

Properties of some univalent functions associated with balloon-shaped domain

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Abstract. This paper introduces a novel subclass of analytic functions defined by the product of a modified sigmoid function and the lemniscate Bernoulli function. We initiate the study by deriving initial coefficient bounds for functions within this subclass, followed by an investigation into several key analytic properties. Specifically, we establish the Fekete–Szegő inequality and analyze Hankel determinants of various orders. Furthermore, the study provides estimates for the logarithmic coefficients and establishes bounds for both the inverse coefficients and the logarithmic inverse coefficients of functions in the subclass. This comprehensive analysis offers a significant contribution to the theory of analytic function subclasses associated with the products of special functions.

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1. Introduction


Let \mathcal{A} denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{A}$, then f has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let \mathcal{S} denote the class of all univalent (i.e., one-to-one) functions in \mathcal{A} . For given analytic functions f and g in \mathbb{D} , we say f is subordinate to g in \mathbb{D} and write $f \prec g$ if there exists an analytic function w with the property $w(0) = 0$ and $|w(z)| < 1$ such

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that $f(z) = g(w(z)), (z \in \mathbb{D})$. Moreover, if the function g is univalent in \mathbb{D} , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. A function $f \in \mathcal{A}$ is called starlike (convex respectively) if $f(\mathbb{D})$ is starlike with respect to the origin (convex respectively). Let \mathcal{S}^* and \mathcal{C} denote the class of starlike and convex functions in \mathcal{S} respectively. It is well known that a function $f \in \mathcal{A}$ is in \mathcal{S}^* if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad \text{for } z \in \mathbb{D}.$$

Similarly, a function $f \in \mathcal{A}$ is in \mathcal{C} if and only if

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad \text{for } z \in \mathbb{D}.$$

From the above it is easy to see that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is said to be close-to-convex with argument α and with respect to g if

$$\operatorname{Re} \left(e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.2}$$

Using the notion of subordination between two analytic functions, Ma and Minda [19] introduced the more general class $\mathcal{S}^*(\phi)$ by

$$\mathcal{S}^*(\phi) := \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec \phi(z) \right\},$$

where ϕ is a regular function with a positive real part in \mathbb{D} , with $\phi(0) = 1$ and $\phi'(0) > 0$. In addition, the function ϕ maps \mathbb{D} onto a star-shaped region with respect to $\phi(0) = 1$ and is symmetric with respect to the real axis. Varying the function ϕ we can obtain several familiar subclasses of the class $\mathcal{S}^*(\phi)$, and for details see [16] and references within.

The classic Fekete–Szegő problem [11] involves finding the exact limits of the functional $|a_3 - \mu a_2^2|$ for a compact-function family or $f \in \mathcal{A}$ with any $\mu \in \mathbb{C}$; for further details, one may refer to [27].

Pommerenke provided the following *Hankel determinant* in [21],[22], denoted by $\mathcal{H}_{q,n}(f)$, which contains the coefficients of a function $f \in \mathcal{S}$:

$$\mathcal{H}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

with $q, n \in \mathbb{N} := \{1, 2, \dots\}$. Therefore, by altering the parameters q and n we obtain the following Hankel determinants:

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad \mathcal{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

That denote the first and the second order Hankel determinants. There are a few references in the literature to the Hankel determinant for functions in the general family \mathcal{S} . The best-known sharp inequality for the function $f \in \mathcal{S}$ is $\mathcal{H}_{2,n}(f) \leq \kappa\sqrt{n}$,

where κ is a constant, and it is due to Hayman ([12] Theorem 1). Additionally, for the class \mathcal{S} , it was found in [20] that

$$|\mathcal{H}_{2,2}(f)| \leq \kappa, \quad \text{where } 1 \leq \kappa \leq \frac{11}{3} \simeq 3.66 \dots$$

The precise bounds of Hankel determinants for a given family of functions have piqued the interest of several mathematicians. For the three well-known subfamilies of the set \mathcal{S} that are \mathcal{C} , \mathcal{S}^* , and \mathcal{R} (convex, starlike, and functions of a bounded turning, respectively), Janteng et al. ([13],[14]) computed the sharp bounds $|\mathcal{H}_{2,2}(f)|$. These bounds are provided by

$$|\mathcal{H}_{2,2}(f)| \leq \begin{cases} \frac{1}{8}, & \text{for } f \in \mathcal{C}, \\ 1, & \text{for } f \in \mathcal{S}^*, \\ \frac{4}{9}, & \text{for } f \in \mathcal{R}. \end{cases}$$

Moreover, the sharp bounds of this determinant for a few subclasses \mathcal{S}^* and \mathcal{K} were found in [17] and subsequently studied in [9]. This problem was solved for various families of bi-univalent functions in [2, 4, 15]. The logarithmic coefficients of $f \in \mathcal{S}$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \tag{1.3}$$

where γ_n are known as the logarithmic coefficients. The logarithmic coefficients γ_n play a central role in the theory of univalent functions. Very few exact upper bounds for γ_n seem to have been established. The significance of this problem in the context of the Bieberbach conjecture was pointed out by Milin in his conjecture. Milin conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^n \sum_{k=1}^m \left(k |\gamma_k|^2 - \frac{1}{k} \right) \leq 0, \tag{1.4}$$

which led de Branges, by proving this conjecture, to the proof of the Bieberbach conjecture [6]. More attention has been given to the results in an average sense [7, 8] than the exact upper bounds for $|\gamma_n|$. For the Koebe function $k(z) = \frac{z}{(1-z)^2}$, the logarithmic coefficients are $\gamma_n = \frac{1}{n}$. Since the Koebe function $k(z)$ plays the role of extremal function for most of the extremal problems in the class \mathcal{S} , it is expected that $|\gamma_n| \leq \frac{1}{n}$ holds for functions in \mathcal{S} . But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class \mathcal{S} with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ ([7, Theorem 8.4]). Now equating the coefficients of (1.3) we obtain

$$\begin{aligned} \gamma_1 &= \frac{a_2}{2}, & \gamma_2 &= \frac{1}{2} \left(a_3 - \frac{a_2^2}{2} \right), & \gamma_3 &= \frac{1}{2} \left(a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right), \\ \gamma_4 &= \frac{1}{2} \left(a_5 - a_2 a_4 - \frac{1}{2} a_3^2 + a_2^2 a_3 - \frac{1}{4} a_2^4 \right). \end{aligned} \tag{1.5}$$

