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Coefficient bounds and Fekete-Szegö inequality for a unified subclass of m-fold symmetric bi-univalent functions

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Abstract. In this paper, we introduce a new and unified subclass of m-fold symmetric bi-univalent functions by subordinating to generalized Janowski function, in the open unit disc $\mathbb{E} = \{z :| z |< 1\}$. Bounds for the initial coefficients and Fekete-Szegö inequality for the functions in this class are studied. Particular cases of the results derived here, are also discussed.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f having Taylor-Maclaurin series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

defined in the unit disc $\mathbb{E} = \{z :| z |< 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Further, the class of functions $f \in \mathcal{A}$ and univalent in $\mathbb{E}$, is denoted by $\mathcal{S}$.
By $\mathcal{U}$, we denote the class of Schwarz functions of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc $\mathbb{E}$ and satisfy the conditions $u(0) = 0$ and $| u(z) |< 1$.

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The fundamental classes of starlike functions and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively, and given by

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E} \right\},$$

and

$$\mathcal{K} = \left\{ f \in \mathcal{A} : Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in \mathbb{E} \right\}.$$

It is obvious that the classes $\mathcal{S}^*$ and $\mathcal{K}$ are related as $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$. This relation was established by Alexander [1] and is known as Alexander relation.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of close-to-convex functions if there exists a convex function $h \in \mathcal{K}$ such that $Re\left(\frac{f'(z)}{h'(z)}\right) > 0$ or equivalently there exists a starlike function $g \in \mathcal{S}^*$ such that $Re\left(\frac{zf'(z)}{g(z)}\right) > 0$. This class was introduced by Kaplan [20].

Further, Noor [30] introduced the class $\mathcal{C}^*$ of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be quasi-convex if there exists a convex function $h \in \mathcal{K}$ such that

$$Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, z \in \mathbb{E}.$$

Every quasi-convex function is convex. Obviously, $f \in \mathcal{C}^*$ if and only if $zf' \in \mathcal{C}$.

Sakaguchi [33] established the class $\mathcal{S}_s^*$ of functions $f \in \mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0.$$

The functions in the class $\mathcal{S}_s^*$ are called starlike functions with respect to symmetric points. Das and Singh [11] proved that $\frac{f(z) - f(-z)}{2}$ is a starlike function in $\mathbb{E}$. So it is obvious that the class $\mathcal{S}_s^*$ is contained in the class $\mathcal{C}$ of close-to-convex functions.

Later on, Das and Singh [11] introduced the class $\mathcal{K}_s$ of the functions $f \in \mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right) > 0.$$

The functions in the class $\mathcal{K}_s$ are called convex functions with respect to symmetric points. It is easy to verify that $f \in \mathcal{K}_s$ if and only if $zf' \in \mathcal{S}_s^*$. Further, some more subclasses of Sakaguchi-type functions were investigated by several authors including [21, 31, 36, 37].

For $0 \leq \alpha < 1$ and $s, t \in \mathbb{C}$(Complex plane) with $s \neq t$, Frasin [13] introduced and studied the class $\mathcal{S}(\alpha, s, t)$ which consists of the functions $f \in \mathcal{A}$ and satisfy the condition

$$Re\left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right) > \alpha.$$

Analogously, the class $\mathcal{T}(\alpha, s, t)$ consists of the functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left(\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right) > \alpha.$$

Obviously $f \in \mathcal{T}(\alpha, s, t)$ if and only if $zf' \in \mathcal{S}(\alpha, s, t)$. Also the following observations are obvious:
(i) $\mathcal{S}(0, 1, -1) \equiv \mathcal{S}_s^*$.
(ii) $\mathcal{T}(0, 1, -1) \equiv \mathcal{K}_s$.
(iii) $\mathcal{S}(0, 1, 0) \equiv \mathcal{S}^*$.
(iv) $\mathcal{T}(0, 1, 0) \equiv \mathcal{K}$.

