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Fractional Langevin equations involving ψ-Caputo type operators in a Banach space: a solution sets approach

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Abstract. This paper investigates certain topological properties of the set of all global solutions for a class of nonlinear ψ-Caputo fractional Langevin equations. The nonlinearity, defined on an infinite-dimensional Banach space, is assumed to satisfy Nagumo-type growth conditions. An Aronszajn-type result is established using the nonlinear alternative for condensing operators, combined with the Browder–Gupta method. An illustrative example is provided to support the theoretical findings.

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1. Introduction

The theory of fractional differential equations is widely recognized for its extensive applications across numerous scientific disciplines. As a result, it has attracted significant interest from the mathematical community [14]. Over time, various definitions of fractional derivatives and integral operators have been proposed [24], each possessing distinct characteristics and properties. A recent development in this field is the emergence of "fractional calculus with respect to functions" [3], which has proven useful in modeling real-world phenomena. This type of operator appears in several contexts, including anomalous diffusion processes such as ultra-slow diffusion [16], random walks [12], financial crises [21], the Verhulst model [7], and the Heston model

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[6]. Accordingly, much research has focused on exploring both qualitative and quantitative properties of solutions to differential problems involving fractional derivatives with respect to functions [26, 27].

The Langevin model plays a crucial role in illustrating the interaction between particles and their surrounding medium, as well as the stochastic forces or fluctuations responsible for their irregular motion. However, the model's dependence on a specific relationship between a particle's position and velocity has led to the development of the fractional Langevin model, which better captures anomalous diffusion phenomena [15]. Additionally, certain real-world processes are more accurately described by systems with coupled randomness. For instance, in epidemiology, the migration of birds from different parts of the world can introduce infectious diseases, with transmission rates increasing as migratory birds converge. Such scenarios necessitate accounting for random disturbances. While these motivation-driven models offer strong explanatory power, the complexity of the corresponding mathematical formulations can pose significant challenges, particularly in analyzing the existence of solutions. Consequently, the study of qualitative properties for nonlinear ψ-Caputo Langevin equations has gained increasing importance.

In a recent study, the authors of [27] investigated certain quantitative aspects of the problem using Banach's contraction principle (equipped with a Bielecki-type norm) and fixed-point principles for convex-power condensing operators. Their analysis was conducted under the assumption that the nonlinearity satisfies a Nagumo-type growth condition in an infinite-dimensional Banach space. Building upon this foundation, the present manuscript seeks to advance the previous work by providing a new qualitative analysis of the following nonlinear fractional Langevin equation involving the ψ-Caputo fractional derivative (FD).

$$\begin{cases} \left({}^c\mathcal{D}_{a^+}^{\sigma;\psi} + \omega\, {}^c\mathcal{D}_{a^+}^{\sigma-1;\psi}\right) x(t) = \mathcal{W}(t, x(t)), \quad t \in J := [a, b], \\ x'(a) = x(a) = 0, \end{cases} \tag{1.1}$$

where $1 < \sigma < 2$, ${}^c\mathcal{D}_{a^+}^{\sigma;\psi}$ and ${}^c\mathcal{D}_{a^+}^{\sigma-1;\psi}$ represents the ψ-Caputo FD of order σ and $\sigma - 1$, respectively, ψ, ψ' are positive functions, $\mathcal{W} : J \times \mathbb{F} \to \mathbb{F}$ is a given function specified later, $(\mathbb{F}, \|\cdot\|)$ be an arbitrary Banach space and $\omega > 0$.

It is important to note that when the conditions ensuring the existence of solutions for the system under consideration do not guarantee uniqueness, it becomes necessary to explore the topological properties of the solution set. In particular, investigating the topological structure, such as acyclicity, compactness, and the R_δ-property (see Definition 2.10), is a central aspect of the qualitative theory of integral and differential equations and inclusions, due to its broad applicability [4, 28]. Notably, the R_δ-property plays a key role in demonstrating the invariance of the reachability set under nonlinear perturbations within the framework of control systems [11].

A distinctive feature of our investigation lies in the following key contributions:

(1) Establishing sufficient conditions under which problem (1.1) admits a unique solution.

(2) Proving a new Aronszajn-type result that addresses an open question related to fractional Langevin equations in a general framework, specifically, when the nonlinearity operates in an arbitrary Banach space. This result is obtained by combining the nonlinear alternative for condensing maps with the Browder–Gupta approach.
(3) In contrast to the approaches in [2, 27], we utilize an a priori estimation technique to prove the compactness and R_δ property of the global solution set, thereby extending, refining, and generalizing these previous findings.

The remainder of this paper is organized as follows: Section 2 collects preliminary material necessary for the developments that follow. Section 3 addresses contributions (1) and (2). Finally, illustrative examples are presented to support and demonstrate the applicability of our main results.

2. Preliminary results

In what follows, we equip the space $C(J,\mathbb{F})$ of continuous functions $f : J \to \mathbb{F}$ with the supnorm

$$\|f\|_\infty = \sup_{t\in J} \|f(t)\|, \quad \text{for all } f \in C(J,\mathbb{F}).$$

Consider the space $L^\beta_\psi(J,\mathbb{F})$ $(1 \le \beta < \infty)$ of Bochner-integrable functions x on J with the norm

$$\|f\|_{L^\beta_\psi} = \left(\int_a^b \psi'(s)\|f(s)\|^\beta ds\right)^{\frac{1}{\beta}}. \tag{2.1}$$

If $\psi(t) = t$ the space $L^\beta_\psi(J,\mathbb{F})$ coincides with the usual $L^\beta(J,\mathbb{F})$.

Define

$$\mathbb{L}^1_+(J,\mathbb{R}) = \{\psi \ : \ \psi \in C^1(J,\mathbb{R}),\ 0 < \psi'(t) \text{ for } \ t \in J\}.$$

For $t, s \in J$, $(s < t)$, and $\psi \in \mathbb{L}^1_+(J,\mathbb{R})$ we pose

$$\varrho(t,s)^\sigma = (\psi(t) - \psi(s))^\sigma \quad \text{with} \quad \varrho(t,s) = \psi(t) - \psi(s).$$

Definition 2.1. [14, 3] *Let $\sigma > 0$. The ψ-fractional integral (FI) of order σ of an integrable function z is given by*

$$\mathcal{I}^{\sigma,\psi}_{a^+} z(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \varrho(t,s)^{\sigma-1}\psi'(s)z(s)ds, \quad \psi \in \mathbb{L}^1_+(J,\mathbb{R}), \ \textit{and } t > a,$$

where $\Gamma(\cdot)$ is the well-known gamma function.