The famous Koebe 1/4-theorem ensures that, for each univalent function f defined in \mathbb{D} , its inverse f^{-1} exists at least on a disc of radius 1/4 with Taylor’s series of the form representation

$$f^{-1}(\omega) = \omega + h_2\omega^2 + h_3\omega^3 + h_4\omega^4 + \dots. \tag{1.6}$$

Using the representation $f(f^{-1}(\omega)) = \omega$, we obtain

$$\begin{aligned} h_2 &= -a_2, & h_3 &= -a_3 + 2a_2^2, & h_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ h_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.7}$$

In 1959, Sakaguchi [26] introduced the class of starlike functions with respect to symmetric points as:

$$S_S^* := \left\{ f \in \mathcal{S} : \Re \left(\frac{2zf'(z)}{f(z) - f(-z)} \right) > 0; z \in \mathbb{D} \right\}.$$

These functions are also known as Sakaguchi functions which are close-to-convex as well as univalent. In 2004, making use of subordination between two analytic functions Ravichandran [24] introduced a unified class $S_s^*(\phi)$ as follows:

$$S_S^*(\phi) = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z); z \in \mathbb{D} \right\},$$

where $\phi(z) = 1 + \sum_{n=1}^{\infty} D_n z^n$ is univalent starlike function with respect to 1 which maps \mathbb{D} onto a symmetric region with respect to real axis in the right half plane. A good amount of literature is available for finding the upper bounds of coefficient functional for several subclasses of the class S_s^* . For details see [16] and reference within. Motivated by aforementioned works, we introduce the following subclass.

Definition 1.1. A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class S_*^r if the below condition holds true:

$$S_*^r = \left\{ f \in \mathcal{S} : \frac{(1+r)zf'(z)}{f(z) - rf(-z)} \prec \frac{2\sqrt{1+z}}{1+e^{-z}}; z \in \mathbb{D} \right\}. \tag{1.8}$$

Provided the denominator $f(z) - rf(-z) \neq 0$.

- Starlike Functions (S_*^0): By setting $r = 0$, the denominator simplifies to $f(z)$ and the scaling factor $(1+r)$ becomes unity. This reduces the condition to the cornerstone of geometric function theory:

$$\frac{zf'(z)}{f(z)} \prec \frac{2\sqrt{1+z}}{1+e^{-z}}, \quad z \in \mathbb{D}.$$

In this state, the function maps the unit disk onto a domain that is starlike with respect to the origin.

- Sakaguchi-Type Functions (S_*^1): By setting $r = 1$, the expression transforms into the functional defined by Sakaguchi:

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{2\sqrt{1+z}}{1+e^{-z}}, \quad z \in \mathbb{D}.$$

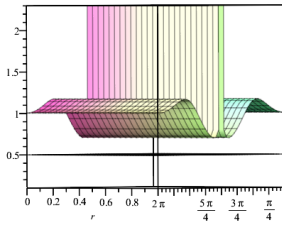
Functions satisfying this condition are known as starlike with respect to symmetric points.

Next, we need to demonstrate that the chosen class is well-defined and non-empty. Additionally, the class is characterized by the product of a sigmoid function and a Bernoulli-shaped domain. It is symmetric with respect to the origin and satisfies the Ma-Minda condition.

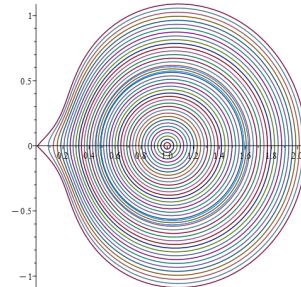
Remark 1.2. (i) Note that $p(z) := \frac{2\sqrt{1+z}}{1+e^{-z}}$ is chosen correctly, because $p(0) = 1$ and $p'(z) = \frac{2e^{-z}z+3e^{-z}+1}{\sqrt{1+z}(1+e^{-z})^2}$, hence $p'(0) = 1 \neq 0$. Also, we can see in the Figure 1a that

$$\begin{aligned}
 P(z) &= \operatorname{Re} \frac{zp'(z)}{p(z) - p(0)} = \operatorname{Re} \frac{zp'(z)}{p(z) - 1} \\
 &= \operatorname{Re} \left(-\frac{(2e^{-z}z + 3e^{-z} + 1)z}{(-2\sqrt{1+z} + 1 + e^{-z})\sqrt{1+z}(1 + e^{-z})} \right) > 0, \quad z \in \mathbb{D},
 \end{aligned}$$

and using this fact together with $p'(0) = 1 \neq 0$ it follows that $p(z) = \frac{2\sqrt{1+z}}{1+e^{-z}}$ is also a starlike (univalent) function in \mathbb{D} and because $p(\bar{z}) = \overline{p(z)}$, $z \in \mathbb{D}$, the domain $p(\mathbb{D})$ is symmetric with respect to the real axis see Figure 1b.



(A) The image of $P(re^{it})$, $r \in [0, 1], t \in [0, 2\pi]$



(B) The image of $p(\mathbb{D})$ (Balloon-Shaped Domain)

FIGURE 1. Figures for Remark 1.2 (i).

(ii) The class is well chosen if we could prove that it isn't empty. First, to show that the class is non-empty for $r = 0$, take $f(z) = z + 0.3z^2 + 0.2z^3 \in \mathcal{A}$ such that

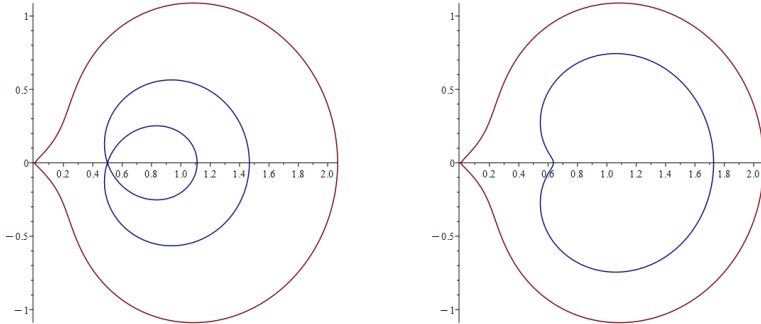
$$\phi(z) = \frac{zf'(z)}{f(z)} = \frac{0.6z^2 + 0.6z + 1.0}{0.2z^2 + 0.3z + 1.0}, z \in \mathbb{D}.$$

It can be seen from Figure 2a that the class is non-empty.

