative $\delta \to B'(\delta)z$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 2.2. *If the condition $(R_1), (R_2)$ hold, then the problem (2.1) has a unique resolvent operator.*

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.3. [2, 5] *Let $\Im$ be a Banach space and $\Omega_\Im$ the bounded subsets of $\Im$. The Kuratowski measure of noncompactness is the map $\varkappa : \Omega_\Im \to [0, \infty)$ defined by*

$$\varkappa(\widehat{\mathfrak{Z}}) = \inf\{\epsilon > 0 : \widehat{\mathfrak{Z}} \subseteq \bigcup_{i=1}^{n} \widehat{\mathfrak{Z}}_i \textit{ and } diam(\widehat{\mathfrak{Z}}_i) \leq \epsilon\}; \textit{ here } \widehat{\mathfrak{Z}} \in \Omega_\Im,$$

and verifies:

- $\varkappa(\widehat{\mathfrak{Z}}) = 0 \Leftrightarrow \bar{\widehat{\mathfrak{Z}}}$ *is compact ($\widehat{\mathfrak{Z}}$ is relatively compact);*
- $\varkappa(\widehat{\mathfrak{Z}}) = \varkappa(\bar{\widehat{\mathfrak{Z}}})$*;*
- $\mathfrak{Z} \subset \widehat{\mathfrak{Z}} \Rightarrow \varkappa(\mathfrak{Z}) \leq \varkappa(\widehat{\mathfrak{Z}})$;
- $\varkappa(\mathfrak{Z} + \widehat{\mathfrak{Z}}) \leq \varkappa(\mathfrak{Z}) + \varkappa(\widehat{\mathfrak{Z}})$;
- $\varkappa(c\widehat{\mathfrak{Z}}) = |c|\varkappa(\widehat{\mathfrak{Z}}); c \in \mathbb{R}$;
- $\varkappa(conv\widehat{\mathfrak{Z}}) = \varkappa(\widehat{\mathfrak{Z}})$.

Here, $\bar{\widehat{\mathfrak{Z}}}$ and $conv\widehat{\mathfrak{Z}}$ denote the closure and convex hull of bounded set $\widehat{\mathfrak{Z}}$, respectively.

Lemma 2.4. *Let $\mathcal{D}$ be a bounded subset of $\Im$. Then, for each $\epsilon > 0$, there exists a sequence $\{U_n\}_{n=1}^{\infty} \subset \mathcal{D}$ such that*

$$\varkappa(\mathcal{D}) \leq 2\varkappa(\{U_n\}_{n=1}^{\infty}) + \epsilon.$$

Lemma 2.5. *If $\{U_n\}_{n=0}^{\infty} \subset L^1$ is uniformly integrable, then the function $\delta \to \varkappa(\{U_n(\delta)\}_{n=0}^{\infty})$ is measurable and*

$$\varkappa\left(\left\{\int_0^\delta U_n(s)ds\right\}\right) \leq 2\int_0^\delta \varkappa(\{U_n(s)\}_{n=0}^{\infty})ds.$$

For any arbitrary nonempty bounded subset D of the space $\widehat{\Bbbk}$ and a function h, we define

$$\eta^\infty(h, \epsilon) = \sup\{\|h(\delta) - h(s)\|; \delta, s \in \Re, |\delta - s| \leq \epsilon\},$$

and

$$\eta^\infty(D, \epsilon) = \sup\{\eta^\infty(h, \epsilon); h \in D\}, \eta_0^\infty(D) = \lim_{\epsilon\to 0} \eta^\infty(D, \epsilon).$$

Now we consider

$$\bar{\varkappa}_\infty(D) = \lim_{T\to\infty} \sup\{\varkappa(D(\delta)) : \delta \in [0, T]\},$$

$$diam D(\delta) = \sup\{\|x(\delta) - y(\delta)\|_\Im : x, y \in D\},$$

$$c(D) = \lim_{\delta \to \infty} diam D(\delta).$$

Finally, we can consider the following function $\varkappa^*$ define by:

$$\varkappa^*(D) = \eta_0^{\infty}(D) + \bar{\varkappa}_{\infty}(D) + c(D).$$

The function $\varkappa^*$ is a measure of noncompactness in the space $\widehat{\Bbbk}$, for more details see for instance [3].

Now we give the definition of a Meir-Keeler contraction.

Definition 2.6. [28] *Let (X, d) be a metric space. A mapping N on X is called a Meir-Keeler contraction if, for each $\epsilon > 0$, there exists $\lambda > 0$, such that*

$$\epsilon \leq d(x, y) < \epsilon + \lambda \Rightarrow d(Nx, Ny) < \epsilon, \text{ for all } x, y \in X.$$

Definition 2.7. [1] *Let Y be a nonempty subset of Banach space $\Im$, and μ be a measure of noncompactness on $\Im$. An operator $N : Y \to Y$ is called a Meir-Keeler condensing operator if, for each $\epsilon > 0$, there exists $\lambda > 0$, such that $\epsilon \leq \mu(\Omega) < \epsilon + \lambda \Rightarrow \mu(N\Omega) < \epsilon$, for any bounded set Ω of Y.*

Now we give the fixed point theorem with respect a Meir-Keeler condensing operator.

Theorem 2.8. [1] *Let $\mathcal{D}$ be a nonempty, bounded, closed, convex subset of a Banach space $\Im$. and μ be an arbitrary measure of noncompactness on $\Im$. If $N : \mathcal{D} \to \mathcal{D}$ is a continuous and Meir-Keeler condensing operator, then N has at least one fixed point and the set of all fixed points of N in $\mathcal{D}$ is compact.*

In this segment, we will utilize an axiomatic definition of the phase space $\mathfrak{Q}$ as introduced by Hale and Kato in [22]. We will adhere to the terminology employed in [24]. Consequently, $(\mathfrak{Q}, |\cdot|_{\mathfrak{Q}})$ will constitute a seminormed linear space of functions that map $(-\infty, 0]$ into $\Im$ and adhere to the specified axioms:

(A_1) If $\mathfrak{w} : (-\infty, \zeta) \to \Im, \zeta > 0$, is continuous on $\Re$ and $\mathfrak{w}_0 \in \mathfrak{Q}$, then for every $\delta \in \Re$:
(i) $\mathfrak{w}_\delta \in \mathfrak{Q}$;
(ii) $\exists\ \widehat{l_1} > 0$ such that $|\mathfrak{w}(\delta)| \leq \widehat{l_1} \|\mathfrak{w}_\delta\|_{\mathfrak{Q}}$;
(iii) $\exists\ \widehat{l_2}(\cdot), \widehat{l_3}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $\mathfrak{w}$ with $\widehat{l_2}$ continuous and bounded, and $\widehat{l_3}$ locally bounded where:

$$\|\mathfrak{w}_\delta\|_{\mathfrak{Q}} \leq \widehat{l_2}(\delta) \sup\{\ |\mathfrak{w}(\varrho)| : 0 \leq \varrho \leq \delta\} + \widehat{l_3}(\delta) \|\mathfrak{w}_0\|_{\mathfrak{Q}}.$$

(A_2) For the function $\mathfrak{w}$ in (A_1), $\mathfrak{w}_\delta$ is a $\mathfrak{Q}-$valued continuous function on $\Re$.
(A_3) The space $\mathfrak{Q}$ is complete.