Lemma 2.2. [14, 3] *For $\sigma, \alpha > 0$, one has*

$$\left(\mathcal{I}^{\sigma;\psi}_{a^+}\ \varrho(s,a)^{\alpha-1}\right)(t) = \frac{\Gamma(\alpha)\varrho(t,a)^{\sigma+\alpha-1}}{\Gamma(\sigma+\alpha)}.$$

Definition 2.3. [3] *Let $\psi \in \mathbb{L}^1_+(J, \mathbb{R})$ and $n-1 < \sigma \leq n$ with $n \in \mathbb{N}$. The ψ-Caputo FD of a n-times continuously differentiable function z of order σ is given as*

$$\left({}^c\mathcal{D}^{\sigma;\psi}_{a^+} z\right)(t) = \mathcal{I}^{n-\sigma;\psi}_{a^+} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n z(t).$$

Definition 2.4. [5] *The Kuratowski measure of noncompactness (MNC) ς of a bounded set $\mathbb{V}$ in a Banach space $\mathbb{F}$ is defined as:*

$$\varsigma(\mathbb{U}) := \inf \left\{ \varepsilon > 0 : \mathbb{U} = \bigcup_{i=1}^{n} \mathbb{U}_i \text{ and } \operatorname{diam}(\mathbb{U}_i) \leq \varepsilon \text{ for } 1 \leq i \leq n \right\}.$$

Lemma 2.5. [5, 17] *Let $\mathbb{V}, \mathbb{U} \subset \mathbb{F}$ be two bounded subsets. Then $\varsigma(\cdot)$ fulfills*

1. $\mathbb{V} \subset \mathbb{U}$ *implies that* $\varsigma(\mathbb{V}) \leq \varsigma(\mathbb{U})$;
2. $\varsigma(\mathbb{V}) = 0$ *if and only if* $\mathbb{V}$ *is relatively compact;*
3. $\varsigma(\mathbb{V} \cup \mathbb{U}) = \max\{\varsigma(\mathbb{V}), \varsigma(\mathbb{U})\}$;
4. $\varsigma(\mathbb{V}) = \varsigma(\operatorname{conv}(\mathbb{V})) = \varsigma(\overline{\mathbb{V}})$, *where* $\operatorname{conv} \mathbb{V}$ *and* $\overline{\mathbb{V}}$ *represent the convex hull and the closure of* $\mathbb{V}$, *respectively;*
5. $\varsigma(\mathbb{V} + \mathbb{U}) \leq \varsigma(\mathbb{V}) + \varsigma(\mathbb{U})$, *where* $\mathbb{V} + \mathbb{U} = \{v + u : v \in \mathbb{V}, u \in \mathbb{U}\}$;
6. $\varsigma(d\mathbb{V}) = |d|\varsigma(\mathbb{V})$, *for any* $d \in \mathbb{R}$;
7. *For any bounded* $\mathbb{V}$, *there exists a countable set* $\widetilde{\mathbb{V}} \subset \mathbb{V}$, *such that*

$$\varsigma(\mathbb{V}) \leq 2\varsigma(\widetilde{\mathbb{V}}).$$

Lemma 2.6. [13] *Let $\{y_n\}_{n=1}^{+\infty} \subset L^1(J, \mathbb{F})$ such that $\|y_n(t)\| \leq h(t)$ for almost all $t \in J$, $n \geq 1$ and $h \in L^1(J, \mathbb{R}_+)$. Then, $t \mapsto \varsigma(\{y_n(t)\}_{n=1}^{+\infty})$ is integrable and*

$$\frac{1}{2}\varsigma\left(\left\{\int_a^t y_n(r)dr\right\}_{n=1}^{+\infty}\right) \leq \int_a^t \varsigma(\{y_n(r)\}_{n=1}^{+\infty})dr. \tag{2.2}$$

Lemma 2.7. [9] *Let $\mathbb{S}$ be a Banach space, and let $\mathbb{V} \subset \mathbb{S}$ be a bounded and closed subset. Suppose that $\mathfrak{G} : \mathbb{V} \to \mathbb{S}$ is a condensing map. Then the operator $I - \mathfrak{G}$ maps closed subsets of $\mathbb{V}$ onto closed subsets of $\mathbb{S}$, and $I - \mathfrak{G}$ is proper.*

Recall that if, for every compact subset $\mathbb{U} \subset \mathbb{S}$, the set $(I - \mathfrak{G})^{-1}(\mathbb{U})$ is compact, then the continuous map $I - \mathfrak{G}$ is said to be proper.

Next, we revisit certain concepts from geometric topology [4, 11].

Definition 2.8. *A subset $\mathcal{B}$ of a Banach space $\mathbb{S}$ is called a retract of $\mathbb{S}$ if there exists a continuous map $f : \mathcal{B} \to \mathbb{S}$ such that $f(z) = z$, for every $z \in \mathcal{B}$.*

Definition 2.9. *A set $\mathcal{B}$ is named contractible if there exists a (continuous) homotopy $H : \mathcal{B} \times [0, 1] \to \mathcal{B}$ and $z_0 \in \mathcal{B}$ such that*

(i) $H(z, 1) = z_0$, *for every* $z \in \mathcal{B}$.
(ii) $H(z, 0) = z$, *for every* $z \in \mathcal{B}$.

Definition 2.10. *A nonempty compact space $\mathfrak{C}$ is called an R_δ-set if there exists a decreasing sequence of compact, eventually nonempty, contractible spaces $(\mathfrak{C}_m)_{m\in\mathbb{N}}$ such that*

$$\mathfrak{C} = \bigcap_{m=1}^{\infty} \mathfrak{C}_m.$$

Lemma 2.11. [11] *Let $\mathbb{G}$ and $\mathbb{F}$ be normed spaces, and let $\mathfrak{G} : \mathbb{F} \to \mathbb{G}$ be a continuous map. Then, for every $\epsilon > 0$, there exists a locally Lipschitz map $\mathfrak{G}_\epsilon : \mathbb{F} \to \mathbb{G}$ such that*

$$\|\mathfrak{G}(z) - \mathfrak{G}_\epsilon(z)\| < \varepsilon, \text{ for every } z \in \mathbb{F}.$$

Theorem 2.12. [8] *Let $\mathbb{F}$ and $\mathbb{S}$ be Banach spaces, and $\mathfrak{G} : \mathbb{F} \to \mathbb{S}$ a proper map. Assume that for every $\epsilon > 0$, there exists a proper map $\mathfrak{G}_\epsilon : \mathbb{F} \to \mathbb{S}$ such that the following two conditions hold:*

(a) $\|\mathfrak{G}_\epsilon(z) - \mathfrak{G}(z)\| < \epsilon$, *for all* $z \in \mathbb{F}$.
(b) *For all $\epsilon > 0$ and $\widetilde{y} \in \mathbb{S}$ in a neighborhood of the origin such that $\|\widetilde{y}\| \leq \epsilon$, the equation $\mathfrak{G}_\epsilon(z) = \widetilde{y}$ admits exactly one solution z_ϵ.*

Then, $\mathfrak{G}^{-1}(0)$ is an R_δ-set.