(iii) Now for $r = 1$ the class is non empty, if we take $f(z) = z + 0.3z^2 + 0.1z^3 \in \mathcal{A}$ such that

$$\psi(z) = \frac{2zf'(z)}{f(z) - f(-z)} = \frac{0.6z^2 + 1.2z + 2.0}{0.2z^2 + 2.0}, z \in \mathbb{D}.$$

It can be seen from Figure 2b. That the class is non-empty.



(A) The images of $\phi(e^{it})$ (blue) and $p(e^{it})$ (red), $t \in [0, 2\pi)$, (for $r=0$)

(B) The images of $\psi(e^{it})$ (blue) and $p(e^{it})$ (red), $t \in [0, 2\pi)$, (for $r=1$)

FIGURE 2. Figures for Remark 1.2 (ii) and (iii).

The corresponding extremal function for the defined class is given by

$$f(z) = z + \frac{(1+r)z^2}{1+3r} + \frac{(9-5r)z^3}{16(1+3r)} + \frac{(1+r)(31-27r)z^4}{48(5r+3)(1+3r)} + \frac{(7r^2-70r+31)z^5}{192(5r+3)(1+3r)} + \dots$$

Now for $r = 0$ the corresponding extremal function is

$$f(z) = z + z^2 + \frac{9}{16}z^3 + \frac{31}{144}z^4 + \frac{31}{576}z^5 + \dots$$

Now for $r = 1$ the corresponding extremal function is

$$f(z) = z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{1}{192}z^4 - \frac{1}{192}z^5 + \dots$$

In this study we adopt a novel methodology centered on establishing a direct correspondence between the coefficients of functions within a specific class and those of their associated Carathéodory functions. In various instances, this framework allows for the intuitive prediction of precise functional estimates, streamlining the subsequent computational process — a characteristic particularly evident within the class under investigation. By integrating the foundational lemmas with refined analytical techniques and more rigorous calculus, we have established enhanced bounds, the majority of which are sharp. The following lemmas serve as the essential theoretical groundwork required to substantiate these findings.

2. Preliminaries

Let us define by \mathcal{P} the well-known Carathéodory class i.e., the family of holomorphic functions d in \mathbb{D} that satisfies the condition $\text{Re } d(z) > 0, z \in \mathbb{D}$, and of the form

$$d(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, z \in \mathbb{D}. \tag{2.1}$$

We need the following lemmas in order to prove our results.

Lemma 2.1. *Let $d \in \mathcal{P}$ be of the form (2.1).*

(i) *Then, for $n \geq 1$*

$$|d_n| \leq 2. \tag{2.2}$$

The inequality holds for all $n \geq 1$ if and only if $d(z) = \frac{1 + \lambda z}{1 - \lambda z}, |\lambda| = 1$.

(ii) *Also, if $\mu \geq 0$ then*

$$|d_{n+k} - \mu d_n d_k| \leq 2 \max \{1; |2\mu - 1|\} = \begin{cases} 2, & \text{if } 0 \leq \mu \leq 1, \\ 2|2\mu - 1|, & \text{otherwise.} \end{cases} \tag{2.3}$$

If $0 < \mu < 1$, the inequality is sharp for the function $d(z) = \frac{1 + z^{n+k}}{1 - z^{n+k}}$. In the other cases, the inequality is sharp for the function $d(z) = \frac{1 + z}{1 - z}$.

Note that the inequality (2.2) is the well-known result of the Carathéodory Lemma [5] (see also [23, Corollary 2.3, p. 41] and [7, Carathéodory Lemma, p. 41]). Inequality (2.3) represents Lemma 2.3 of [25], that for $\mu = 1$ was proved in a more general form in [18, Lemma 1, p. 546]. We emphasize that the inequality (2.3) remains valid for all $\mu \in \mathbb{C}$ as it was proved in [10, Theorem 1]

Lemma 2.2. [3, Lemma 2.2.] *If $d \in \mathcal{P}$ has the form (2.1), then*

$$|\alpha d_1^3 - \beta d_1 d_2 + \gamma d_3| \leq 2(|\alpha| + |\beta - 2\alpha| + |\alpha - \beta + \gamma|). \tag{2.4}$$

Lemma 2.3. [1, Lemma 3, p. 66] *Let $d \in \mathcal{P}$ and has the expansion of the form (2.1). If $B \in [0, 1]$ with $B(2B - 1) \leq D < B$, then*

$$|d_3 - 2Bd_1 d_2 + Dd_1^3| \leq 2. \tag{2.5}$$

3. Coefficient estimates and Fekete-Szegő inequality

In this section, we aim to examine the upper bounds of the first four initial coefficients, along with the Fekete–Szegő functional $|a_3 - \mu a_2^2|$, for the considered class S_*^r . The focus is placed on deriving appropriate upper estimates for these coefficients within the framework of the class S_*^r .

Theorem 3.1. *Let the function $f \in \mathcal{A}$ be of the form (1.1) belongs to the class S_*^r . Then*

$$|a_2| \leq \frac{1+r}{1+3r}, \quad |a_3| \leq \frac{9-5r}{16(1+3r)}, \quad |a_4| \leq \frac{|(57r-5)(1+r)| + (1+r)(519r+197)}{192(1+3r)(5r+3)},$$

$$|a_5| \leq \frac{|227r^2 + 10r - 13| + 6349r^2 + 3926r + 253}{768(1+3r)(5r+3)} + \frac{1}{4}. \tag{3.1}$$

Proof. If the function $f \in \mathcal{A}$ of the form (1.1) belongs to the class S_*^r , then by Definition 1.1 there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1, (z \in \mathbb{D})$ such that

$$\frac{(1+r)zf'(z)}{f(z) - rf(-z)} = \frac{2\sqrt{1+w(z)}}{1+e^{-w(z)}}. \tag{3.2}$$

Writing the analytic function w in terms of $d \in \mathcal{P}$, that is

$$d(z) = \frac{1+w(z)}{1-w(z)} = 1 + d_1z + d_2z^2 + \dots, \quad z \in \mathbb{D}.$$