Denote

$$\widehat{l_2} = \sup\{\widehat{l_2}(\delta) : \delta \in \Re\},$$

and

$$\widehat{l_3} = \sup\{\widehat{l_3}(\delta) : \delta \in \Re\}.$$

3. Existence of mild solutions

Definition 3.1. *We say that a function $z \in \Bbbk$ is a mild solution of problem (1.1) if $z(\delta) = \varpi(\delta), \delta \in (-\infty, 0]$ and*

$$z(\delta) = R(\delta)\varpi(0) + \int_0^\delta R(\delta - s)\Psi\big(s, z_{\rho(s,z_s)}, \int_0^s \zeta(s, \theta, z_{\rho(\theta, z_\theta)})\, d\theta\big)ds, \, a.e. \quad \delta \in \Re. \tag{3.1}$$

Set

$$\Lambda(\rho^-) = \{\rho(\varrho, \varpi) : (\varrho, \varpi) \in \Re \times \mathfrak{Q},\ \rho(\varrho, \varpi) \leq 0\}.$$

Let $\rho : \Re \times \mathfrak{Q} \to \mathbb{R}$ be continuous and:

(EC_ϖ) The function $\delta \to \varpi_\delta$ is continuous from $\Lambda(\rho^-)$ into $\mathfrak{Q}$ and there exists a continuous and bounded function $\mho^\varpi : \Lambda(\rho^-) \to (0, \infty)$ where

$$\|\varpi_\delta\| \leq \mho^\varpi(\delta)\|\varpi\| \quad \text{for every } \delta \in \Lambda(\rho^-).$$

Remark 3.2. The condition (EC_ϖ), is frequently verified by functions continuous and bounded. For more details, see for instance [24].

Lemma 3.3. [23] *If $z : \mathbb{R} \to \Im$ is a function such that $z_0 = \varpi$, then*

$$\|z_\varrho\|_\mathfrak{Q} \leq (\widehat{l_3} + \mho^\varpi)\|\varpi\|_\mathfrak{Q} + \widehat{l_2} \sup\{|z(\chi)|; \chi \in [0, max\{0, \varrho\}]\},\ \ \varrho \in \Lambda(\rho^-) \cup \Re,$$

where $\mho^\varpi = \sup_{\delta \in \Lambda(\rho^-)} \mho^\varpi(\delta)$.

Let us introduce the following hypotheses:

(EC_1) $A : D(A) \subset \Im \to \Im$ is the infinitesimal generator of a uniformly continuous C_0 semigroup $(\Upsilon(\delta))_{\delta \geq 0}$.

(EC_2) The conditions $(R_1), (R_2)$ hold, and there exist constants $M \geq 1, \alpha > 0$ such that

$$\sup\{\|R(\delta)\|_{B(\Im)} : \delta \geq 0\} \leq Me^{-\alpha\delta}.$$

(EC_3) The function $\Psi : \Re \times \mathfrak{Q} \times \Im \longrightarrow \Im$ is Carathéodory and satisfies the following conditions:

i) There exists a function $k_1 \in L^1(\Re)$, such that :

$$\|\Psi(\delta, x, y) - \Psi(\delta, x^*, y^*)\| \leq k_1(\delta)(\|x - x^*\| + \|y - y^*\|), \delta \in \Re, x, x^* \in \mathfrak{Q}, y, y^* \in \Im.$$

ii) For each $\delta \in \Re$, we have and

$$\lim_{\delta \to \infty} \sup_{\delta \in \Re} \int_0^\delta e^{-\alpha(\delta - s)} k_1(s)ds = 0.$$

iii) For each bounded set $\mathcal{D}_1 \subset \mathfrak{Q}$ and $\mathcal{D}_2 \subset \Im$, and each $\delta \in \Im$ we have

$$\varkappa(\Psi(\delta, \mathcal{D}_1, \mathcal{D}_2)) \leq k_1(\delta) \left[\sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_1(\varrho)) + \varkappa(\mathcal{D}_2) \right].$$

(EC_4) i) There exists a function $k_2 \in L^1(\Re)$ such that:

$$\|\zeta(\delta, s, z) - \zeta(\delta, s, z^*)\| \leq k_2(s)\|z - z^*\|,$$

for each $(\delta, s) \in \Gamma$ and for all $z, z^* \in \mathfrak{Q}$.

ii) For each bounded set $\mathcal{D} \subset \mathfrak{Q}$, we have

$$\varkappa(\zeta(\delta, s, \mathcal{D}) \leq k_2(\delta) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}(\varrho)).$$

Remark 3.4. We denote by

$$k_i^* = \|k_i\|_{L^1(\Re)}$$

for

$$i = 1, 2,\ \Psi^* = \sup_{\delta \in \Re} \int_0^\delta \Psi(s, 0, 0) ds,\ \zeta^* = \sup\{\|\zeta(\delta, s, 0)\|,\ (\delta, s) \in \Gamma\}.$$

Theorem 3.5. *Assume that* $(EC_1) - (EC_4), (EC_\varpi)$ *hold. If* $8M\widehat{l_2}k_1^*(1 + k_2^*) < 1$*, then the problem (1.1) has at least one mild solution on* $\mathbb{k}$*.*

Proof. We transform the problem (1.1) into a fixed point problem. Consider the operator $\Xi : \mathbb{k} \to \mathbb{k}$ defined by :

$$\Xi(z) := \begin{cases} \varpi(\delta), \quad \text{if } \delta \in (-\infty, 0], \\ \\ R(\delta)\varpi(0) + \displaystyle\int_0^\delta R(\delta - s)\Psi\big(s, z_{\rho(s, z_s)}, \int_0^s \zeta(s, \theta, z_{\rho(\theta, z_\theta)}) d\theta\big) ds, \quad \text{if } \delta \in \Re. \end{cases}$$

Let $\vartheta(\cdot) : \mathbb{R} \to \Im$ be the function defined by:

$$\vartheta(\delta) = \begin{cases} \varpi(\delta), & \text{if } \delta \in (-\infty, 0]; \\ \\ R(\delta)\ \varpi(0), & \text{if } \delta \in \Re. \end{cases}$$

Then $\vartheta_0 = \varpi$. For each $h \in \mathbb{k}$ with $h(0) = 0$, we denote by $\overline{h}$ the function

$$\overline{h}(\delta) = \begin{cases} 0; & \text{if } \delta \in (-\infty, 0]; \\ \\ h(\delta); & \text{if } \delta \in \Re. \end{cases}$$

If z satisfies (3.1), we can decompose it as $z(\delta) = h(\delta) + \vartheta(\delta),\ \delta \in \Re$, which implies $z_\delta = h_\delta + \vartheta_\delta$ for every $\delta \in \Re$ and the function $h(\cdot)$ satisfies

$$h(\delta) = \int_0^\delta R(\delta - s)\Psi\big(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)},$$
$$\int_0^s \zeta(s, \theta, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta\big) ds.$$