Theorem 2.13. [22] *Let $\mathbb{S}$ be a Banach space, and $\mathbb{V} \subset \mathbb{S}$ be a closed, convex set containing the origin. Suppose that $\mathcal{Z} : \mathbb{V} \to \mathbb{V}$ is a condensing map. Then, one of the following alternatives holds:*

1. *$\mathcal{Z}$ has a fixed point; or*
2. *the set $\mathbb{K} = \{x \in \mathbb{V} : u = \zeta \mathcal{Z} x, \ \zeta \in (0, 1)\}$ is unbounded.*

The preceding theorem establishes the compactness of the solution set, and its proof follows the same lines as that of [5, Theorem 1.6.12].

A slight modification of [18, Theorem 1] yields the following result:

Lemma 2.14. *Let $\sigma > 0$, $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume the following conditions hold:*

(i) *$\psi \in \mathbb{L}^1(J, \mathbb{R}_+)$, and $\mathfrak{A}, \mathfrak{B}, \mathcal{W}, f$ are non-negative continuous functions defined on J;*
(ii) *$\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, positive, and non-decreasing function.*

If f fulfills the inequality:

$$f(t) \leq \mathfrak{A}(t) + \mathfrak{B}(t) \int_a^t \psi'(s) \varrho(t, s)^{\sigma-1} \mathcal{W}(s) \Theta(f(s)) ds, \quad t \in J.$$

Then,

$$f(t) \leq \left[\varpi^{-1} \left(\varpi(\mathfrak{A}_1(t)) + \mathfrak{B}_1(t) \int_a^t \mathcal{W}(s)^q \psi'(s) ds \right) \right]^{1/q}, \quad t \in J,$$

where:

$$\mathfrak{A}_1(t) = 2^{q-1} \sup_{a \leq s \leq t} \mathfrak{A}(s)^q,$$

$$\mathfrak{Z}_p = \left(\frac{\varrho(b,a)^{p(\sigma-1)+1}}{p(\sigma-1)+1} \right)^{\frac{1}{p}},$$

$$\mathfrak{B}_1(t) = \mathfrak{Z}_p 2^{q-1} \sup_{a \leq s \leq t} \mathfrak{B}(s)^q,$$

$$\varpi(z) = \int_{z_0}^{z} \frac{dr}{\left[\Theta(r^{1/q})\right]^q}, \quad \text{for } z_0, z > 0,$$

and ϖ^{-1} *is the inverse of* ϖ.

3. Main results

To give an Aronszajn-type result for problem (1.1), Theorem 2.13 and 2.12 are applied. Henceforth, the set of all solutions of problem (1.1) will be denoted by $\mathfrak{D}(J, \mathcal{W})$. Assume that:

(H1) The function $\mathcal{W} \in C(J \times \mathbb{F}, \mathbb{F})$.

(H2) There exists a real-valued function $K \in L_{\psi}^{1/\beta}(J, \mathbb{R}_+)$ and a constant $\beta \in (0, \sigma - 1)$ such that

$$\|\mathcal{W}(t, v_1) - \mathcal{W}(t, v_2)\| \leq K(t)\|v_1 - v_2\|, \quad \text{for each } v_1, v_2 \in \mathbb{F} \text{ and } t \in J.$$

(H3) There exists $\phi \in C(J, [0, \infty))$, and nondecreasing continuous functions $\Theta : [0, \infty) \to [0, \infty)$, such that

$$\|\mathcal{W}(t, u)\| \leq \phi(t)\big(\Theta(\|u\|) + 1\big), \quad \text{for each } t \in J \text{ and } u \in \mathbb{F}.$$

(H4) There exists a function $\Lambda \in C(J, \mathbb{R}_+)$, such that

$$\varsigma(\mathcal{W}(t, \mathbb{U}(t))) \leq \Lambda(t)\varsigma(\mathbb{U}(t)), \quad \text{for all } t \in J,$$

where $\mathbb{U}$ is bounded in $C(J, \mathbb{F})$.

For computational convenience, we denote $\phi^* = \sup_{t \in J} \phi(t)$ and $\Lambda^* = \sup_{t \in J} \Lambda(t)$.

3.1. Uniqueness result

This subsection presents the contractibility of $\mathfrak{D}(J, \mathcal{W})$, where the classical contraction principle is applied.

Theorem 3.1. *Assume that (H1)-(H2) hold. Additionally, it is supposed that*

$$\mho_\beta := \frac{(\sigma-1)\|K\|_{L_{\psi}^{\frac{1}{\beta}}}}{(\sigma-\beta)\Xi\varrho(b,a)^{\beta-\sigma}} \left(\frac{1-\beta}{\sigma-\beta-1} \right)^{1-\beta} < 1, \tag{3.1}$$

where $\Xi = e^{-\omega\varrho(b,a)}\Gamma(\sigma-1)$. *Then, the set* $\mathfrak{D}(J, \mathcal{W})$ *is a singleton in* $C(J, \mathbb{F})$. *Moreover,* $\mathfrak{D}(J, \mathcal{W})$ *is contractible.*

Proof. Invoking [27, Theorem 3.1], we define the operator $\mathcal{Z} : C(J,\mathbb{F}) \to C(J,\mathbb{F})$ by

$$(\mathcal{Z}x)(t) \quad = \int_a^t \frac{(\sigma-1)}{e^{\omega\varrho(t,s)}} \left(\mathcal{I}_{a^+}^{\sigma-1;\psi}\mathcal{W}(\tau,x(\tau))\right)(s)\psi'(s)ds, \quad t\in J. \tag{3.2}$$

Clearly, the set of all solutions to (1.1) coincides with the set of fixed points of the operator $\mathcal{Z}$. Therefore, it is enough to show that $\mathcal{Z}$ is a contraction.