To present the transition from the Schwarz function $w(z)$ to the coefficients of the Carathéodory function $d(z)$ we get

$$w(z) = \frac{1}{2}d_1z + \left(\frac{1}{2}d_2 - \frac{1}{4}d_1^2\right)z^2 + \left(\frac{1}{8}d_1^3 - \frac{1}{2}d_1d_2 + \frac{1}{2}d_3\right)z^3$$

$$+ \left(\frac{1}{2}d_4 - \frac{1}{2}d_1d_3 - \frac{1}{4}d_2^2 - \frac{1}{16}d_1^4 + \frac{3}{8}d_1^2d_2\right)z^4 + \dots, \quad z \in \mathbb{D}.$$

Now substituting $w(z)$ in the right-hand side of (3.2) we get

$$\frac{2\sqrt{1+w(z)}}{1+e^{-w(z)}} = 1 + \frac{1}{2}d_1z + \left(\frac{d_2}{2} - \frac{7d_1^2}{32}\right)z^2 + \left(\frac{1}{2}d_3 - \frac{7}{16}d_1d_2 + \frac{17}{192}d_1^3\right)z^3$$

$$+ \left(\frac{1}{2}d_4 - \frac{7}{16}d_1d_3 - \frac{7}{32}d_2^2 + \frac{17}{64}d_2d_1^2 - \frac{203}{6144}d_1^4\right)z^4 + \left(-\frac{7}{16}d_1d_4 - \frac{7}{16}d_3d_2 + \frac{17}{64}d_3d_1^2\right)z^5$$

$$+ \frac{17}{64}d_1d_2^2 - \frac{203}{1536}d_2d_1^3 + \frac{1}{2}d_5 + \frac{733}{61440}d_1^5z^5 + \dots. \tag{3.3}$$

Expanding the left-hand side of (3.2) in a Maclaurin series with respect to z , we obtain

$$\frac{(1+r)zf'(z)}{f(z) - rf(-z)} = 1 + \left(\frac{a_2(1+3r)}{1+r}\right)z + \left(\frac{(2r^2+4r+2)a_3}{(1+r)^2} + \frac{(3r^2-2r-1)a_2^2}{(1+r)^2}\right)z^2$$

$$+ \left(\frac{(5r^3+13r^2+11r+3)a_4}{(1+r)^3} + \frac{(-r^3-5r^2-7r-3)a_2a_3}{(1+r)^3} + \frac{(3r^3-5r^2+r+1)a_2^3}{(1+r)^3}\right)z^3$$

$$+ \left(\frac{(4r^4+16r^3+24r^2+16r+4)a_5}{(1+r)^4} + \frac{(8r^4+12r^3-4r^2-12r-4)a_2a_4}{(1+r)^4}\right)z^4$$

$$- \frac{(2r^4+8r^3+12r^2+8r+2)a_3^2}{(1+r)^4} - \frac{4(r-1)}{(1+r)}a_2^2a_3 + \frac{(3r+1)(r-1)^3}{(1+r)^4}a_2^4z^4 + \dots. \tag{3.4}$$

Comparing the coefficients of (3.3) and (3.4) we get

$$a_2 = \frac{d_1(1+r)}{2(1+3r)}. \tag{3.5}$$

$$a_3 = \frac{d_2}{4} - \frac{(29r-1)d_1^2}{64(1+3r)} = \frac{1}{4} \left(d_2 - \frac{(29r-1)}{16(1+3r)}d_1^2 \right). \tag{3.6}$$

$$a_4 = \frac{(57r-5)(1+r)d_1^3}{384(1+3r)(5r+3)} - \frac{3(1+19r)(1+r)d_2d_1}{2(1+3r)(5r+3)} + \frac{(1+r)d_3}{2(5r+3)}. \tag{3.7}$$

$$a_5 = -\frac{(227r^2+10r-13)d_1^4}{3072(1+3r)(5r+3)} + \frac{(119r^2+42r-1)d_1^2d_2}{128(1+3r)(5r+3)} - \frac{(137r^2+82r+5)d_3d_1}{64(1+3r)(5r+3)} - \frac{(1080r^2+1008r+216)d_2^2}{3072(1+3r)(5r+3)} - \frac{(-5760r^2-5376r-1152)d_4}{3072(1+3r)(5r+3)}. \tag{3.8}$$

Now taking modulus in (3.5) and applying (2.2) of Lemma 2.1 we get bound for $|a_2|$. After that taking modulus on both sides of (3.6) and applying (2.3) of Lemma 2.1 we get the desired estimation for $|a_3|$. Now taking modulus in (3.7) and applying (2.4) of Lemma 2.2 we get our desired estimation for $|a_4|$. Now rearranging the terms of a_5 we get

$$a_5 = - \left[\left(\frac{(227r^2+10-13)d_1^3}{3072(1+3r)(5r+3)} - \frac{(119r^2+42r-1)d_1d_2}{128(1+3r)(5r+3)} + \frac{(137r^2+82r+5)d_3}{64(1+3r)(5r+3)} \right) d_1 - \frac{1}{8} \left(d_4 - \frac{3d_2^2}{16} \right) \right]. \tag{3.9}$$

Now taking modulus on both sides of (3.9) and applying triangle inequality we get

$$|a_5| \leq \left| \frac{(227r^2+10r-13)d_1^3}{3072(1+3r)(5r+3)} - \frac{(119r^2+42r-1)d_1d_2}{128(1+3r)(5r+3)} + \frac{(137r^2+82r+5)d_3}{64(1+3r)(5r+3)} \right| |d_1| + \frac{1}{8} \left| d_4 - \frac{3d_2^2}{16} \right|. \tag{3.10}$$

Using (2.4) for $\alpha = \frac{(227r^2+10r-13)}{3072(1+3r)(5r+3)}$, $\beta = \frac{(119r^2+42r-1)}{128(1+3r)(5r+3)}$, $\gamma = \frac{(137r^2+82r+5)}{64(1+3r)(5r+3)}$ in the first part of (3.10) and applying (2.3) of Lemma 2.1 for $\mu = \frac{3}{16}$ in the second part of (3.10) we get

$$|a_5| \leq \frac{|227r^2+10r-13|+6349r^2+3926r+253}{768(1+3r)(5r+3)} + \frac{1}{4}.$$

Here we get the desired estimation for $|a_5|$. □

The next theorem gives the bound of Fekete- Szegő functional for the class S_*^r .