Let f and g be two analytic functions in $\mathbb{E}$. Then f is said to be subordinate to g, if there exists a Schwarz function $u \in \mathcal{U}$ such that $f(z) = g(u(z))$. If f is subordinated to g, then it is written as $f \prec g$. Further, if g is univalent in $\mathbb{E}$, then $f \prec g$ implies $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Univalent functions are one-one and so are invertible. Also the inverse functions need not be defined on the entire unit disc $\mathbb{E}$. The Koebe one-quarter theorem [12] ensures that the image of $\mathbb{E}$ under the functions $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. It is obvious that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z (z \in \mathbb{E}),$$

and

$$f(f^{-1}(w)) = w\left(\mid w \mid < r_0(f) : r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{E}$ if both f and f^{-1} are univalent in $\mathbb{E}$. The class of functions $f \in \mathcal{A}$ which are bi-univalent in $\mathbb{E}$, is denoted by Σ. Some examples of functions in the class Σ are $\frac{z}{1-z}$, $-log(1-z)$, $\frac{1}{2}log\left(\frac{1+z}{1-z}\right)$ with the corresponding inverse functions $\frac{e^w-1}{e^w}$, $\frac{w}{1+w}$, $\frac{e^{2w}-1}{e^{2w}+1}$. But, the well known Koebe function $f(z) = \frac{z}{(1-z)^2}$ is not a member of Σ.

Estimating the Taylor-Maclaurin coefficients a_n, is an important problem in geometric function theory as it provides information about the geometric properties of the functions in $\mathcal{A}$. Lewin [22] was the first, who investigated the class Σ and proved that $|a_2| < 1.51$. Further, Brannan and Clunie [7] conjectured that $|a_2| < \sqrt{2}$.

Subsequently, non-sharp bounds for $|a_2|$ and $|a_3|$ for various sub-classes of Σ were studied by several authors in [10, 14, 17, 19, 24, 25, 26, 28, 29, 37, 41, 42, 43, 44, 46, 47] and more recently by Singh and Singh [39, 40].

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, Polatoglu et al. [32] introduced the class $\mathcal{P}(A, B; \alpha)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with $\mathcal{P}(A, B)$, which is a subclass of $\mathcal{A}$ introduced by Janowski [18].

For $m \in \mathbb{N}$, a domain $\mathbb{D}$ is said to be m-fold symmetric if a rotation of $\mathbb{D}$ about the origin through an angle $\frac{2\pi}{m}$, carries $\mathbb{D}$ on itself. It follows that a function f analytic in $\mathbb{E}$ is said to be m-fold symmetric if

$$f\left(e^{\frac{2\pi}{m}} z\right) = e^{\frac{2\pi}{m}} f(z).$$

We denote the class of m-fold symmetric univalent functions by $\mathcal{S}_m$. A function $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}.$$

Srivastava et al. [45] defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for every $m \in \mathbb{N}$. Furthermore, for the normalized form of f, they obtained the series expansion of f^{-1} as follows:

$$g(w) = f^{-1}(w) = w - a_{m+1} w^{m+1} + ((m+1)a_{m+1}^2 - a_{2m+1}) w^{2m+1} + ...,$$

where $f^{-1} = g$. We denote the class of m-fold symmetric bi-univalent functions by Σ_m. For $m = 1$, the class Σ_m coincides with Σ. Various examples of m-fold symmetric bi-univalent functions are $\left(\dfrac{z^m}{1 - z^m}\right)^{\frac{1}{m}}$, $[-log(1 - z^m)]^{\frac{1}{m}}$, $\dfrac{1}{2} log\left(\dfrac{1 + z^m}{1 - z^m}\right)^{\frac{1}{m}}$, with the corresponding inverse functions $\left(\dfrac{w^m}{1 + w^m}\right)^{\frac{1}{m}}$, $\left(\frac{e^{w^m} - 1}{e^{w^m}}\right)^{\frac{1}{m}}$, $\left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1}\right)^{\frac{1}{m}}$. Some important subclasses of m-fold symmetric bi-univalent functions were studied by a few researchers including Al-Khafaji [3], Altinkya and Yalcin [5], Bulut [9], Hussain et al. [15], Ibrahim et al. [16], Sakar and Tasar [34] and Senthil and Keerthi [35].