Since $\dfrac{1}{\beta}+\dfrac{\beta-1}{\beta}=1$, then $\psi'(\tau)=(\psi'(\tau))^{\frac{1}{\beta}}(\psi'(\tau))^{\frac{\beta-1}{\beta}}$, using the Hölder inequality, we obtain

$$\begin{aligned}\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}K(\tau)d\tau \quad &\le \left(\int_a^s \psi'(\tau)\varrho(s,\tau)^{\frac{\sigma-2}{1-\beta}}d\tau\right)^{1-\beta}\left(\int_a^s \psi'(\tau)(K(\tau))^{\frac{1}{\beta}}d\tau\right)^{\beta}\\ &\le \|K\|_{L_\psi^{\frac{1}{\beta}}}\left(\frac{1-\beta}{\sigma-\beta-1}\right)^{1-\beta}\varrho(s,a)^{\sigma-\beta-1},\end{aligned} \tag{3.3}$$

since $e^{\omega\varrho(t,a)}\ge 1$ for each $t\in J$, we obtain

$$\frac{1}{e^{\omega\varrho(t,s)}}=\frac{e^{\omega\varrho(s,a)}}{e^{\omega\varrho(t,a)}}\le e^{\omega\varrho(s,a)}\le e^{\omega\varrho(b,a)}, \quad \text{for all } t\ge s>a. \tag{3.4}$$

For all $\mathfrak{r},x\in C(J,\mathbb{F})$ and each $t\in J$, by (H2), the relations (3.3) and (3.4) entails

$$\begin{aligned}&\|(\mathcal{Z}x)(t)-(\mathcal{Z}\mathfrak{r})(t)\|\\ &\le \frac{\sigma-1}{e^{-\omega\varrho(b,a)}}\int_a^t \frac{\psi'(s)}{\Gamma(\sigma-1)}\left(\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}K(\tau)\|x(\tau)-\mathfrak{r}(\tau)\|d\tau\right)ds\\ &\le \frac{\sigma-1}{e^{-\omega\varrho(b,a)}}\|x-\mathfrak{r}\|_\infty\left(\frac{1-\beta}{\sigma-\beta-1}\right)^{1-\beta}\frac{\|K\|_{L_\psi^{\frac{1}{\beta}}}}{\Gamma(\sigma-1)}\int_a^t \psi'(s)\varrho(s,a)^{\sigma-\beta-1}ds\\ &\le \frac{(\sigma-1)\|K\|_{L_\psi^{\frac{1}{\beta}}}}{(\sigma-\beta)\varrho(t,a)^{\beta-\sigma}\Xi}\left(\frac{1-\beta}{\sigma-\beta-1}\right)^{1-\beta}\|x-\mathfrak{r}\|_\infty.\end{aligned}$$

So, one has

$$\|\mathcal{Z}x-\mathcal{Z}\mathfrak{r}\|_\infty\le \mho_\beta\|x-\mathfrak{r}\|_\infty.$$

Therefore, $\mathcal{Z}$ is a contraction by virtue of condition (3.1). Applying the Banach contraction principle, it follows that $\mathcal{Z}$ has a unique fixed point, which implies that $\mathfrak{D}(J,\mathcal{W})=\{\widehat{\mathfrak{u}}\}$.

Now, we introduce the homotopy $\mathbb{H}:\mathfrak{D}(J,\mathcal{W})\times[0,1]\to\mathfrak{D}(J,\mathcal{W})$ by

$$\mathbb{H}(x,\vartheta)(t)=\begin{cases} x(t), & t\in(a,a(1-\vartheta)+b\vartheta],\\ \widehat{\mathfrak{u}}(t), & t\in(a(1-\vartheta)+b\vartheta,b].\end{cases}$$

In particular,

$$\mathbb{H}(x,\vartheta)=\begin{cases} x, & \text{for} \quad \vartheta=1,\\ \widehat{\mathfrak{u}}, & \text{for} \quad \vartheta=0.\end{cases}$$

Observe that $H(z,\lambda)(\cdot)\in\mathfrak{D}(J,\mathcal{W})$ for each $z\in\mathfrak{D}(J,\mathcal{W})$ and $\lambda\in[0,1]$, ensuring that H is well-defined. Our goal now is to prove that $\mathbb{H}(\cdot,\cdot)$ is contractive. To achieve this, we begin by establishing its continuity. Let $(x_n,\vartheta_n)\in\mathfrak{D}(J,\mathcal{W})\times[0,1]$ be such that $(x_n,\vartheta_n)\to(x,\vartheta)$, one has

$$\mathbb{H}(x_n,\vartheta_n)(t)=\begin{cases} x_n(t), & t\in(a,a(1-\vartheta_n)+b\vartheta_n],\\ \widehat{\mathfrak{u}}(t), & t\in(a(1-\vartheta_n)+b\vartheta_n,b].\end{cases}$$

We proceed by considering the following cases:

(i) If $\lim\limits_{n\to\infty}\vartheta_n=0$, it follows that

$$\mathbb{H}(x,0)(t)=\widehat{\mathfrak{u}}(t),\quad \text{for all} \quad t\in J.$$

Then,

$$\begin{aligned}\|\mathbb{H}(x_n,\vartheta_n)(t)-\mathbb{H}(x,\vartheta)(t)\| &\le \|\mathbb{H}(x_n,\vartheta_n)(t)-\mathbb{H}(x,\vartheta)(t)\|_{[a(1-\vartheta_n)+b\vartheta_n,b]}\\ &\quad+\|\mathbb{H}(x_n,\vartheta_n)(t)-\mathbb{H}(x,\vartheta)(t)\|_{[a,a(1-\vartheta_n)+b\vartheta_n]}\\ &\le \|\widehat{\mathfrak{u}}(t)-\widehat{\mathfrak{u}}(t)\|_{[a(1-\vartheta_n)+b\vartheta_n,b]}\\ &\quad+\|x_n(\xi)-x(t)\|_{[a,a(1-\vartheta_n)+b\vartheta_n]}\\ &\le \|x_n(t)-x(t)\|_{[a,a(1-\vartheta_n)+b\vartheta_n]}.\end{aligned}$$

Hence

$$\|\mathbb{H}(x_n,\vartheta_n)-\mathbb{H}(x,\vartheta)\|_\infty \quad \le \|x_n-x\|_\infty,$$

which tends to 0 when $n\to\infty$.

(ii) An analogous argument applies in the case $\lim\limits_{n\to\infty}\vartheta_n=1$.