Theorem 3.2. *If $f \in S_*^r$ has the form (1.1), then for any complex number μ we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{2} \max \left\{ 1; \left| \frac{2\mu(1+r)^2}{(1+3r)^2} + \frac{5r-9}{8(1+3r)} \right| \right\}.$$

Proof. If $f \in S_*^r$, making use of (3.5) and (3.6) we obtain

$$a_3 - \mu a_2^2 = \frac{1}{4} \left[d_2 - \left(\frac{29r - 1}{16(1 + 3r)} + \frac{(1 + r)^2 \mu}{(1 + 3r)^2} \right) d_1^2 \right] \tag{3.11}$$

Taking the modulus on both sides of (3.11) and applying (2.3) of Lemma 2.1 we get the desired estimation for this. \square

Letting $\mu = 1$ in Theorem 3.2 we obtain the following result.

Corollary 3.3. *If $f \in S_*^r$ has the form (1.1), then*

$$|a_3 - a_2^2| = |\mathcal{H}_{2,1}(f)| \leq \frac{1}{2} \max \left\{ 1; \left| \frac{31r^2 + 10r + 7}{8(1 + 3r)^2} \right| \right\}.$$

4. Hankel determinant bounds for the class S_*^r

In this section we investigate the upper bound of Hankel determinant of order two for the functions that belong to the class S_*^r .

Theorem 4.1. *If the function $f \in \mathcal{A}$ given by (1.1) belongs to the class S_*^r , then*

$$|\mathcal{H}_{2,2}(f)| \leq \frac{44649r^3 + 50245r^2 + 16715r + 1927}{1536(1 + 3r)^2(5r + 3)}. \tag{4.1}$$

Proof. From (3.8), (3.9) and (3.10) it follows that

$$a_2 a_4 - a_3^2 = - \left(\frac{(11703r^3 + 4955r^2 - 1259r + 89) d_1^3}{12288(5r + 3)(1 + 3r)^2} - \frac{(359r^3 + 235r^2 - 11r - 7) d_2 d_1}{128(5r + 3)(1 + 3r)^2} - \frac{d_3(1 + r)^2}{4(1 + 3r)(5r + 3)} \right) d_1 - \frac{d_2^2}{16}. \tag{4.2}$$

Taking modulus and applying triangle inequality in (4.2), we get

$$|a_2 a_4 - a_3^2| \leq \left| \frac{(11703r^3 + 4955r^2 - 1259r + 89) d_1^3}{12288(5r + 3)(1 + 3r)^2} - \frac{(359r^3 + 235r^2 - 11r - 7) d_1 d_2}{128(5r + 3)(1 + 3r)^2} \right| d_1 + \frac{d_2^2}{16}. \tag{4.3}$$

Making use of Lemmas 2.1, 2.2 in the right-hand side of (4.3) we get

$$|a_2 a_4 - a_3^2| \leq \frac{44649r^3 + 50245r^2 + 16715r + 1927}{1536(1 + 3r)^2(5r + 3)}.$$

Hence we get the desired estimation. \square

Theorem 4.2. *If the function $f \in \mathcal{A}$ given by (1.1) belongs to the class S_*^r , then*

$$|a_4 - a_2 a_3| \leq \frac{(1 + r) [303r^2 + 144r - 7] + (561r^2 + 432r + 103)}{96(1 + 3r)^2(5r + 3)}. \tag{4.4}$$

Proof. If $f \in S_*^r$ has the form (1.1), using the relations (3.8)–(3.10) we get

$$a_4 - a_2a_3 = \frac{(1+r)(303r^2 + 144r - 7)d_1^3}{192(5r+3)(1+3r)^2} - \frac{d_1d_2(29r+7)(1+r)}{16(5r+3)(1+3r)} + \frac{d_3(1+r)}{10r+6}. \tag{4.5}$$

Considering the modulus on both sides of (4.5) and applying (2.4) of Lemma 2.2 we get

$$\begin{aligned} |a_4 - a_2a_3| &\leq \frac{(1+r)|303r^2 + 144r - 7|}{96(5r+3)(1+3r)^2} + \frac{(1+r)(561r^2 + 432r + 103)}{96(1+3r)^2(5r+3)} \\ &= \frac{(1+r)[|303r^2 + 144r - 7| + (561r^2 + 432r + 103)]}{96(1+3r)^2(5r+3)}. \end{aligned}$$

Here we obtained the required estimation. □

5. Logarithmic coefficient estimates for the class S_*^r

In this section, we determine upper bounds estimates for the first four logarithmic coefficients of the functions f that belong to the class S_*^r .

Theorem 5.1. *If $f \in S_*^r$ given by (1.1), then*

$$\begin{aligned} |\gamma_1| &\leq \frac{(1+r)}{2(1+3r)}, & |\gamma_2| &\leq \frac{1}{4} \max \left\{ 1; \left| \frac{23r^2 - 6r - 1}{8(1+3r)^2} \right| \right\}, \\ |\gamma_3| &\leq \frac{1+r}{2(5r+3)}, & |\gamma_4| &\leq \frac{71r+33}{32(5r+3)}. \end{aligned} \tag{5.1}$$

Proof. Substituting the values of a_2, a_3, a_4 and a_5 from (3.5)–(3.8) in the relation (1.3) gives

$$\gamma_1 = \frac{(1+r)}{4(1+3r)}d_1. \tag{5.2}$$

$$\gamma_2 = \frac{d_2}{8} - \frac{(95r^2 + 42r + 7)d_1^2}{128(1+3r)^2}. \tag{5.3}$$

$$\gamma_3 = \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)d_1^3}{384(1+3r)^3(5r+3)} - \frac{d_1d_2(29r+7)(1+r)}{32(5r+3)(1+3r)} + \frac{d_3(1+r)}{20r+12}. \tag{5.4}$$

$$\begin{aligned} \gamma_4 &= -\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)d_1^4}{16384(1+3r)^4(5r+3)} \\ &+ \frac{d_1^2d_2(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} - \frac{3d_1d_3(17r+7)}{640r+384} - \frac{7d_2^2}{256} + \frac{d_4}{16}. \end{aligned} \tag{5.5}$$