Getting inspired from the research work mentioned above, now we define a new and generalized subclass $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A, B; s, t)$ of Σ_m. This class is a generalization of many earlier established classes. By giving particular values to the parameters introduced in this class, several known results established by various authors, can be easily obtained. Each new class and its associated results add to the theoretical understanding of bi-univalent functions and their geometric properties. Here we used the concept of subordination, which provides a powerful and flexible method to establish various interesting results of the subclasses of univalent and bi-univalent functions, in the unit disc $\mathbb{E}$. So the class proposed here is very significant and plays an important role in motivating the other researchers to study some more generalized subclasses of m-fold symmetric bi-univalent functions. In this paper, we establish the non-sharp bounds for the Taylor-Maclaurin coefficients such as $|a_{m+1}|, |a_{2m+1}|$ and Fekete-Szegö inequality for the defined class.

Definition 1.1. *A function $f \in \Sigma_m$ is said to be in the class $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ if the following conditions are satisfied:*

$$(1-\alpha)\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)}+\alpha\frac{(s-t)[(zf'(z))'\,]^{\lambda}}{(f(sz)-f(tz))'}\prec\left(\frac{1+[B+(A-B)(1-\eta)]z^m}{1+Bz^m}\right)^{\beta},$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-g(tw)}+\alpha\frac{(s-t)[(wg'(w))'\,]^{\lambda}}{(g(sw)-g(tw))'}\prec\left(\frac{1+[B+(A-B)(1-\eta)]w^m}{1+Bw^m}\right)^{\beta},$$

where $s,t \in \mathbb{C}$ with $s \neq t, |t| \leq 1,\ 0 \leq \alpha \leq 1,\ \lambda \geq 0,\ 0 < \beta \leq 1,\ 0 \leq \eta < 1,\ z \in \mathbb{E}$, $w \in \mathbb{E}$ and $g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + ((m+1)a_{m+1}^2 - a_{2m+1})w^{2m+1} + \ldots$.

The following particular cases are obvious:
(i) For $m = 1$, the class $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ reduces to $\mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$.
(ii) For $m = 1, \eta = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ agrees with $\mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta}(A,B;s,t)$, the class studied by Singh et al. [44].
(iii) On putting $m = 1, \eta = 0, \lambda = 1, s = 1, t = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ coincides with $\mathcal{M}_{\Sigma}^{s}(\beta,\alpha;A,B)$, the class studied by Singh [38].
(iv) By substituting $m = 1, \alpha = 0, \eta = 0, A = 1, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ reduces to $\mathcal{S}_{\Sigma}^{\lambda}(s,t,\beta)$, the class studied by Mazi and Opoola [24].
(v) For $0 \leq \gamma < 1$, on putting $m = 1, \alpha = 0, \eta = 0, \beta = 1, A = 1-2\gamma, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ agrees with $\mathcal{S}_{\Sigma}^{\lambda}(s,t,\gamma)$, the class studied by Mazi and Opoola [24].
(vi) On Substituting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, \alpha = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ reduces to $\mathcal{S}_{\Sigma_m}^{\beta}$, the class studied by Altinkya and Yalcin [5].
(vii) By putting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1-2\gamma, B = -1, \alpha = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ reduces to $\mathcal{S}_{\Sigma_m}^{\gamma}$, the class studied by Altinkya and Yalcin [5].
(viii) For $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, m = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ coincides with $\mathcal{M}_{\Sigma}(\beta,\alpha)$, the class studied by Li and Wang [23].
(ix) For $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, m = 1, \alpha = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ reduces to $\mathcal{M}_{\Sigma}(\beta,1)$, the class studied by Li and Wang [23].
(x) On Putting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, m = 1, \alpha = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ agrees with $\mathcal{S}_{\Sigma}^{*}(\beta)$, the class studied by Brannan and Taha [8].
(xi) For $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1-2\gamma, B = -1, m = 1, \alpha = 0, \beta = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ reduces to $\mathcal{S}_{\Sigma}^{*}(\gamma)$, the class of bi-starlike functions of order γ studied by Brannan and Taha [8].