Set

$$\widehat{\mathbb{k}}_0 = \{h \in \widehat{\mathbb{k}} : h(0) = 0\},$$

and let

$$\|h\|_{\widehat{\mathbb{k}}_0} = \sup\{|h(\delta)| : \delta \in \Re\},\ h \in \widehat{\mathbb{k}}_0.$$

$\widehat{\mathbb{k}}_0$ is a Banach space with the norm $\|.\|_{\widehat{\mathbb{k}}_0}$.
We define the operator $\mathcal{A}:\widehat{\mathbb{k}}_0\rightarrow\widehat{\mathbb{k}}_0$ by:

$$\mathcal{A}(h)(\delta):=\begin{cases} 0, \quad \text{if } \delta\leq 0,\\ \displaystyle\int_0^\delta R(\delta-s)\Psi\big(s,h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)},\\ \displaystyle\int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds, \text{if } \delta\in\Re, \end{cases}$$

The operator $\mathcal{A}$ maps $\widehat{\mathbb{k}}_0$ into $\widehat{\mathbb{k}}_0$, for each $\delta\in\Re$ we have

$$\begin{aligned} &|\mathcal{A}(h)(\delta)|\\ &\leq \|\int_0^\delta R(\delta-s)[\Psi\big(s,h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)},\\ &\int_0^s\zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)}\big)d\theta\big)\\ &\quad-\Psi(s,0,0)+\Psi(s,0,0)]ds\|\\ &\leq M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\|h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)}\|ds\\ &\quad+M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\|\int_0^s\big(\zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)}\big)\\ &\quad-\zeta(s,\theta,0)+\zeta(s,\theta,0)\big)d\theta\|ds\\ &\quad+M\int_0^\delta e^{-\alpha(\delta-s)}\|\Psi(s,0,0)\|ds. \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{A}(h)(\delta)|\leq{}& M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\|h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)}\|ds\\ &+M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\int_0^s k_2(\theta)\|h_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)}\|d\theta ds\\ &+M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\int_0^s\|\zeta(s,\theta,0)\|d\theta ds\\ &+M\int_0^\delta e^{-\alpha(\delta-s)}\|\Psi(s,0,0)\|ds. \end{aligned}$$

Then,

$$\begin{aligned} |\mathcal{A}(h)(\delta)|\leq{}& M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)(\widehat{l_2}|h(s)|+(\widehat{l_3}+\mho^{\varpi}+\widehat{l_2}Me^{-\alpha s}\widehat{l_1})\|\varpi\|_{\mathfrak{Q}}))ds\\ &+M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\int_0^s k_2(\theta)(\widehat{l_2}|h(\theta)|\\ &\quad+(\widehat{l_3}+\mho^{\varpi}+\widehat{l_2}Me^{-\alpha\theta}\widehat{l_1})\|\varpi\|_{\mathfrak{Q}})d\theta ds \end{aligned}$$

$$+ M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s \|\zeta(s,\theta,0)\| d\theta ds$$
$$+ M \int_0^\delta e^{-\alpha(\delta-s)} \|\Psi(s,0,0)\| ds.$$

Let $C_1 = (\widehat{l_3} + \mho^{\varpi} + \widehat{l_2} M \widehat{l_1})\|\varpi\|_{\mathfrak{Q}}$. Then, we have

$$|\mathcal{A}(h)(\delta)| \leq Mk_1^*\widehat{l_2}\|h\|_{\widehat{\mathbb{k}}_0} + MC_1k_1^* + Mk_1^*k_2^*\widehat{l_2}\|h\|_{\widehat{\mathbb{k}}_0}$$
$$+ MC_1k_1^*k_2^* + Mk_1^*\zeta^* + M\Psi^* := C^* \leq C^*.$$

Hence, $\mathcal{A}(h)(\delta) \in \widehat{\mathbb{k}}_0$.

Let $\mathfrak{r} > 0$ be such that $\mathfrak{r} \geq \dfrac{MC_1k_1^* + MC_1k_1^*k_2^* + Mk_1^*\zeta^* + M\Psi^*}{1 - Mk_1^*\widehat{l_2}(1 + k_2^*)}$, and $\sigma_{\mathfrak{r}}$ be the closed ball in $\widehat{\mathbb{k}}_0$ centered at the origin and of radius $\mathfrak{r}$. Let $z \in \sigma_{\mathfrak{r}}$ and $\delta \in \Re$. Then,

$$|\mathcal{A}(z)(\delta)| \leq \quad Mk_1^*\widehat{l_2}\mathfrak{r} + MC_1k_1^* + Mk_1^*k_2^*\widehat{l_2}\mathfrak{r} + MC_1k_1^*k_2^* + Mk_1^*\zeta^* + M\Psi^*$$

Thus,

$$\||\mathcal{A}(z)(\delta)\|_{\widehat{\mathbb{k}}_0} \leq \mathfrak{r},$$

then $\mathcal{A}(\sigma_{\mathfrak{r}}) \subset \sigma_{\mathfrak{r}}$.

The proof can be given by following steps.

Step 1: $\mathcal{A}$ is continuous.
Let $\{h^n\}_{n\in\mathbb{N}}$ be a sequence such that $h^n \to h$ in $\sigma_{\mathfrak{r}}$. Firstly, we are going to study the convergence of the sequences $(h^n_{\rho(s,h^n_s)})_{n\in\mathbb{N}}, s \in \Re$.

If $s \in \Re$ is such that $\rho(s,h_s) > 0$, then we have,

$$\begin{aligned}\|h^n_{\rho(s,h^n_s)} - h_{\rho(s,h_s)}\|_{\mathfrak{Q}} &\leq \|h^n_{\rho(s,h^n_s)} - h_{\rho(s,h^n_s)}\|_{\mathfrak{Q}} + \|h_{\rho(s,h^n_s)} - h_{\rho(s,h_s)}\|_{\mathfrak{Q}} \\ &\leq \widehat{l_2}\|h_n - h\|_{\sigma_{\mathfrak{r}}} + \|h_{\rho(s,h^n_s)} - h_{\rho(s,h_s)}\|_{\mathfrak{Q}},\end{aligned}$$

which proves that $h^n_{\rho(s,h^n_s)} \to h_{\rho(s,h_s)}$ in $\mathfrak{Q}$ as $n \to \infty$ for every $s \in \Re$ such that $\rho(s,h_s) > 0$. Similarly, is $\rho(s,h_s) < 0$, we get

$$\|h^n_{\rho(s,h^n_s)} - h_{\rho(s,h_s)}\|_{\mathfrak{Q}} = \|\phi^n_{\rho(s,h^n_s)} - \phi_{\rho(s,h_s)}\|_{\mathfrak{Q}} = 0,$$

which also shows that $h^n_{\rho(s,h^n_s)} \to h_{\rho(s,h_s)}$ in $\mathfrak{Q}$ as $n \to \infty$ for every $s \in \Re$ such that $\rho(s,h_s) < 0$. Using the previous arguments, we can prove that $h^n_{\rho(s,h_s)} \to \phi$ for every $s \in \Re$ such that $\rho(s,h_s) = 0$.