Now taking modulus in (5.2) and applying (2.2) of Lemma 2.1 we get bound for $|\gamma_1|$. After that taking modulus on both sides of (5.3) and applying (2.3) of Lemma 2.1 we

get the desired estimation for $|\gamma_2|$. Now taking modulus in (5.4) and applying (2.4) of Lemma 2.2 we get

$$|\gamma_3| \leq \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)}{192(1+3r)^3(5r+3)} + \frac{(1+r)(617r^3 + 583r^2 + 215r + 25)}{96(1+3r)^3(5r+3)} + \frac{(1+r)(409r^3 + 587r^2 + 223r + 29)}{192(1+3r)^3(5r+3)} = \frac{1+r}{2(5r+3)}.$$

Here we obtained the desired estimation for $|\gamma_3|$. Now rearranging the terms of γ_4 we get

$$\begin{aligned} \gamma_4 = & -d_1 \left(\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)}{16384(1+3r)^4(5r+3)} d_1^3 \right. \\ & \left. - \frac{(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} d_1 d_2 + \frac{3(17r+7)}{640r+384} d_3 \right) + \left(\frac{d_4}{16} - \frac{7d_2^2}{256} \right). \end{aligned} \tag{5.6}$$

Now taking modulus on both sides of (5.6) and applying triangle inequality we get

$$\begin{aligned} |\gamma_4| \leq & |d_1| \left| \frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)}{16384(1+3r)^4(5r+3)} d_1^3 \right. \\ & \left. - \frac{(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} d_1 d_2 + \frac{3(17r+7)}{640r+384} d_3 \right| + \frac{1}{16} \left| d_4 - \frac{7}{16} d_2^2 \right|. \end{aligned} \tag{5.7}$$

After that applying (2.4) of Lemma 2.2 in first part and (2.3) of Lemma 2.1 in (5.7) we get

$$|\gamma_4| \leq \frac{3(17r+7)}{32(5r+3)} + \frac{1}{8} = \frac{71r+33}{32(5r+3)}.$$

Here we get our desired estimations. □

6. Inverse coefficient bounds for the class S_*^r

In this section, we will estimate the upper bounds of the first four inverse coefficient bounds for the function belonging to the class S_*^r .

Theorem 6.1. *If $f \in S_*^r$ is given by (1.1) and its inverse f^{-1} has the form (1.6), then*

$$|h_2| \leq \frac{1+r}{1+3r}, \tag{6.1}$$

$$|h_3| \leq \frac{1}{2} \max \left\{ 1; \frac{47r^2 + 42r + 23}{8(1+3r)^2} \right\}, \tag{6.2}$$

$$|h_4| \leq \frac{(1+r)(4905r^3 + 5607r^2 + 2583r + 633 + |741r^3 + 1083r^2 + 315r - 59|)}{96(1+3r)^3(5r+3)}, \tag{6.3}$$

$$|h_5| \leq \frac{|371247r^5 + 733389r^4 + 563094r^3 + 202138r^2 + 24299r - 3863|}{1536(1+3r)^4(5r+3)} + \frac{(2r+1)(195r^2 + 206r + 79)}{8(5r+3)(1+3r)^2} + \frac{19}{32}. \tag{6.4}$$

Proof. Substitute the values of a_2, a_3, a_4 and a_5 from (3.5)-(3.8) into (1.7) we get

$$h_2 = -\frac{d_1(1+r)}{2+6r}. \tag{6.5}$$

$$h_3 = \frac{(119r^2 + 90r + 31)d_1^2}{64(1+3r)^2} - \frac{d_2}{4}. \tag{6.6}$$

$$h_4 = -\frac{(1+r)}{(5r+3)} \left[\frac{(4119r^3 + 4641r^2 + 1881r + 335)d_1^3}{192(1+3r)^3} - \frac{d_1d_2(69r+31)}{16(1+3r)} + \frac{d_3}{2} \right]. \tag{6.7}$$

$$h_5 = \frac{(906609r^5 + 2078931r^4 + 2013354r^3 + 991142r^2 + 250165r + 28727)d_1^4}{12288(1+3r)^4(5r+3)} - \frac{d_1^2d_2(1479r^3 + 2269r^2 + 1305r + 259)}{64(5r+3)(1+3r)^2} + \frac{d_1d_3(233r^2 + 274r + 101)}{64(5r+3)(1+3r)} + \frac{27d_2^2}{128} - \frac{d_4}{8}. \tag{6.8}$$

Now taking modulus in (6.5) and applying (2.2) of Lemma 2.1 we get bound for $|h_2|$. After that taking modulus on both sides of (6.6) and applying (2.3) of Lemma 2.1 we get the desired estimation for $|h_3|$. Now taking modulus in (6.7) and applying (2.4) of Lemma 2.2 we get our desired estimation for $|h_4|$. Now rearranging the terms of h_5

we get

$$\begin{aligned}
 h_5 = & \left(\frac{(906609r^5 + 2078931r^4 + 2013354r^3 + 991142r^2 + 250165r + 28727) d_1^3}{12288 (1 + 3r)^4 (5r + 3)} \right. \\
 & - \frac{d_1 d_2 (1479r^3 + 2269r^2 + 1305r + 259)}{64 (5r + 3) (1 + 3r)^2} + \frac{d_3 (233r^2 + 274r + 101)}{64 (5r + 3) (1 + 3r)} \Big) d_1 \\
 & - \frac{1}{8} \left(d_4 - \frac{27}{16} d_2^2 \right). \tag{6.9}
 \end{aligned}$$

Now taking modulus on both sides of (6.9) and applying triangle inequality we get

$$\begin{aligned}
 |h_5| \leq & \left| \frac{(906609r^5 + 2078931r^4 + 2013354r^3 + 991142r^2 + 250165r + 28727) d_1^3}{12288 (1 + 3r)^4 (5r + 3)} \right. \\
 & - \frac{d_1 d_2 (1479r^3 + 2269r^2 + 1305r + 259)}{64 (5r + 3) (1 + 3r)^2} + \frac{d_3 (233r^2 + 274r + 101)}{64 (5r + 3) (1 + 3r)} \Big| |d_1| \\
 & + \frac{1}{8} \left| d_4 - \frac{27}{16} d_2^2 \right|. \tag{6.10}
 \end{aligned}$$