(xii) For $s = 1, t = 0, \eta = 0, A = 1 - 2\gamma, B = -1, m = 1, \alpha = 1, \beta = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A, B; s, t)$ reduces to $\mathcal{B}_{\Sigma}(\gamma, 1)$, the class of bi-convex functions of order γ studied by Li and Wang [23].
(xiii) By substituting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A, B; s, t)$ agrees with $\mathcal{M}_{\Sigma_m}(\beta, \alpha, 1)$, the class established by Motamednezhad et al. [27].
(xiv) On putting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1 - 2\gamma, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A, B; s, t)$ agrees with $\mathcal{M}_{\Sigma_m}(\gamma, \alpha, 1)$, the class studied by Motamednezhad et al. [27].

In the sequel, we lay down once for all that $s, t \in \mathbb{C}$ with $s \neq t, |t| \leq 1$, $0 \leq \alpha \leq 1$, $\lambda \geq 0$, $0 < \beta \leq 1$, $0 \leq \eta < 1$, $-1 \leq B < A \leq 1$, $z \in \mathbb{E}$, $w \in \mathbb{E}$ and $g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + ((m+1)a_{m+1}^2 - a_{2m+1})w^{2m+1} + ...$
For deriving our main results, we need the following lemma:

Lemma 1.2. [6] *If* $p(z) = \dfrac{1 + [B + (A - B)(1 - \eta)]u(z)}{1 + Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, $u(z) \in \mathcal{U}$, *then*

$$|p_n| \leq (A - B)(1 - \eta), n \geq 1.$$

2. Coefficient estimates for the function class $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A, B; s, t)$

Theorem 2.1. *If* $f \in \mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A, B; s, t)$, *then*

$$|a_{m+1}| \leq \frac{\beta\sqrt{2(A - B)(1 - \eta)}}{\sqrt{\Delta}}, \tag{2.1}$$

and

$$|a_{2m+1}| \leq \frac{\beta(A-B)(1-\eta)}{(1+2\alpha m)\Big[\lambda(2m+1)-\big(\frac{s^{2m+1}-t^{2m+1}}{s-t}\big)\Big]} + \frac{(A - B)^2(1 - \eta)^2\beta^2(m + 1)}{2(1 + \alpha m)^2\Big[\lambda(m + 1) - \big(\frac{s^{m+1}-t^{m+1}}{s-t}\big)\Big]^2}, \tag{2.2}$$

where

$$\begin{aligned}\Delta =& \beta\big[2(1 + 2\alpha m + \alpha m^2)\big(\frac{s^{m+1} - t^{m+1}}{s - t}\big)\big(\frac{s^{m+1} - t^{m+1}}{s - t} - \lambda(m + 1)\big) \\ &- (1 + 2\alpha m)(m + 1)\big(\frac{s^{2m+1} - t^{2m+1}}{s - t}\big) + \lambda(m + 1)\big((1 + 2\alpha m)(2m + 1) \\ &+ (\lambda - 1)(m + 1)(1 + 2\alpha m + \alpha m^2)\big)\big] - (\beta - 1)(1 + \alpha m)^2 \\ &\left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t}\right)\right]^2.\end{aligned} \tag{2.3}$$