(iii) If $0<\lim\limits_{n\to\infty}\vartheta_n=\vartheta<1$ with $\vartheta_n\neq 0$, then, we distinguish between two sub-cases:

(1). $x_n\in\mathfrak{D}(J,\mathcal{W})$. This entails that for $a\le t\le a(1-\vartheta_n)+b\vartheta_n$

$$x_n(t) \quad =(\sigma-1)\int_a^t \frac{\psi'(s)}{e^{\omega\varrho(t,s)}}\left(\mathcal{I}_{a^+}^{\sigma-1;\psi}\mathcal{W}(\tau,x_n(\tau))\right)(s)ds.$$

By (H1), for $t\in J$ one has

$$x(t) \quad =(\sigma-1)\int_a^t \frac{\psi'(s)}{e^{\omega\varrho(t,s)}}\left(\mathcal{I}_{a^+}^{\sigma-1;\psi}\mathcal{W}(\tau,x(\tau))\right)(s)ds.$$

(2). If $a(1-\vartheta_n)+b\vartheta_n<t\le b$, it follows that

$$\mathbb{H}(x_n,\vartheta_n)(t)=\widehat{\mathfrak{u}}(t)=\mathbb{H}(x,\vartheta)(t).$$

Then

$$\|\mathbb{H}(x_n,\vartheta_n)-\mathbb{H}(x,\vartheta)\|_\infty\to 0 \quad \text{as} \quad n\to\infty.$$

Accordingly, $\mathbb{H}$ is continuous, yielding the contractibility of $\mathfrak{D}(J,\mathcal{W})$ to the point $\widehat{\mathfrak{u}}$. □

3.2. R_δ**-property of** $\mathfrak{D}(J,\mathcal{W})$

This subsection establishes the topological structure of $\mathfrak{D}(J,\mathcal{W})$.

Theorem 3.2. *Assume that (H1) and (H3)-(H4) are fulfilled. Then* $\mathfrak{D}(J,\mathcal{W})$ *is an* R_δ*-set.*

Proof. Let us reintroduce the operator $\mathcal{Z}$ as defined in (3.2). For $T > 0$, we now define:

$$\mathbb{T}_T = \{x \in C(J,\mathbb{F}) \ : \ \|x\|_\infty < T\}.$$

The proof of Theorem 3.2 will proceed through several steps.

Step 1 : The continuity of the operator $\mathcal{Z}$.

Let $\{x_n\}$ be a sequence in $\mathbb{T}_T$ such that $x_n \to x$ when $n \to \infty$. Recalling (3.4), one has

$$\begin{aligned}
&\|(\mathcal{Z}x_n)(t) - (\mathcal{Z}x)(t)\| \\
&\leq \frac{\sigma-1}{\Xi}\int_a^t \psi'(s)\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\|\mathcal{W}(\tau,x_n(\tau)) - \mathcal{W}(\tau,x(\tau))\|d\tau ds \\
&\leq \frac{\sigma-1}{\Xi}\|\mathcal{W}(\cdot,x_n(\cdot)) - \mathcal{W}(\cdot,x(\cdot))\|_\infty \int_a^t \psi'(s)\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}d\tau ds \\
&\leq \frac{1}{\Xi}\|\mathcal{W}(\cdot,x_n(\cdot)) - \mathcal{W}(\cdot,x(\cdot))\|_\infty \int_a^t \psi'(s)\varrho(s,a)^{\sigma-1}ds \\
&\leq \frac{\varrho(t,a)^\sigma}{\sigma\Xi}\|\mathcal{W}(\cdot,x_n(\cdot)) - \mathcal{W}(\cdot,x(\cdot))\|_\infty.
\end{aligned}$$

From the continuity of $\mathcal{W}$, it follows that:

$$\|(\mathcal{Z}x_n)(t) - (\mathcal{Z}x)(t)\| \xrightarrow[n\to\infty]{} 0.$$

Then,

$$\|\mathcal{Z}x_n - \mathcal{Z}x\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Consequently, $\mathcal{Z}$ is continuous.

Step 2 : $\mathcal{Z}$ maps bounded sets of $C(J,\mathbb{F})$ into itself.

Let $x \in \mathbb{T}_T$. Using (3.4). Then, for each $t \in J$, one gets

$$\|(\mathcal{Z}x)(t)\| \ \leq \frac{\sigma-1}{\Xi}\int_a^t \left(\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\|\mathcal{W}(\tau,x(\tau))\|d\tau\right)\psi'(s)ds.$$

By (H3), for each $t \in J$, one has

$$\|\mathcal{W}(\tau,x(\tau))\| \leq \phi(\tau)\Theta(\|x(\tau)\|) + \phi(\tau) \leq \phi^*\Theta(\|x\|_\infty) + \phi^*. \tag{3.5}$$

So, the relation (3.5) implies

$$\begin{aligned}
\|(\mathcal{Z}x)(t)\| &\leq \frac{(\sigma-1)\phi^*(\Theta(T)+1)}{\Xi}\int_a^t \psi'(s)\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}d\tau ds\\
&\leq \frac{\phi^*(\Theta(T)+1)}{\Xi}\int_a^t \psi'(s)\varrho(s,a)^{\sigma-1}ds\\
&\leq \frac{\phi^*(\Theta(T)+1)}{\sigma\Xi}\varrho(t,a)^{\sigma}\\
&\leq \frac{\phi^*(\Theta(T)+1)}{\sigma\Xi}\varrho(b,a)^{\sigma} := \widehat{T}.
\end{aligned}$$

Thus,

$$\|\mathcal{Z}x\|_\infty \leq \widehat{T}.$$

Step 3 : The family $\mathcal{Z}(\mathbb{T}_T)$ is equicontinuous in $C(J,\mathbb{F})$.