$$\begin{aligned}&|\mathcal{A}(h^n)(\delta) - \mathcal{A}(h)(\delta)| \\ &\leq \| \int_0^\delta R(\delta-s)\Psi\big(s, h^n_{\rho(s,h^n_s+\vartheta_s)} + \vartheta_{\rho(s,h^n_s+\vartheta_s)}, \\ &\quad \int_0^s \zeta(s,\theta,h^n_{\rho(\theta,h^n_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h^n_\theta+\vartheta_\theta)})d\theta\big)ds\end{aligned}$$

$$
\begin{aligned}
&- \int_0^\delta R(\delta-s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)},\\
&\int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds\|\\
\leq{}& M\int_0^\delta e^{-\alpha(\delta-s)}\|\Psi\big(s, h^n_{\rho(s,h^n_s+\vartheta_s)} + \vartheta_{\rho(s,h^n_s+\vartheta_s)},\\
&\int_0^s \zeta(s,\theta,h^n_{\rho(\theta,h^n_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h^n_\theta+\vartheta_\theta)})d\theta\big)ds\\
&- \Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)}, \int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds\|.
\end{aligned}
$$

Then by (EC_3) and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(h^n) - \mathcal{A}(h)\|_{\widehat{\mathbb{k}}_0} \to 0, \text{ as } n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

Step 2: $\mathcal{A}(\sigma_{\mathfrak{r}})$ is equicontinuous. Let $B = [0, b]$ be a compact of $[0, +\infty)$, for $b > 0$.
Let $\delta_1, \delta_2 \in B$, with $\delta_2 > \delta_1$, we have

$$
\begin{aligned}
&|\mathcal{A}(h)(\delta_1) - \mathcal{A}(h)(\delta_2)|\\
&\leq |\int_0^{\delta_1} R(\delta_1-s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)},\\
&\quad \int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds\|\\
&\quad - \int_0^{\delta_2} R(\delta_2-s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)},\\
&\quad \int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds\|\\
&\leq \int_0^{\delta_1} \|R(\delta_1-s) - R(\delta_2-s)\||\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)},\\
&\quad \int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)\\
&\quad - \Psi(s,0,0) + \Psi(s,0,0)|ds\\
&\quad + \int_{\delta_1}^{\delta_2} \|R(\delta_2-s)\||\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)},\\
&\quad \int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)\\
&\quad - \Psi(s,0,0) + \Psi(s,0,0)|ds\\
&\leq \int_0^{\delta_1} \|R(\delta_1-s) - R(\delta_2-s)\|k_1(s)(\widehat{l_2}\mathfrak{r} + C_1)ds
\end{aligned}
$$

$$+\int_0^{\delta_1}\|R(\delta_1-s)-R(\delta_2-s)\|\|\Psi(s,0,0)\|ds$$
$$+\int_0^{\delta_1}\|R(\delta_1-s)-R(\delta_2-s)\|k_1(s)\int_0^s k_2(\theta)(\widehat{l}_2\mathfrak{r}+C_1)d\theta ds$$
$$+\int_0^{\delta_1}\|R(\delta_1-s)-R(\delta_2-s)\|k_1(s)\int_0^s \|\zeta(s,\theta,0)\|d\theta ds$$
$$+M\int_{\delta_1}^{\delta_2}e^{-\alpha(\delta_2-s)}k_1(s)(\widehat{l}_2\mathfrak{r}+C_1)ds$$
$$+M\int_{\delta_1}^{\delta_2}e^{-\alpha(\delta_2-s)}\|\Psi(s,0,0)\|ds$$
$$+M\int_{\delta_1}^{\delta_2}e^{-\alpha(\delta_2-s)}k_1(s)\int_0^s k_2(\theta)(\widehat{l}_2\mathfrak{r}+C_1)d\theta ds$$
$$+M\int_{\delta_1}^{\delta_2}e^{-\alpha(\delta_2-s)}k_1(s)\int_0^s \|\zeta(s,\theta,0)\|d\theta ds.$$

By (EC_1), $|\mathcal{A}(h)(\delta_1)-\mathcal{A}(h)(\delta_2)|\to 0$ as $\delta_1\to\delta_2$. Then $\mathcal{A}(\sigma_{\mathfrak{r}})$ is equicontinuous.

Step 3: We show that $\mathcal{A}:\sigma_{\mathfrak{r}}\to\sigma_{\mathfrak{r}}$ is a Meir-Keeler condensing operator.
For $\mathcal{D}\subset\sigma_{\mathfrak{r}},\mathcal{A}(\mathcal{D})$ is bounded. Then there exists a countable set $\mathcal{D}_{\prime}=\{h_n\}_{n=1}^{\infty}\subset\mathcal{D}$.

Claim 1. Using the fact that $\mathcal{A}(\sigma_{\mathfrak{r}})$ is equicontinuous and by the definition of $\eta_0^{\infty}(\cdot)$, we get that
$$\eta_0^{\infty}(\mathcal{A}(\mathcal{D}))=0.$$

Claim 2. We have
$$\begin{aligned}\varkappa(\mathcal{A}(\mathcal{D})(\delta))\le 2\varkappa(\{&\int_0^{\delta}R(\delta-s)\Psi\big(s,h^n_{\rho(s,h^n_s+\vartheta_s)}+\vartheta_{\rho(s,h^n_s+\vartheta_s)},\\
&\int_0^s\zeta(s,\theta,h^n_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h^n_\theta+\vartheta_\theta)})d\theta\big)ds\}_{n=0}^{\infty})+\epsilon\\
&\le 4\widehat{l}_2M\int_0^{\delta}e^{-\alpha(\delta-s)}k_1(s)(\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))(s)\\
&\quad+\varkappa(\int_0^s\zeta(s,\theta,h^n_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h^n_\theta+\vartheta_\theta)})d\theta\big)ds)+\epsilon\\
&\le 4\widehat{l}_2M\int_0^{\delta}e^{-\alpha(\delta-s)}k_1(s)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))ds\\
&\quad+8\widehat{l}_2M\int_0^{\delta}e^{-\alpha(\delta-s)}k_1(s)(\int_0^s k_2(\theta)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))d\theta)ds+\epsilon\\
&\le 8\widehat{l}_2M\int_0^{\delta}e^{-\alpha(\delta-s)}k_1(s)(1+k_2^*)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))ds+\epsilon.\end{aligned}$$

Then since ϵ is arbitrary, we get

$$\varkappa(\mathcal{A}(\mathcal{D})(\delta)) \leq 8M\widehat{l}_2 k_1^*(1+k_2^*) \sup_{\varrho\in(-\infty,0]} \varkappa(\mathcal{D}_0(\varrho)).$$

Then, $\bar{\varkappa}_\infty(\mathcal{A}\mathcal{D}) \leq 8\widehat{l}_2 M k_1^*(1+k_2^*)\bar{\varkappa}_\infty(\mathcal{D})$.