Proof. Applying the principle of subordination in Definition 1.1, it yields

$$\begin{aligned}&(1-\alpha)\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)}+\alpha\frac{(s-t)[(zf'(z))^{'}]^{\lambda}}{(f(sz)-f(tz))'}\\&=\left(\frac{1+[B+(A-B)(1-\eta)](u(z))^{m}}{1+B(u(z))^{m}}\right)^{\beta}\\&=[p(z)]^{\beta},u\in\mathcal{U},\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}&(1-\alpha)\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-g(tw)}+\alpha\frac{(s-t)[(wg'(w))^{'}]^{\lambda}}{(g(sw)-g(tw))'}\\&=\left(\frac{1+[B+(A-B)(1-\eta)](v(w))^{m}}{1+B(v(w))^{m}}\right)^{\beta}\\&=[q(w)]^{\beta},v\in\mathcal{U},\end{aligned}\tag{2.5}$$

where $p(z)=1+p_mz^m+p_{2m}z^{2m}+...$ and $q(w)=1+q_mw^m+q_{2m}w^{2m}+....$

Expanding (2.4), it takes the form
$1+(1+\alpha m)\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]a_{m+1}z^m$
$+\Big[(1+2\alpha m)\big[\lambda(2m+1)-\big(\frac{s^{2m+1}-t^{2m+1}}{s-t}\big)\big]a_{2m+1}$
$+(1+2\alpha m+\alpha m^2)\left[\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)^2-\lambda(m+1)\left[\frac{s^{m+1}-t^{m+1}}{s-t}-\frac{(\lambda-1)(m+1)}{2}\right]\right]a_{m+1}^2\Big]z^{2m}+$
$\dots$

$$=1+\beta p_mz^m+\left[\beta p_{2m}+\frac{\beta(\beta-1)}{2}p_m^2\right]z^{2m}+\cdots.\tag{2.6}$$

Similarly on expanding (2.5), we obtain
$1-(1+\alpha m)\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]a_{m+1}w^m$
$+\Big[-(1+2\alpha m)\big[\lambda(2m+1)-\big(\frac{s^{2m+1}-t^{2m+1}}{s-t}\big)\big]a_{2m+1}+(1+m)(1+2\alpha m)\big[\lambda(2m+1)-\big(\frac{s^{2m+1}-t^{2m+1}}{s-t}\big)\big]a_{m+1}^2+(1+2\alpha m+\alpha m^2)\left[\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)^2-\lambda(m+1)\left[\frac{s^{m+1}-t^{m+1}}{s-t}-\frac{(\lambda-1)(m+1)}{2}\right]\right]a_{m+1}^2\Big]w^{2m}+...$

$$=1+\beta q_mw^m+\left[\beta q_{2m}+\frac{\beta(\beta-1)}{2}q_m^2\right]w^{2m}+\cdots.\tag{2.7}$$

Equating the coefficients of z^m and z^{2m} in (2.6), we obtain

$$(1+\alpha m)\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]a_{m+1}=\beta p_m,\tag{2.8}$$

$$(1+2\alpha m)\big[\lambda(2m+1)-(\frac{s^{2m+1}-t^{2m+1}}{s-t})\big]a_{2m+1}+(1+2\alpha m+\alpha m^2)$$
$$\left[\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)^2-\lambda(m+1)\left[\frac{s^{m+1}-t^{m+1}}{s-t}-\frac{(\lambda-1)(m+1)}{2}\right]\right]a_{m+1}^2 \tag{2.9}$$
$$=\beta p_{2m}+\frac{\beta(\beta-1)}{2}p_m^2.$$

Equating the coefficients of w^m and w^{2m} in (2.7), it gives

$$-(1+\alpha m)\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]a_{m+1}=\beta q_m, \tag{2.10}$$

$$-(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]a_{2m+1}+(1+m)(1+2\alpha m)$$
$$\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]a_{m+1}^2+(1+2\alpha m+\alpha m^2)$$
$$\left[\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)^2-\lambda(m+1)\left[\frac{s^{m+1}-t^{m+1}}{s-t}-\frac{(\lambda-1)(m+1)}{2}\right]\right]a_{m+1}^2$$
$$=\beta q_{2m}+\frac{\beta(\beta-1)}{2}q_m^2. \tag{2.11}$$

(2.8) and (2.10) together give,

$$p_m=-q_m. \tag{2.12}$$

Further, squaring and adding (2.8) and (2.10), it yields

$$2(1+\alpha m)^2\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]^2a_{m+1}^2=\beta^2(p_m^2+q_m^2). \tag{2.13}$$