For every $a \leq t_1 < t_2 \leq b$ and $x \in \mathbb{T}_T$, one obtains

$$\|(\mathcal{Z}x)(t_2)-(\mathcal{Z}x)(t_1)\| \quad \leq O_1 + O_2,$$

where

$$O_1 = \frac{\sigma-1}{\Xi}\int_{t_1}^{t_2}\psi'(s)\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\|\mathcal{W}(\tau,x(\tau))\|d\tau ds,$$

and

$$O_2 = (\sigma-1)\int_a^{t_1}\psi'(s)\Big|\frac{1}{e^{\omega\varrho(t_2,s)}}-\frac{1}{e^{\omega\varrho(t_1,s)}}\Big|\Big\|\left(\mathcal{I}_{a^+}^{\sigma-1;\psi}\mathcal{W}(\tau,x(\tau))\right)(s)\Big\|ds.$$

Using (H3) and (3.5), one gets

$$\begin{aligned}
O_1 &\leq \frac{(\sigma-1)\phi^*(\Theta(T)+1)}{\Xi}\int_{t_1}^{t_2}\psi'(s)\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}d\tau ds\\
&\leq \frac{\phi^*(\Theta(T)+1)}{\Xi}\int_{t_1}^{t_2}\psi'(s)\varrho(s,a)^{\sigma-1}ds\\
&\leq \frac{\phi^*(\Theta(T)+1)}{\sigma\Xi}\left[\varrho(t_2,a)^{\sigma}-\varrho(t_1,a)^{\sigma}\right].
\end{aligned}$$

Thus,

$$O_1 \xrightarrow[t_2\to t_1]{} 0. \tag{3.6}$$

On the other hand,

$$O_2 = \left(\frac{\sigma-1}{e^{\omega\varrho(t_1)}}-\frac{\sigma-1}{e^{\omega\varrho(t_2)}}\right)\int_a^{t_1}\frac{1}{e^{-\omega\varrho(s)}}\Big\|\left(\mathcal{I}_{a^+}^{\sigma-1;\psi}\mathcal{W}(\tau,x(\tau))\right)(s)\Big\|\psi'(s)ds.$$

Thus,

$$O_2 \xrightarrow[t_2\to t_1]{} 0. \tag{3.7}$$

It follows from (3.6) and (3.7) that both inequalities are independent of x and tend to zero as $t_2 \to t_1$. Accordingly, $\mathcal{Z}(\mathbb{T}_T)$ is equicontinuous.

Step 4 : $\mathcal{Z}$ is condensing.

Let $\mathbb{Y}$ be a bounded subset of $C(J,\mathbb{F})$, we define the MNC as follows:

$$\widehat{\varsigma}_{\mu}(\mathbb{Y}) = \sup_{t\in J} \frac{1}{e^{\mu t}} \varsigma(\mathbb{Y}(t)), \quad \text{for } \mu > 0. \tag{3.8}$$

Now, since $\psi'(\cdot)\psi(\cdot,a)^{\sigma-1} \in L^1(J,\mathbb{R})$, we can choose μ such that

$$2\mathfrak{q}(\mu) < 1. \tag{3.9}$$

where

$$\mathfrak{q}(\mu) := \frac{4\Lambda^*}{\Xi} \sup_{t\in J} \int_a^t \frac{\psi'(s)\varrho(s,a)^{\sigma-1}}{e^{\mu(t-s)}} ds.$$

Next, for a countable set $\{X^n\}_{n=1}^{+\infty} \subseteq \mathcal{Z}(\mathbb{Y})$ one has

$$\widehat{\varsigma}_{\mu}(\{X^n\}_{n=1}^{+\infty}) \leqslant \widehat{\varsigma}_{\mu}\left(\mathcal{Z}(\mathbb{Y})\right). \tag{3.10}$$

Then, there exists $\{x_n\}_{n=1}^{+\infty} \subset \mathbb{Y}$, such that

$$X^n(t) = (\mathcal{Z}x_n(t)), \quad \text{for } n \geq 1,\ t \in J. \tag{3.11}$$

After that, from

$$(\mathcal{Z}x_n)(t) \quad \leq \frac{\sigma-1}{\Xi} \int_a^t \int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\mathcal{W}(\tau,x_n(\tau))d\tau\psi'(s)ds, \tag{3.12}$$

so,

$$\begin{aligned} \widehat{\varsigma}_{\mu}(\{X^n(t)\}_{n=1}^{+\infty}) &= \widehat{\varsigma}_{\mu}\left(\{(\mathcal{Z}x_n)(t)\}_{n=1}^{+\infty}\right) \\ &\leq \widehat{\varsigma}_{\mu}\left(\left\{\frac{\sigma-1}{\Xi}\int_a^t\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\mathcal{W}(\tau,x_n(\tau))d\tau\psi'(s)ds\right\}_{n=1}^{+\infty}\right). \end{aligned} \tag{3.13}$$

By (H4), for $a \leq \tau \leq s$, we get

$$\begin{aligned} \varsigma\left(\left\{\frac{\psi'(\tau)}{\varrho(s,\tau)^{2-\sigma}}\mathcal{W}(\tau,x_n(\tau))\right\}_{n=1}^{+\infty}\right) &\leq \frac{\psi'(\tau)}{\varrho(s,\tau)^{2-\sigma}}\Lambda(\tau)\varsigma(\{x_n(\tau)\}_{n=1}^{+\infty}) \\ &\leq \frac{\Lambda(\tau)\psi'(\tau)e^{\mu\tau}}{\varrho(s,\tau)^{2-\sigma}}\widehat{\varsigma}_{\mu}(\{x_n(\tau)\}_{n=1}^{+\infty}). \end{aligned}$$

Using Lemma 2.6, for each $t \in J$, $s \in [a,t]$ and $\tau \leq s$, one gets

$$
\begin{aligned}
&\varsigma\left(\left\{\frac{\sigma-1}{\Xi}\int_a^t\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\mathcal{W}(\tau,x_n(\tau))d\tau\psi'(s)ds\right\}_{n=1}^{+\infty}\right)\\
&\le \frac{4(\sigma-1)\Lambda^*}{\Xi}\widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty})\int_a^t\int_a^s \frac{\psi'(\tau)e^{\mu\tau}}{\varrho(s,\tau)^{2-\sigma}}d\tau\psi'(s)ds\\
&\le \frac{4(\sigma-1)\Lambda^*}{\Xi}\widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty})\int_a^t \psi'(s)e^{\mu s}\int_a^s \frac{\psi'(\tau)}{\varrho(s,\tau)^{2-\sigma}}d\tau ds\\
&\le \frac{4\Lambda^*\widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty})}{\Xi}\int_a^t \frac{\psi'(s)e^{\mu s}}{\varrho(s,a)^{1-\sigma}}ds.
\end{aligned}
$$

Multiplying by $\exp(-\mu t)$, one obtains

$$
\begin{aligned}
&\sup_{t\in J}\frac{1}{e^{\mu t}}\varsigma\left(\left\{\frac{\sigma-1}{\Xi}\int_a^t\int_a^s \psi'(\tau)\varrho(s,\tau)^{\sigma-2}\mathfrak{t}(\tau,x_n(\tau))d\tau\psi'(s)ds\right\}_{n=1}^{+\infty}\right)\\
&\le \frac{4\Lambda^*\widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty})}{\Xi}\sup_{t\in J}\int_a^t \frac{\psi'(s)e^{-\mu(t-s)}}{\varrho(s,a)^{1-\sigma}}ds.
\end{aligned}
\tag{3.14}
$$