Claim 3. We have

$$\begin{aligned}
&|h(\delta)-h^*(\delta)| \\
&\leq \| \int_0^\delta R(\delta-s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)}, \\
&\quad \int_0^s \zeta(s,\theta,h_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big) \\
&\quad - \int_0^\delta R(\delta-s)\Psi\big(s, h^*_{\rho(s,h^*_s+\vartheta_s)}+\vartheta_{\rho(s,h^*_s+\vartheta_s)}, \\
&\quad \int_0^s \zeta(s,\theta,h^*_{\rho(\theta,h^*_\theta+\vartheta_\theta)}+\vartheta^*_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)\| \\
&\leq M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\|h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)}-h^*_{\rho(s,h^*_s+\vartheta_s)}-\vartheta_{\rho(s,h^*_s+\vartheta_s)}\|ds \\
&\quad + M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\int_0^s k_2(\theta)\|h_{\rho(\theta,h_\theta+\vartheta_\theta)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)} \\
&\quad - h^*_{\rho(\theta,h^*_\theta+\vartheta_\theta)}-\vartheta_{\rho(\theta,h^*_\theta+\vartheta_\theta)}\|d\theta ds \\
&\leq M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\widehat{l}_2|h(s)-h^*(s)|ds \\
&\quad + M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\int_0^s k_2(\theta)\widehat{l}_2|h(\theta)-h^*(\theta)|d\theta ds.
\end{aligned}$$

Then,

$$c(\mathcal{A}\mathcal{D}) \leq (M\widehat{l}_2 k_1^*(1+k_2^*))c(\mathcal{D}).$$

From Claims 1,2 and 3, we deduce that

$$\varkappa^*(\mathcal{A}(\mathcal{D})) \leq 8(M\widehat{l}_2 k_1^*(1+k_2^*))\varkappa^*(\mathcal{D}).$$

For a fixed $\epsilon > 0$, set

$$\lambda = \frac{1-8\widehat{l}_2 M k_1^*(1+k_2^*)}{8\widehat{l}_2 M k_1^*(1+k_2^*)}\epsilon.$$

We get that

$$\epsilon \leq \varkappa^*((\mathcal{D}) < \epsilon+\lambda \Rightarrow \varkappa^*(\mathcal{A}(\mathcal{D})) < \epsilon.$$

That means $\mathcal{A}$ is a Meir-Keeler condensing operator, then using Theorem 2.8 we can say that the operator $\mathcal{A}$ has at least fixed point, so the operator $\Xi : \Bbbk \to \Bbbk$ has a fixed point $z(\delta) = h(\delta)+\vartheta(\delta)$ and the set of all fixed points of Ξ in $\Bbbk$ is compact.

□

4. Attractivity

In this section, we are going to study the attractivity of solutions.

Definition 4.1. [16] *We say that the solutions of problem* (1.1) *are locally attractive if there exists a closed ball* $\overline{B}(\xi^*, \mathfrak{r}^*)$ *in the space* $\Bbbk$, *where* $\xi^* \in \Bbbk$, *such that for any solutions* ξ *and* $\widetilde{\xi}$ *of* (1.1) *belonging to* $\overline{B}(\xi^*, \mathfrak{r}^*)$, *the following convergence holds:*

$$\lim_{s \to +\infty} (\xi(s) - \widetilde{\xi}(s)) = 0.$$

When the limit is uniform with respect to $\overline{B}(\xi^*, \mathfrak{r}^*)$, *we say that the solutions of (1.1) are uniformly locally attractive.*

Theorem 4.2. *Suppose that the hypotheses* $(EC_1)-(EC_4), (EC_{\varpi})$ *hold. If* $8M\widehat{l_2}k_1^*(1+k_2^*) < 1$, *Then the solutions of problem (1.1) are uniformly locally attractive.*

Proof. Let h^* be a solution of (1.1), for $h \in B(h^*, \mathfrak{r}^*)$, by $(EC_1) - (EC_4)$, we have

$$\begin{aligned}
&\|\mathcal{A}(h)(\delta) - h^*(\delta)\| \\
&\quad = \|\mathcal{A}(h)(\delta) - \mathcal{A}(h^*)(\delta)\| \\
&\quad \leq M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \widehat{l_2} |h(s) - h^*(s)| ds \\
&\qquad + M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \int_0^s k_2(\theta) \widehat{l_2} |h(\theta) - h^*(\theta)| d\theta ds \\
&\quad \leq M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \widehat{l_2} \mathfrak{r}^* ds + M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \int_0^s k_2(\theta) \widehat{l_2} \mathfrak{r}^* d\theta ds \\
&\quad \leq (M\widehat{l_2} k_1^*(1 + k_2^*)) \mathfrak{r}^* \\
&\quad \leq \mathfrak{r}^*.
\end{aligned}$$

Then $\mathcal{A}(B^*_{\mathfrak{r}}) \subset B^*_{\mathfrak{r}}$.
For each $h, \widetilde{h} \in B(h^*, \mathfrak{r}^*), \delta \in [0, +\infty)$, we have :

$$\begin{aligned}
&\|h(\delta) - \widetilde{h}(\delta)\| \\
&\quad = \|\mathcal{A}(h)(\delta) - \mathcal{A}(\widetilde{h})(\delta)\| \\
&\quad \leq M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \widehat{l_2} |h(s) - \widetilde{h}(s)| ds \\
&\qquad + M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \int_0^s k_2(\theta) \widehat{l_2} |h(\theta) - \widetilde{h}(\theta)| d\theta ds \\
&\quad \leq M\widehat{l_2} \mathfrak{r}^* \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) ds + M\widehat{l_2} \mathfrak{r}^* k_2^* \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) ds.
\end{aligned}$$

We conclude that $\|h(\delta) - \widetilde{h}(\delta)\| \to 0$, as $\delta \to \infty$. □

5. Controllability

Definition 5.1. *The system* (1.2) *is said to be controllable if for every initial function* $\varpi(\delta) \in \mathfrak{Q}$, *and* $\widehat{z} \in \Im$ *and for* $\gamma \in \Re$, *there is a control* $z \in L^2([0,\gamma],\Im)$, *such that the mild solution* $z(\cdot)$ *of the problem* (1.2) *satisfies the condition* $z(\gamma) = \widehat{z}$.

Let us introduce the following hypotheses:

(EC_5) (i)For some $\gamma \in \mathbb{N}$, the linear operator $\mathfrak{W} : L^2([0,\gamma],H) \to \Im$ is defined by

$$\mathfrak{W}z = \int_0^\gamma R(\gamma - s)\mathcal{C}u(s)ds,$$

has a pseudo-invertible operator $\widetilde{\mathfrak{W}}^{-1}$ which takes values in $L^2([0,\gamma],H)/Ker\mathfrak{W}$
(ii) There exists positive constants $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_1$ such that :

$$\|C\| \leq \mathcal{M}_1 \textit{ and } \|\widetilde{\mathfrak{W}}^{-1}\| \leq \mathcal{M}_2.$$

(iii) There exists a function $k_3 \in L^1(\Re)$, and a positive constant k_4 such that for any bounded sets $\mathcal{D} \subset \Im$, and $\mathcal{D}' \subset H$, we have

$$\varkappa((\widetilde{\mathfrak{W}}^{-1}\mathcal{D})(\delta)) \leq k_3(\delta)\varkappa(\mathcal{D}),$$

and

$$\varkappa(\mathcal{C}(\mathcal{D}')) \leq k_4\varkappa(\mathcal{D}'(\delta)),$$

where $\delta \in \Re$.