After that applying (2.4) of Lemma (2.2) in first part and (2.3) of Lemma 2.1 in (6.10) we obtain

$$\begin{aligned}
 |h_5| \leq & \frac{|371247r^5 + 733389r^4 + 563094r^3 + 202138r^2 + 24299r - 3863|}{1536 (1 + 3r)^4 (5r + 3)} \\
 & + \frac{(2r + 1) (195r^2 + 206r + 79)}{8 (5r + 3) (1 + 3r)^2} + \frac{19}{32}.
 \end{aligned}$$

Hence the Theorem 6.1 is proved. □

7. Logarithmic inverse coefficient bounds for the class S_*^r

Coefficient problems represent a fundamental aspect of geometric function theory, offering a bridge between the analytic characterization of a function and its geometric properties. In particular, the study of logarithmic coefficients γ_n has gained prominence due to their utility in sharp growth estimates and their historical role in the development of the Milin and de Branges theorems. While the standard Taylor-Maclaurin coefficients a_n and the associated Fekete-Szegő functionals have been extensively documented for the class S_*^r , the corresponding logarithmic inverse coefficients remain comparatively less explored. Let $f \in S_*^r$ and let f^{-1} be the inverse function defined in a neighborhood of the origin. The logarithmic inverse coefficients Γ_n are generated by the following series expansion:

$$\log \left(\frac{f^{-1}(w)}{w} \right) = 2 \sum_{n=1}^{\infty} \Gamma_n w^n, \quad |w| < \rho. \tag{7.1}$$

where $\rho \geq 1/4$ denotes the radius of the Bloch disk. Differentiating and using (1.7), we obtain

$$\begin{aligned} \Gamma_1 &= -\frac{1}{2}h_2, & \Gamma_2 &= -\frac{1}{2}h_3 + \frac{3}{4}h_2^2, & \Gamma_3 &= -\frac{1}{2}h_4 + 2h_2h_3 - \frac{5}{3}h_2^3, \\ \Gamma_4 &= -\frac{1}{2}h_5 + \frac{5}{2}h_2h_4 - \frac{15}{2}h_2^2h_3 + \frac{5}{4}h_2h_3^2 + \frac{35}{8}h_2^4. \end{aligned} \tag{7.2}$$

In this section, we provide a systematic investigation into this problem, establishing upper bounds for the initial four coefficients $|\Gamma_1|, |\Gamma_2|, |\Gamma_3|$ and $|\Gamma_4|$.

Theorem 7.1. *If $f \in S_*^r$ given by (1.1), then*

$$\begin{aligned} |\Gamma_1| &\leq \frac{(1+r)}{2(1+3r)}, & |\Gamma_2| &\leq \frac{1}{4} \max \left\{ 1; \left| \frac{23r^2 - 6r - 1}{8(1+3r)^2} \right| \right\}, & |\Gamma_3| &\leq \frac{1+r}{2(5r+3)}, \\ |\Gamma_4| &\leq \frac{71r+33}{32(5r+3)}. \end{aligned} \tag{7.3}$$

Proof. Substituting the values of h_2, h_3, h_4 and h_5 from (1.7) in the relation (7.2) gives

$$\Gamma_1 = \frac{d_1(1+r)}{4(1+3r)}. \tag{7.4}$$

$$\Gamma_2 = \frac{d_2}{8} - \frac{(95r^2 + 42r + 7)d_1^2}{128(1+3r)^2} = \frac{1}{8} \left(d_2 - \frac{(95r^2 + 42r + 7)}{16(1+3r)^2} d_1^2 \right). \tag{7.5}$$

$$\Gamma_3 = \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)d_1^3}{384(1+3r)^3(5r+3)} - \frac{d_1d_2(29r+7)(1+r)}{32(5r+3)(1+3r)} + \frac{d_3(1+r)}{20r+12}. \tag{7.6}$$

$$\begin{aligned} \Gamma_4 &= -\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)d_1^4}{16384(1+3r)^4(5r+3)} \\ &+ \frac{d_2d_1^2(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} - \frac{3d_1d_3(17r+7)}{640r+384} + \frac{d_4}{16} - \frac{7d_2^2}{256}. \end{aligned} \tag{7.7}$$

Now taking modulus in (7.4) and applying (2.2) of Lemma 2.1 we get bound for $|\Gamma_1|$. After that taking modulus on both sides of (7.5) and applying (2.3) of Lemma 2.1 we get the desired estimation for $|\Gamma_2|$. Now taking modulus in (7.6) and applying (2.4) of Lemma (2.2) we get

$$\begin{aligned} |\Gamma_3| &\leq \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)}{192(1+3r)^3(5r+3)} + \frac{(1+r)(617r^3 + 583r^2 + 215r + 25)}{96(1+3r)^3(5r+3)} \\ &+ \frac{(1+r)(409r^3 + 587r^2 + 223r + 29)}{192(1+3r)^3(5r+3)} = \frac{1+r}{2(5r+3)}. \end{aligned}$$

Here we get the desired estimation for $|\Gamma_3|$. Now rearranging the terms of Γ_4 we get

$$\Gamma_4 = -d_1 \left(\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203) d_1^3}{16384 (1 + 3r)^4 (5r + 3)} - \frac{d_1 d_2 (1381r^3 + 1401r^2 + 495r + 51)}{512 (5r + 3) (1 + 3r)^2} + \frac{3d_3 (17r + 7)}{640r + 384} \right) + \frac{1}{16} \left(d_4 - \frac{7}{16} d_2^2 \right). \tag{7.8}$$

Now taking modulus on both sides of (7.8) and applying triangle inequality we get

$$\Gamma_4 \leq |d_1| \left| \frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203) d_1^3}{16384 (1 + 3r)^4 (5r + 3)} - \frac{d_1 d_2 (1381r^3 + 1401r^2 + 495r + 51)}{512 (5r + 3) (1 + 3r)^2} + \frac{3d_3 (17r + 7)}{640r + 384} \right| + \frac{1}{16} \left| d_4 - \frac{7}{16} d_2^2 \right|. \tag{7.9}$$

After that applying (2.4) of Lemma (2.2) in first part and (2.3) of Lemma 2.1 in (7.9) we obtain

$$\begin{aligned} |\Gamma_4| &\leq \frac{74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203}{4096 (1 + 3r)^4 (5r + 3)} \\ &+ \frac{124643r^5 + 214521r^4 + 147806r^3 + 50482r^2 + 8751r + 613}{2048 (1 + 3r)^4 (5r + 3)} \\ &+ \frac{205261r^5 + 373911r^4 + 267138r^3 + 95726r^2 + 17217r + 1259}{4096 (1 + 3r)^4 (5r + 3)} + \frac{1}{8} \\ &= \frac{3(17r + 7)}{32(5r + 3)} + \frac{1}{8} = \frac{71r + 33}{32(5r + 3)}. \end{aligned}$$

Hence we get our required estimations for Theorem 7.1. □

Concluding remarks

In this paper, we successfully introduced and comprehensively investigated a novel subclass of analytic functions uniquely defined through the product of a modified sigmoid function and the lemniscate Bernoulli function. Our systematic analysis began by establishing precise initial coefficient bounds for functions belonging to this new subclass.