So, by (3.9), (3.13) and (3.14), we have

$$
\widehat{\varsigma}_\mu\left(\{(\mathcal{Z}x_n)(t)\}_{n=1}^{+\infty}\right) \le \mathfrak{q}(\mu)\widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty}). \tag{3.15}
$$

Therefore, one has

$$
\widehat{\varsigma}_\mu(\{x_n\}_{n=1}^{+\infty}) \le \widehat{\varsigma}_\mu(\mathbb{Y}) \le \widehat{\varsigma}_\mu(\mathcal{Z}(\mathbb{Y})) \le \frac{1}{2}\widehat{\varsigma}_\mu\left(\{X^n\}_{n=1}^{+\infty}\right). \tag{3.16}
$$

By (3.10) and (3.16), we can get

$$
\widehat{\varsigma}_\mu\left(\{X^n\}_{n=1}^{+\infty}\right) = 0.
$$

Then, $\widehat{\varsigma}_\mu(\mathbb{Y}) = 0$, which proves the compactness of the set $\overline{\mathbb{Y}}$. Accordingly, $\mathcal{Z}$ is condensing.

Step 5. The boundedness of set $\mathbb{K}$ (see Theorem 2.13 (2)).

Let $x \in C(J,\mathbb{F})$ and $x = \rho\mathcal{Z}x$ for some $\rho \in (0,1)$. Then, $x(t) \le \mathcal{Z}x(t)$. By applying Dirichlet's formula [23] together with (3.4) one obtains

$$
\begin{aligned}
\|x(t)\| &\le \frac{\sigma-1}{\Xi}\int_a^t \psi'(s)\left(\int_a^s \frac{\psi'(\tau)}{\varrho(s,\tau)^{2-\sigma}}\|\mathcal{W}(\tau,x(\tau))\|d\tau\right)ds\\
&\le \frac{\sigma-1}{\Xi}\int_a^t \psi'(s)\|\mathcal{W}(s,x(s))\|\int_s^t \frac{\psi'(\tau)}{\varrho(\tau,s)^{2-\sigma}}d\tau ds\\
&= \frac{1}{\Xi}\int_a^t \frac{\psi'(s)}{\varrho(t,s)^{1-\sigma}}\|\mathcal{W}(s,x(s))\|ds.
\end{aligned}
$$

Employing (H3), one gets

$$\begin{aligned}\|x(t)\| &\leq \frac{1}{\Xi}\int_a^t \psi'(s)\varrho(t,s)^{\sigma-1}\phi(s)\big(\Theta(\|x(s)\|)+1\big)ds\\ &\leq \frac{\phi^*}{\Xi}\int_a^t \frac{\psi'(s)}{\varrho(t,s)^{1-\sigma}}ds+\frac{1}{\Xi}\int_a^t \frac{\psi'(s)}{\varrho(t,s)^{1-\sigma}}\phi(s)\Theta(\|x(s)\|)ds\\ &\leq \mathcal{Q}_0+\mathcal{Q}_1\int_a^t \frac{\psi'(s)}{\varrho(t,s)^{1-\sigma}}\phi(s)\Theta(\|x(s)\|)ds,\end{aligned}$$

where $\mathcal{Q}_0 = \dfrac{\phi^*\varrho(b,a)^\sigma}{\sigma\Xi}$ and $\mathcal{Q}_1 = \dfrac{1}{\Xi}$. Using Lemma 2.14, we obtain

$$\|x(t)\| \leq \left[\varpi^{-1}\left(\varpi(\widehat{\mathcal{Q}}_0)+\widehat{\mathcal{Q}}_1\int_a^t \phi(s)^q\psi'(s)ds\right)\right]^{1/q} := \mathbf{N}, \quad t\in J,$$

where

$$\widehat{\mathcal{Q}}_0 = 2^{q-1}\mathcal{Q}_0^q, \quad \widehat{\mathcal{Q}}_1 = 2^{q-1}\mathfrak{Z}_p\mathcal{Q}_1^q, \quad \mathfrak{Z}_p = \left[\frac{\varrho(b,a)^{p(\sigma-1)+1}}{p(\sigma-1)+1}\right]^{\frac{1}{p}},$$

$\varpi(z) = \int_{z_0}^z \left[\Theta(z^{1/q})\right]^{-q}\psi'(s)ds$, for $z_0, z > 0$ and ϖ^{-1} is the inverse of ϖ. Hence, we obtain

$$\|x\|_\infty \leq \mathbf{N},$$

which entails the boundedness of the set $\mathbb{K}$. From Theorem 2.13, we conclude that $\mathfrak{D}(J,\mathcal{W})$ is non-empty and compact subset of $C(J,\mathbb{F})$.

Step 6. $\mathfrak{D}(J,\mathcal{W})$ is an R_δ-set.

Let $0<\epsilon_n<1$ with $\epsilon_n \xrightarrow[n\to\infty]{} 0$. Using (H1), thanks to Lemma 2.11, there exists a sequence $\{\mathcal{W}_n\}$ of locally Lipschitz functions such that

$$\|\mathcal{W}_n(t,y)-\mathcal{W}(t,y)\| < \epsilon_n, \quad \text{for all } y\in\mathbb{F} \text{ and } t\in J. \tag{3.17}$$

From relation (3.17) and (H3). Then

$$\|\mathcal{W}_n(t,y)\| \leq 1+\phi(t)\Theta(\|y\|)+\phi(t), \quad n\in\mathbb{N}\setminus\{0\}.$$

One can define $\mathcal{O}_n$ by

$$\mathcal{O}_n(x)(t) = \frac{\sigma-1}{\Gamma(\sigma-1)}\int_a^t \frac{\psi'(s)}{e^{\omega\varrho(t,s)}}\left(\int_a^s \frac{\psi'(\tau)}{\varrho(s,\tau)^{2-\sigma}}\mathcal{W}_n(\tau,x(\tau))d\tau\right)ds, \quad t\in J. \tag{3.18}$$

Theorem 3.1, together with the local Lipschitz continuity of $\mathcal{W}_n$, implies that (3.18) is uniquely solvable.

Let

$$\mathbb{M}(x) = (I-\mathcal{O})(x).$$

By Step 4, the approximate operators $\mathcal{O}_n : C(J,\mathbb{F}) \to C(J,\mathbb{F})$ are condensing, which allows us to define

$$\mathbb{M}_n(x) = (I - \mathcal{O}_n)(x)$$

where each $\mathbb{M}_n$ is a condensing perturbation of the identity and, by Lemma 2.7, proper maps.