Remark 5.2. We denote by $k_3^* = \|k_3\|_{L^1(\Re)}$.

Theorem 5.3. *Suppose that the hypotheses* $(EC_1) - (EC_5), (EC_\varpi)$ *hold and*

$$8M\widehat{l}_2(k_1^* + k_2^*k_3^*)(1 + \gamma\mathcal{M}_2\mathcal{M}_1M) < 1.$$

Then problem (1.2) *is controllable.*

Proof. We define a control

$$u_z = \widetilde{\mathfrak{W}}^{-1}\big(\widehat{z} - R(\gamma)\varpi(0) - \int_0^\gamma R(\gamma - s)\Psi\big(s, z_{\rho(s,z_s)}, \int_0^s \zeta(s,\theta,z_{\rho(\theta,z_\theta)}d\theta)ds\big)\big)(\delta).$$

Now we shall show that the operator $\Xi : \Bbbk \to \Bbbk$ defined by :

$$\Xi(z) := \begin{cases} \varpi(\delta), \quad \text{if } \delta \in (-\infty,0], \\ \\ R(\delta)\varpi(0) + \displaystyle\int_0^\delta R(\delta - s)\Psi\big(s, z_{\rho(s,z_s)}, \int_0^s \zeta(s,\theta,z_{\rho(\theta,z_\theta)})d\theta\big)ds \\ + \displaystyle\int_0^\delta R(\delta - s)\mathcal{C}u_z(s)ds, \quad \text{if } \delta \in \Re, \end{cases}$$

Let $\vartheta(\cdot) : \mathbb{R} \to \Im$ be the function defined by:

$$\vartheta(\delta) = \begin{cases} \varpi(\delta), & \text{if } \delta \in (-\infty,0], \\ R(\delta)\ \varpi(0), & \text{if } \delta \in \Re. \end{cases}$$

Then $\vartheta_0 = \varpi$. For each $h \in \mathbb{k}$ with $h(0) = 0$, we denote by $\overline{h}$ the function

$$\overline{h}(\delta) = \begin{cases} 0; & \text{if } \delta \in (-\infty, 0], \\ h(\delta); & \text{if } \delta \in \Re. \end{cases}$$

If z satisfies (3.1), we can decompose it as $z(\delta) = h(\delta) + \vartheta(\delta),\ \delta \in \Re$, which implies $z_\delta = h_\delta + \vartheta_\delta$ for every $\delta \in \Re$ and the function $h(\cdot)$ satisfies

$$h(\delta) = \int_0^\delta R(\delta - s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)},$$
$$\int_0^s \zeta(s, \theta, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds + \int_0^\delta R(\delta - s)\mathcal{C}u_{h+\vartheta}(s)ds, \delta \in \Re.$$

Set

$$\widehat{\mathbb{k}}_0 = \{h \in \widehat{\mathbb{k}} : h(0) = 0\},$$

and let

$$\|h\|_{\widehat{\mathbb{k}}_0} = \sup\{|h(\delta)| : \delta \in \Re\},\ h \in \widehat{\mathbb{k}}_0.$$

$\widehat{\mathbb{k}}_0$ is a Banach space with the norm $\|.\|_{\widehat{\mathbb{k}}_0}$.
We define the operator $\mathcal{A} : \widehat{\mathbb{k}}_0 \to \widehat{\mathbb{k}}_0$ by:

$$\mathcal{A}(h)(\delta) := \begin{cases} 0, \quad \text{if } \delta \leq 0, \\ \displaystyle\int_0^\delta R(\delta - s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)}, \\ \displaystyle\int_0^s \zeta(s, \theta, h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds, \\ \displaystyle+ \int_0^\delta R(\delta - s)\mathcal{C}u_{h+\vartheta}(s)ds, \quad \text{if } \delta \in \Re, \end{cases}$$

The operator $\mathcal{A}$ maps $\widehat{\mathbb{k}}_0$ into $\widehat{\mathbb{k}}_0$, for each $\delta \in \Re$ we have

$$\begin{aligned}
&|u_{h+\vartheta}(s)| \\
&\leq |\widetilde{\mathfrak{W}}^{-1}\big(\widehat{z} - R(\gamma)\varpi(0) + \int_0^\gamma R(\gamma - \delta)\Psi\big(\delta, z_{\rho(\delta,z_\delta)}, \int_0^\delta \zeta(\delta, \theta, z_{\rho(\theta,z_\theta)})d\theta\big)dt\big)(s)| \\
&\leq \mathcal{M}_2[|\widehat{z}| + M\|\varpi\| + \int_0^\gamma \|R(\gamma - \delta)\|k_1(\delta)\|h_{\rho(\delta,h_\delta+\vartheta_\delta)} + \vartheta_{\rho(\delta,h_\delta+\vartheta_\delta)}\|dt \\
&\quad + \int_0^\gamma \|R(\gamma - \delta)\Psi(\delta, 0, 0)\|d\delta \\
&\quad + \int_0^\gamma \|R(\gamma - \delta)\|k_1(\delta) \int_0^\delta k_2(\theta)\|h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)}\|d\theta d\delta \\
&\quad + \int_0^\gamma \|R(\gamma - \delta)\|k_1(\delta) \int_0^\delta \zeta(\delta, \theta, 0)\|d\theta d\delta](s) \\
&\leq \mathcal{M}_2[|\widehat{z}| + M\widehat{l_1}\|\mathfrak{w}_\delta\|_{\mathfrak{Q}} + Mk_1^* \int_0^\gamma (\widehat{l_2}|h(\delta)| + (\widehat{l_3} + \mho^\varpi + \widehat{l_2}Me^{-\alpha s}\widehat{l_1})\|\varpi\|_{\mathfrak{Q}}))d\delta
\end{aligned}$$