Adding (2.9) and (2.11), we obtain

$$\left[(1+m)(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]+2(1+2\alpha m+\alpha m^2)\right.$$
$$\left.\left[(\frac{s^{m+1}-t^{m+1}}{s-t})^2-\lambda(m+1)\left(\frac{s^{m+1}-t^{m+1}}{s-t}-\frac{(\lambda-1)(m+1)}{2}\right)\right]\right]a_{m+1}^2$$
$$=\beta(p_{2m}+q_{2m})+\frac{\beta(\beta-1)}{2}\left(p_m^2+q_m^2\right). \tag{2.14}$$

Using (2.13) in (2.14), we get

$$\left[(1+m)(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]+2(1+2\alpha m+\alpha m^2)\right.$$
$$\left.\left[(\frac{s^{m+1}-t^{m+1}}{s-t})^2-\lambda(m+1)\left(\frac{s^{m+1}-t^{m+1}}{s-t}-\frac{(\lambda-1)(m+1)}{2}\right)\right]\right]a_{m+1}^2$$
$$=\beta(p_{2m}+q_{2m})+\frac{(\beta-1)(1+\alpha m)^2(\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right))^2}{\beta}a_{m+1}^2. \tag{2.15}$$

Simplifying (2.15), it gives

$$a_{m+1}^2 = \frac{\beta^2(p_{2m}+q_{2m})}{\Delta}, \tag{2.16}$$

where Δ is defined in (2.3).
Taking modulus and applying triangle inequality in (2.16), it takes the form

$$|a_{m+1}|^2 \leq \frac{\beta^2(|p_{2m}|+|q_{2m}|)}{\Delta}. \tag{2.17}$$

Applying Lemma 1.2 in (2.17), we can easily obtain (2.1).
Now subtracting (2.11) from (2.9), we get

$$2(1+2\alpha m)\left[\lambda(2m+1)-\left(\tfrac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]a_{2m+1}$$
$$-(1+m)(1+2\alpha m)\left[\lambda(2m+1)-\left(\tfrac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]a_{m+1}^2$$
$$= \beta(p_{2m}-q_{2m})+\frac{\beta(\beta-1)}{2}\left(p_m^2-q_m^2\right). \tag{2.18}$$

Using (2.12) and (2.13) in (2.18), it gives

$$2(1+2\alpha m)\left[\lambda(2m+1)-\left(\tfrac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]a_{2m+1}$$
$$= \beta(p_{2m}-q_{2m})+\frac{(1+m)(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]\beta^2p_m^2}{(1+\alpha m)^2\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]^2}. \tag{2.19}$$

Taking modulus and applying triangle inequality in (2.19), it yields

$$|a_{2m+1}| \leq \frac{\beta(|p_{2m}|+|q_{2m}|)}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}$$
$$+\frac{(m+1)\beta^2|p_m|^2}{2(1+\alpha m)^2\left[\lambda(m+1)-\left(\frac{s^{m+1}-t^{m+1}}{s-t}\right)\right]^2}. \tag{2.20}$$

Using Lemma 1.2 in (2.20), the inequality (2.2) can be easily obtained.

□

On putting $m=1$, Theorem 2.1 gives the following result:

Remark 2.2. If $f \in \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$, then

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)(1-\eta)}}{\sqrt{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}},$$

and

$$|a_3| \leq \frac{\beta(A-B)(1-\eta)}{(1+2\alpha)(3\lambda-s^2-st-t^2)}+\frac{(A-B)^2(1-\eta)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}.$$

By substituting $m=1, \eta=0$ in Theorem 2.1, it gives the following result due to Singh et al. [44]:

Remark 2.3. If $f \in \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta}(A,B;s,t)$, then

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)}}{\sqrt{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}},$$

and

$$|a_3| \leq \frac{\beta(A-B)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}.$$

For $m=1, \eta=0, \lambda=1, s=1, t=-1$, Theorem 2.1 gives the following result due to Singh [38]:

Remark 2.4. If $f \in \mathcal{M}_{\Sigma}^{s}(\beta,\alpha;A,B)$, then

$$|a_2| \leq \frac{\beta\sqrt{A-B}}{\sqrt{2((1+\alpha)^2-\beta\alpha^2)}},$$

and

$$|a_3| \leq \frac{\beta^2(A-B)^2}{4(1+\alpha)^2} + \frac{\beta(A-B)}{2(1+2\alpha)}.$$

By substituting $m=1, \alpha=0, \eta=0, A=1, B=-1$ in Theorem 2.1, it gives the following result due to Mazi and Opoola [24]:

Remark 2.5. If $f \in \mathcal{S}_{\Sigma}^{\lambda}(s,t,\beta)$, then

$$|a_2| \leq \frac{2\beta}{\sqrt{(6\lambda-4\lambda(s+t-\lambda+1)+2st)\beta-(\beta-1)(2\lambda-s-t)^2}},$$

and

$$|a_3| \leq \frac{2\beta}{(3\lambda-s^2-t^2-st)} + \frac{4\beta^2}{(2\lambda-s-t)^2}.$$

By putting $m=1, \alpha=0, \eta=0, \beta=1, A=1-2\gamma, B=-1$ in Theorem 2.1, the following result due to Mazi and Opoola [24], is obvious:

Remark 2.6. If $f \in \mathcal{S}_{\Sigma}^{\lambda}(s,t,\gamma)$, then

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3\lambda-2\lambda(s+t-\lambda+1)+st}},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{(3\lambda-s^2-t^2-st)} + \frac{4(1-\gamma)^2}{(2\lambda-s-t)^2}.$$

For $\lambda=1, s=1, t=0, \alpha=0, \eta=0, A=1, B=-1$, Theorem 2.1 yields the following result due to Altinkya and Yalcin [5]:

Remark 2.7. If $f \in \mathcal{S}_{\Sigma_m}^{\beta}$, then

$$|a_{m+1}| \leq \frac{2\beta}{m\sqrt{\beta+1}},$$

and

$$|a_{2m+1}| \leq \frac{\beta}{m} + \frac{2(m+1)\beta^2}{m^2}.$$

On putting $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ in Theorem 2.1, the following result due to Altinkya and Yalcin [5], can be easily obtained:

Remark 2.8. If $f \in \mathcal{S}_{\Sigma_m}^{\gamma}$, then

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\gamma)}}{m},$$

and

$$|a_{2m+1}| \leq \frac{1-\gamma}{m} + \frac{2(m+1)(1-\gamma)^2}{m^2}.$$

For $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, m = 1, A = 1, B = -1$, Theorem 2.1 yields the following result due to Brannan and Taha [8]:

Remark 2.9. If $f \in \mathcal{S}_{\Sigma}^{*}(\beta)$, then

$$|a_2| \leq \frac{2\beta}{\sqrt{\beta+1}},$$

and

$$|a_3| \leq 4\beta^2 + \beta.$$

By substituting $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, m = 1, \beta = 1, A = 1 - 2\gamma, B = -1$, Theorem 2.1 agrees with the following result due to Brannan and Taha [8]:

Remark 2.10. If $f \in \mathcal{S}_{\Sigma}^{*}(\gamma)$, then

$$|a_2| \leq \sqrt{2(1-\gamma)},$$

and

$$|a_3| \leq 4(1-\gamma)^2 + (1-\gamma).$$

3. Fekete-Szegö inequality for the function class $\mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$

Theorem 3.1. *If* $f \in \mathcal{S}_{\Sigma_m}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$, *then*

$$|a_{2m+1}-\mu a_{m+1}^2| \le \begin{cases} \dfrac{\beta(1-\eta)(A-B)}{(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}, \\ \text{if } 0 \le |l(\mu)| < \dfrac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]} \\ 2\beta(1-\eta)(A-B)|l(\mu)|, \\ \text{if } |l(\mu)| \ge \dfrac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]} \end{cases} \tag{3.1}$$

where

$$l(\mu) = \frac{\beta(\frac{m+1}{2}-\mu)}{\Delta}, \tag{3.2}$$

and Δ *is defined in (2.3).*

Proof. Using (2.13) in (2.17), we have

$$a_{2m+1} = \frac{\beta(p_{2m}-q_{2m})}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]} + \left(\frac{m+1}{2}\right)a_{m+1}^2. \tag{3.3}$$