Furthermore, we rigorously explored several critical analytic characteristics. Key findings include the establishment of the Fekete–Szegő inequality, alongside a thorough analysis of Hankel determinants of various orders, which provides valuable insight into the coefficient variability. We also successfully derived important estimates for the logarithmic coefficients and determined significant bounds for both the inverse coefficients and their corresponding logarithmic inverse coefficients.

Collectively, the results presented here offer a significant and substantial contribution to the field of geometric function theory. They not only deepen the understanding of coefficient problems within this novel class but also enrich the broader theory of analytic function subclasses associated with the products of special functions. The methods and findings lay a strong foundation for future research, particularly in exploring other differential and integral operators in conjunction with functions defined by similar special functional products.

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References

- [1] Ali, R. M., *Coefficients of the inverse of strongly starlike functions*, Bull. Malays. Math. Sci. Soc. **26**(2003), 63-71.
- [2] Altinkaya, S., Yalçın, S., *Upper bound of second Hankel determinant for bi-Bazilevič functions*, Mediterr. J. Math., **13**(2016), 4081–4090, DOI:10.1007/S00009-016-0733-5.
- [3] Arif, M., Raza, M., Tang, H., Hussain, S., Khan, H., *Hankel determinant of order three for familiar subsets of analytic functions related with sine function*, Open Math., **17**(2019), 1615-1630, DOI:10.1515/math-2019-0132.
- [4] Çağlar, M., Deniz, E., Srivastava, H. M., *Second Hankel determinant for certain subclasses of bi-univalent functions*, Turkish J. Math., **41**(2017), 694–706, DOI:10.3906/mat-1602-25.
- [5] Carathéodory, C., *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*. Math. Ann., **64**(1)(1907), 95-115.
- [6] De Branges, L., *A proof of the Bieberbach conjecture*, Acta Math. **154**(1985), no. 1-2, 137-152, DOI 10.1007/BF02392821.
- [7] Duren, P. L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, **259**(1983).
- [8] Duren, P. L., Leung, Y. J., *Logarithmic coefficients of univalent functions*, J. Anal. Math., **36**(1980), 36–43, DOI 10.1007/BF02798766.
- [9] Ebadian, A., Bulboacă, T., Cho, N. E., Analouei Adegani, E., *Coefficient bounds and differential subordinations for analytic functions associated with starlike function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **114**(2020), pp. 128, DOI:10.1007/s13398-020-00871-x.
- [10] Efraimidis, I., *A generalization of Livingston's coefficient inequalities for functions with positive real part*, J. Math. Anal. Appl. **435**(1)(2016), 369-379, <https://doi.org/10.1016/j.jmaa.2015.10.050>.
- [11] Fekete, M., Szegő, G., *Eine Bemerkung Über ungerade schlichte Funktionen*, J. Lond. Math. Soc., **1**(1933), 85–89.
- [12] Hayman, W. K., *On the second Hankel determinant of mean univalent functions*, Proc. Lond. Math. Soc., **3**(1968), 77–94.

- [13] Janteng, A., Halim, S. A., Darus, M., *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math, **7**(2006), 1–5.
- [14] Janteng, A., Halim, S. A., Darus, M., *Hankel determinant for starlike and convex functions*, J. Inequal. Pure Appl. Math., **1**(2007), 619–625.
- [15] Kanas, S., Analouei Adegani, E., Zireh, A., *An unified approach to second Hankel determinant of bi-subordinate functions*, Mediterr. J. Math., **14**(2007), pp. 233, <https://doi.org/10.1007/s00009-017-1031-6>.
- [16] Khatter, K., Ravichandran, V., Sivaprasad Kumar, S., *Estimates for Initial Coefficients of Certain Starlike Functions with Respect to Symmetric Points*, 2016, 10.1007/978-81-322-3640-5-24.
- [17] Lee, S. K., Ravichandran, V., Supramaniam, S., *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl., **281**(2013), <https://doi.org/10.1186/1029-242X-2013-281>.
- [18] Livingston, A. E., *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc., **21**(1969), 545–552, DOI:10.1090/S0002-9939-1969-0243054-0.
- [19] Ma, W., Minda, C., *A unified treatment of some special classes of functions*, Proceedings of the Conference on Complex Analysis, 1994.
- [20] Obradović, M., Tuneski, N., *Hankel determinants of second and third order for the class S of univalent functions*, Math. Slovaca, **71**(2021), 649–654, DOI:10.1515/ms-2021-0010.
- [21] Pommerenke, C., *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc., **1**(1966), 111–122, <https://doi.org/10.1112/JLMS/S1-41.1.111>.
- [22] Pommerenke, C., *On the Hankel determinants of univalent functions*, Mathematika, **14**(1967), 108–112, <https://doi.org/10.1112/S002557930000807X>.
- [23] Pommerenke, C., *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, Germany, 1975.
- [24] Ravichandran, V., *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyházi, **20**(1)(2004), 31–37.
- [25] Ravichandran, V., Verma, S., *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris, Ser. 1 **353**(2015), 505–510, DOI:10.1016/j.crma.2015.03.003.
- [26] Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan, **11**(1959), 72–75, DOI: 10.2969/jmsj/011110072.
- [27] Srivastava, H. M., Raza, M., Abu Jarad, E. S. A., Srivastava, G., Abujarad, M. H., *Fekete-Szegő inequality for classes of (p, q) -starlike and (p, q) -convex functions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM., **113**(2019), 3563–3584, DOI:10.1007/s13398-019-00713-5.

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