$$
\begin{aligned}
&+ M\Psi^* + Mk_1^* \int_0^\gamma k_2(\theta)(\widehat{l_2}|h(\theta)| + (\widehat{l_3} + \mho^{\varpi} + \widehat{l_2}M\widehat{l_1})\|\varpi\|_{\mathfrak{Q}})d\theta d\delta + Mk_1^*\zeta^*]) \\
\leq\ & \mathcal{M}_2[|\widehat{z}| + M\widehat{l_1}\|\mathfrak{w}_\delta\|_{\mathfrak{Q}} + M\Psi^* + Mk_1^*\zeta^* + nMk_1^*(\widehat{l_2}\|h\| + C_1) \\
&+ nMk_1^*k_2^*(\widehat{l_2}\|h\| + C_1)] := C_2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
|\mathcal{A}(h)(\delta)| \leq \| &\int_0^\delta R(\delta - s)\Psi\big(s, h_{\rho(s,h_s+\vartheta_s)} + \vartheta_{\rho(s,h_s+\vartheta_s)}, \\
&\int_0^s \zeta(s,\theta, h_{\rho(\theta,h_\theta+\vartheta_\theta)} + \vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds\| \\
&+ |\int_0^\delta R(\delta - s)\mathcal{C}u_{h+\vartheta}(s)ds|
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
|\mathcal{A}(h)(\delta)| \leq & Mk_1^*\widehat{l_2}\|h\|_{\widehat{\mathbb{k}}_0} + MC_1k_1^* + Mk_1^*k_2^*\widehat{l_2}\|h\|_{\widehat{\mathbb{k}}_0} + MC_1k_1^*k_2^* \\
&+ \mathcal{M}_2\mathcal{M}_1M[|\widehat{z}| + M\widehat{l_1}\|\mathfrak{w}_\delta\|_{\mathfrak{Q}} + M\Psi^* \\
&+ Mk_1^*\zeta^* + nMk_1^*(\widehat{l_2}\|h\| + C_1) + nMk_1^*k_2^*(\widehat{l_2}\|h\| + C_1)] \\
\leq & C^* + \mathcal{M}_1MC_2 := C_3 \\
\leq & C_3.
\end{aligned}
$$

Hence, $\mathcal{A}(h)(\delta) \in \widehat{\mathbb{k}}_0$.

Let $\mathfrak{r} > 0$ be such that

$$
\mathfrak{r} \geq \frac{\mathcal{M}_2\mathcal{M}_1M[|\widehat{z}| + M\widehat{l_1}\|\mathfrak{w}_\delta\|_{\mathfrak{Q}} + M\Psi^* + Mk_1^*\zeta^* + nMC_1k_1^*(1+k_2^*)] + MC_1k_1^* + MC_1k_1^*k_2^*}{1 - M\widehat{l_2}k_1^*(1+k_1^*k_2^*) - \gamma\mathcal{M}_2\mathcal{M}_1M^2\widehat{l_2}k_1^*(1+k_2^*)},
$$

and $\sigma_{\mathfrak{r}}$ be the closed ball in $\widehat{\mathbb{k}}_0$ centered at the origin and of radius z. Let $z \in \sigma_{\mathfrak{r}}$ and $\delta \in \Re$. Then,

$$
\begin{aligned}
|\mathcal{A}(z)(\delta)| \leq\ & Mk_1^*\widehat{l_2}\mathfrak{r} + MC_1k_1^* + Mk_1^*k_2^*\widehat{l_2}\mathfrak{r} + MC_1k_1^*k_2^* \\
&+ \mathcal{M}_2\mathcal{M}_1M[|\widehat{z}| + M\widehat{l_1}\|\mathfrak{w}_\delta\|_{\mathfrak{Q}} + nMk_1^*(\widehat{l_2}\mathfrak{r} + C_1) \\
&+ M\Psi^* + Mk_1^*\zeta^* + nMk_1^*k_2^*(\widehat{l_2}\mathfrak{r} + C_1)
\end{aligned}
$$

Thus,

$$
\||\mathcal{A}(z)(\delta)\|_{\widehat{\mathbb{k}}_0} \leq \mathfrak{r},
$$

then $\mathcal{A}(\sigma_{\mathfrak{r}}) \subset \sigma_{\mathfrak{r}}$.

The proof can be given by following steps.

Step 1: As in the proofs of Theorem 3.5, we can prove that $\mathcal{A}(h)(\delta)$ is continuous and $\mathcal{A}(\sigma_{\mathfrak{r}})$ is equicontinuous on any compact $B = [0, b]$ of $[0, +\infty)$, for $b > 0$.

Step 2: We show that $\mathcal{A}:\sigma_{\mathfrak{r}}\to\sigma_{\mathfrak{r}}$ is a Meir-Keeler condensing operator. For $\mathcal{D}\subset\sigma_{\mathfrak{r}}, \mathcal{A}(\mathcal{D})$ is bounded. Then there exists a countable set $\mathcal{D}_0=\{h_\gamma\}_{\gamma=1}^{\infty}\subset\mathcal{D}$.

Claim 1. Using the fact that $\mathcal{A}(\sigma_{\mathfrak{r}})$ is equicontinuous and by the definition of $\eta_0^\infty(\cdot)$, we get that $\eta_0^\infty(\mathcal{D})\le\eta_0^\infty(\mathcal{A}\mathcal{D})=0$. Then $\eta_0^\infty(\mathcal{D})=0$.

Claim 2. We have

$$\begin{aligned}
\varkappa(\mathcal{A}(\mathcal{D})(\delta)) &\le 8\widehat{l_2}M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)(1+k_2^*)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))ds+\epsilon\\
&\quad+4Mk_4\int_0^\delta e^{-\alpha(\delta-s)}k_3(s)\\
&\quad\times\varkappa(\{\int_0^n R(\gamma-\delta)\Psi\big(s,z_{\rho(s,z_s)},\int_0^s\zeta(\delta,s,z_{\rho(\theta,z_\theta)})d\theta\big)d\delta ds\}_{\gamma=0}^{\infty})\\
&\le 8\widehat{l_2}M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)(1+k_2^*)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))ds+\epsilon\\
&\quad+4Mk_4\int_0^\delta e^{-\alpha(\delta-s)}k_3(s)nM\widehat{l_2}k_1(s)(1+2k_2^*)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))ds\\
&\le 8\widehat{l_2}M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)(1+k_2^*)+k_4k_3(s)Mnk_1(s)(1+k_2^*)\\
&\quad\times\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))ds.
\end{aligned}$$

Then,

$$\varkappa(\mathcal{A}(\mathcal{D})(\delta)) \quad\le\quad 8M\widehat{l_2}k_1^*(1+k_2^*)+k_4k_3^*Mnk_1^*(1+k_2^*)\sup_{\varrho\in(-\infty,0]}\varkappa(\mathcal{D}_0(\varrho))+\epsilon.$$

We have

$$\bar{\varkappa}_\infty(\mathcal{A}(\mathcal{D}))\le[8M\widehat{l_2}k_1^*(1+k_2^*)+k_4k_3^*Mnk_1^*(1+k_2^*)]\bar{\varkappa}_\infty(\mathcal{D}).$$

Then, $\bar{\varkappa}_\infty(\mathcal{A}(\mathcal{D}))\le L_1\bar{\varkappa}_\infty(\mathcal{D})$.