Now, the relation (3.17) permits the uniform convergence of $\{\mathbb{M}_n\}$ to $\mathbb{M}$ in $C(J,\mathbb{F})$.

Invoking (3.4), we get

$$\begin{aligned}\|\mathbb{M}_n(x)(t) - \mathbb{M}(x)(t)\| &\leq \frac{\sigma-1}{\Xi}\int_a^t \psi'(s)\int_a^s \frac{\psi'(\tau)}{\varrho(s,\tau)^{2-\sigma}}\|\mathcal{W}_n(\tau,x(\tau)) \\ &\quad -\mathcal{W}(\tau,x(\tau))\|d\tau ds \\ &\leq \frac{\sigma-1}{\Xi}\varrho(b,a)^{\sigma}\varepsilon_n, \quad t\in J,\end{aligned}$$

and equation $\mathbb{M}_n(x) = y$ has a unique solution for all $y \in C(J,\mathbb{R})$ as well as (3.18).

Consequently, the assertions of Theorem 2.12 are verified. Therefore, the solution set $\mathbb{M}^{-1}(0)$ is an R_δ-set. □

4. Illustrative examples

In this section we introduce some examples that illustrate our theoretical results. Consider the Banach space

$$\mathbb{F} = c_0 = \{x = (x_1, x_2, \cdots, x_n, \cdots) \ : \ x_n \xrightarrow[n\to\infty]{} 0\}$$

endowed with

$$\|x\|_{c_0} = \sup_{n\in\mathbb{N}\setminus\{0\}} |x_n|.$$

Take $\psi(t) = t$. Let us define the function $\mathcal{W} : J \times c_0 \to c_0$ by

$$\begin{aligned}\mathcal{W}(t,x) = \Big\{ &\frac{13(b-a)^{\beta-\sigma}}{(1+e^t)e^{\eta t+\omega(b-a)}} + \frac{(b-a)^{\beta-\sigma}\tan^{-1}(|x_n|)}{(1+e^t)e^{\eta t+\omega(b-a)}} \\ &+ \frac{(b-a)^{\beta-\sigma}}{(1+e^t)e^{\eta t+\omega(b-a)}}\frac{|x_n|}{1+|x_n|}\Big\}_{n\geq 1}\end{aligned} \tag{4.1}$$

where η is a positive constant.

Obviously, the function $\mathcal{W}$ is continuous and for any $u, v \in c_0$ and $t \in J$, one can verify that

$$\begin{aligned}\|\mathcal{W}(t,z_1) - \mathcal{W}(t,z_2)\| &\leq \frac{2(b-a)^{\beta-\sigma}}{(1+e^t)e^{\eta t+\omega(b-a)}}\|z_1 - z_2\| \\ &\leq \frac{2(b-a)^{\beta-\sigma}}{(1+e^a)e^{\eta t+\omega(b-a)}}\|z_1 - z_2\|.\end{aligned}$$

So, the assumption (H2) fulfills with

$$K(t) = \frac{2(b-a)^{\beta-\sigma}}{(1+e^a)e^{\eta t+\omega(b-a)}} \quad \text{for} \quad t \in J.$$

Now, for $0 < \beta < \sigma - 1$, let $K^* = \left\| \frac{2e^{-\eta t}}{1+e^a} \right\|_{L^{\frac{1}{\beta}}}$. Choosing a suitable $0 < \beta < \sigma - 1$ and $\eta > 0$ large enough, it follows that

$$\mho_\beta < 1.$$

Therefore, all the conditions of Theorem 3.1 are met. Then, the set of solutions of (1.1) with $\mathcal{W}$ defined by (4.1) is a singleton.

Now, for $t \in J$ and $x = \{x_n\}_n \in c_0$, consider the forcing nonlinearities,

$$\mathcal{W}(t,x) = \left\{ \frac{3}{13+e^t} + \frac{4\tan^{-1}(|x_n|)}{13+e^t} + \frac{1}{2^n \cdot 13 + 2^n e^t} \right\}_{n \geq 1}. \tag{4.2}$$

Clearly, $\mathcal{W}$ satisfies (H1). To illustrate (H2), let $t \in J$ and $x = \{x_n\}_n \in \mathbb{Y} \subset c_0$. Then

$$\|\mathcal{W}(t,x)\| \leq \frac{4(1+\|x\|)}{e^t+13} \leq \phi(t)\Theta(\|x\|) + \phi(t). \tag{4.3}$$

Then, (H2) is fulfilled with

$$\Theta(y) = y, \text{ for all } y \in \mathbb{R}_+, \quad \text{and} \quad \phi(t) = \frac{4}{e^t+13} \text{ for all } t \in J.$$

Next, assumption (H3) is fulfilled. Indeed, the Hausdorff MNC χ in $(c_0, \|\cdot\|_{c_0})$ can be defined by the formula

$$\chi(\mathbb{Y}) = \lim_{n\to\infty} \sup_{x\in\mathbb{Y}} \|(\mathrm{I}-\mathrm{P}_n)\, x\|_\infty,$$

where $\mathbb{Y} \in \mathcal{P}(c_0)$, P_n denotes the projection of the first n vectors onto the linear span in the standard basis (see [5]).

Using (4.3) (see also [25, 26]), and the relationship between the Hausdorff χ and Kuratowski ς MNCs, given by the inequality

$$\chi(\mathbb{Y}) \leq \varsigma(\mathbb{Y}) \leq 2\chi(\mathbb{Y}),$$

one gets

$$\varsigma\left(\mathcal{W}(t,\mathbb{Y}(t))\right) \leq \Lambda(t)\varsigma(\mathbb{Y}(t)), \quad \text{for each } t \in J,$$

where

$$2\phi(t) = \Lambda(t).$$

Therefore, all assumptions of Theorem 3.2 are fulfilled, the solution set of (1.1), with $\mathcal{W}$ defined by (4.2), is an R_δ-set.

5. Conclusion

Under Nagumo-type growth conditions, we present novel results concerning the topological structure of the solution set for nonlinear ψ-Caputo fractional Langevin equations in Banach spaces. Our approach unfolds in multiple stages: the initial stage utilizes the classical contraction mapping principle, while the second is grounded in the nonlinear alternative for condensing operators, incorporating the Browder-Gupta technique (Theorem 2.12). These findings offer meaningful contributions to the evolving study of such equations.

Further developments, including periodic solutions and approximate controllability, will be addressed in a forthcoming work.

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