Claim 3. We have

$$\begin{aligned}
&|h(\delta)-h^*(\delta)|\\
&\quad\le|\int_0^\delta R(\delta-s)\Psi\big(s,h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(s,h_s+\vartheta_s)},\\
&\qquad\int_0^s\zeta(s,\theta,h_{\rho(s,h_s+\vartheta_s)}+\vartheta_{\rho(\theta,h_\theta+\vartheta_\theta)})d\theta\big)ds\\
&\qquad+\int_0^\delta R(\delta-s)\mathcal{C}u_{h+\vartheta}(s)ds-\int_0^\delta R(\delta-s)\Psi\big(s,h^*_{\rho(s,h^*_s+\vartheta_s)}+\vartheta_{\rho(s,h^*_s+\vartheta_s)},
\end{aligned}$$

$$\int_0^s \zeta(s,\theta,h^*_{\rho(s,h^*_s+\vartheta_s)}+\vartheta_{\rho(\theta,h^*_\theta+\vartheta_\theta)})d\theta\big)ds|$$
$$+\int_0^\delta R(\delta-s)\mathcal{C}u_{h^*+\vartheta}(s)ds$$
$$\leq M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\widehat{l_2}|h(s)-h^*(s)|ds$$
$$+M\int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\int_0^s k_2(\theta)\widehat{l_2}|h(\theta)-h^*(\theta)|d\theta ds$$
$$+\mathcal{M}_1\mathcal{M}_2M\int_0^\delta e^{-\alpha(\delta-s)}nMk_1(s)\widehat{l_2}|h(s)-h^*(s)|ds$$
$$+\mathcal{M}_1\mathcal{M}_2M\int_0^\delta e^{-\alpha(\delta-s)}nMk_1(s)\int_0^s k_2(\theta)\widehat{l_2}|h(\theta)-h^*(\theta)|d\theta ds.$$

We have $c(\mathcal{AD})\leq[M\widehat{l_2}k_1^*(1+k_2^*)(1+\mathcal{M}_1\mathcal{M}_2Mn)]c(\mathcal{D})$. Then, $c(\mathcal{AD})\leq L_2c(\mathcal{D})$.

From claim 1,2 and 3 we deduce that

$$\varkappa^*(\mathcal{A}(\mathcal{D}))\leq L_3\varkappa^*(\mathcal{D}),$$

where $L_3=max(L_1,L_2)$. So for a given $\epsilon>0$, let $\lambda=\frac{1-L_3}{L_3}\epsilon$. We get that

$$\epsilon\leq\varkappa^*(\mathcal{D})<\epsilon+\lambda\Rightarrow\varkappa^*(\mathcal{A}(\mathcal{D}))<\epsilon.$$

That means $\mathcal{A}$ is a Meir-Keeler condensing operator, then using the Theorem 2.8 we can say that the operator $\mathcal{A}$ has at least fixed point, so the operator $\Xi:\Bbbk\to\Bbbk$ has a fixed point $z(\delta)=h(\delta)+\vartheta(\delta)$ and the set of all fixed points of Ξ in $\Bbbk$ is compact, which mean that the problem (1.2) is controllable.

6. Example

We consider the following partial integro-differential problem

$$\begin{aligned}\frac{\partial}{\partial\delta}h(\delta,x)&=\frac{\partial^2}{\partial x^2}h(\delta,x)+a\frac{\partial}{\partial x}h(\delta,x)+bh(\delta,x)\\&+\int_0^\delta Y(\delta-s)\left[\frac{\partial^2}{\partial x^2}(s,x)+a\frac{\partial}{\partial x}h(s,x)+bh(s,x)\right]\\&+\frac{e^{-\alpha\delta}}{21(\delta^2+2)}\int_{-\infty}^{-\delta}\frac{\sin(\tau)e^{-\Delta(\delta,\tau)-\alpha(\delta-\tau)}}{(\delta+\tau)^2+1}d\tau\\&+\frac{e^{-\alpha\delta}}{21(\delta^2+2)}\int_0^\delta\frac{\ln(1+e^{-\delta})e^{-\alpha(\delta-s)}}{1+2\delta^2+s^2}\int_{-\infty}^{-s}\frac{\cos(\Delta(s,\tau))e^{-\Delta(s,\tau)-\alpha(s-\tau)}}{(s+\tau)^2+1}d\tau ds,\\&x\in[0,1],\ \delta\in\Re,\end{aligned}\tag{6.1}$$

$$h(\delta,0)=h(\delta,1),\ \delta\in\Re,\tag{6.2}$$

$$h(\delta,x)=\phi(\delta),\ \delta\in(-\infty,0],\ x\in[0,1],\tag{6.3}$$

where $Y : \Re \to \mathbb{R}$ is continuous, $\Delta : \Re \times \mathbb{R} \to \mathbb{R}$ a continuous function. Let $\Im = L^2(0,1)$, and A is an operator induced on $\Im$ define as

$$Ah = h'' + ah' + bh, a, b \in \mathbb{R} \text{ and, } D(A) = H^2(0,1) \cap H_0^1(0,1).$$

From [17] we know that A is the infinitesimal generator of analytic C_0 semigroup $(\Upsilon(\delta))_{\delta \geq 0}$ on $\Im$. Since the semigroup generated by A is analytic, then it's norm continuous for $\delta > 0$. So the resolvent operator is operator-norm continuous for $\delta > 0$. We define the operator $B : \Im \to \Im$ as

$$B(\delta)h = Y(\delta)Ah, \quad \text{for}\, \delta \geq 0.$$

We assume that there exist $\beta > \beta_1 > 1$ such that $Y(\delta) < \frac{1}{\beta_1} e^{-\beta\delta}$, for all $\delta \geq 0$. Then, we have $M = 1, \alpha = 1 - \frac{1}{\beta_1}$.
Consider $\mathfrak{Q} = BUC(\mathbb{R}^-, \Im)$, the phase space of bounded uniformly continuous functions endowed with the following norm

$$\|\phi\|_{\mathfrak{Q}} = \sup_{\tau \in (-\infty, 0]} \|\phi(\tau)\|_{\Im}, \phi \in \mathfrak{Q},$$

and $\widehat{l}_2 = \widehat{l}_3 = 1$ let $\rho(\delta, h)(x) = \Delta(\delta, h(\delta + \tau, x))$. See [24] for more details.

We have also

$$\Psi(\delta, f, g) = \frac{e^{-\alpha\delta}}{21(\delta^2 + 2)} \int_{-\infty}^{-\delta} \frac{\sin(\tau) e^{-f(\delta+\tau,x) - \alpha(\delta-\tau)}}{(\delta + \tau)^2 + 1} d\tau + \frac{e^{-\alpha\delta}}{21(\delta^2 + 2)} g(\delta, x),$$

$$\zeta(\delta, s, f) = \frac{\ln(1 + e^{-\delta}) e^{-\alpha(\delta - s)}}{1 + 2\delta^2 + s^2} \int_{-\infty}^{-s} \frac{\cos(f(s + \tau, x)) e^{-f(s+\tau,x) - \alpha(s-\tau)}}{(s + \tau)^2 + 1} d\tau,$$

and

$$k_1 = \frac{e^{-\alpha\delta}}{21(\delta^2 + 2)}, k_2 = \ln(2) = k_2^*, k_1^* = \frac{1}{42}.$$

The problem (1.1) in an abstract formulation of the problem (6.1)-(6.3). Since $(EC_1) - (EC_4), (EC_{\varpi})$ are satisfied, and $8M\widehat{l}_2 k_1^*(1 + k_2^*) < 1$, then the problem (6.1)-(6.3) has at least one mild solution on $\Bbbk$, which is uniformly locally attractive.

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