Making use of (2.16) and (3.3), it yields

$$a_{2m+1}-\mu a_{m+1}^2 = \frac{\beta(p_{2m}-q_{2m})}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]} + \left(\frac{m+1}{2}-\mu\right)\left[\frac{\beta^2(p_{2m}+q_{2m})}{\Delta}\right],$$

where Δ is defined in (2.3).
Further, (3.4) can be expressed as

$$a_{2m+1}-\mu a_{m+1}^2 = \beta\Bigg[\left(l(\mu)+\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}\right)p_{2m} + \left(l(\mu)-\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}\right)q_{2m}\Bigg], \tag{3.4}$$

where $l(\mu)$ is defined in (3.2).
Taking modulus and applying triangle inequality in (3.4), we obtain

$$|a_{2m+1}-\mu a_{m+1}^2| \le \beta\Bigg[\left|l(\mu)+\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}\right||p_{2m}| + \left|l(\mu)-\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}\right||q_{2m}|\Bigg]. \tag{3.5}$$

Using Lemma 1.2, (3.5) yields

$$|a_{2m+1}-\mu a_{m+1}^2|\leq \beta(1-\eta)(A-B)\left[\left|l(\mu)+\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}\right|\right.$$

$$\left.+\left|l(\mu)-\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}\right|\right]. \tag{3.6}$$

For $0\leq |l(\mu)|<\frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}$,

$$|a_{2m+1}-\mu a_{m+1}^2|\leq \frac{\beta(1-\eta)(A-B)}{(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}. \tag{3.7}$$

For $|l(\mu)|\geq \frac{1}{2(1+2\alpha m)\left[\lambda(2m+1)-\left(\frac{s^{2m+1}-t^{2m+1}}{s-t}\right)\right]}$,

$$|a_{2m+1}-\mu a_{m+1}^2|\leq 2\beta(1-\eta)(A-B)|l(\mu)|. \tag{3.8}$$

The proof of Theorem 3.1 is obvious from (3.7) and (3.8).

□

For $m=1$, Theorem 3.1 yields the following result:

Remark 3.2. If $f\in \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$, then

$$|a_3-\mu a_2^2|\leq \begin{cases}\frac{\beta(1-\eta)(A-B)}{(1+2\alpha)\left[3\lambda-\left(s^2+st+t^2\right)\right]}, & \text{if } 0\leq |l(\mu)|<\frac{1}{2(1+2\alpha)\left[3\lambda-\left(s^2+st+t^2\right)\right]},\\ 2\beta(1-\eta)(A-B)|l(\mu)|, & \text{if } |l(\mu)|\geq \frac{1}{2(1+2\alpha)\left[3\lambda-\left(s^2+st+t^2\right)\right]},\end{cases}$$

where

$$l(\mu)=\frac{\beta(1-\mu)}{\Delta_1},$$

and Δ_1 is given by

$$\Delta_1=\beta\left[2(1+3\alpha)(s+t)(s+t-2\lambda)-2(1+2\alpha)\left(s^2+st+t^2\right)+2\lambda(3(1+2\alpha)+2(\lambda-1)(1+3\alpha))\right]$$

$$-(\beta-1)(1+\alpha)^2\left[2\lambda-(s+t)\right]^2. \tag{3.9}$$

On putting $m=1,\eta=0$ in Theorem 3.1, the following result can be easily obtained:

Remark 3.3. If $f\in \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta}(A,B;s,t)$, then

$$|a_3-\mu a_2^2|\leq \begin{cases}\frac{\beta(A-B)}{(1+2\alpha)\left[3\lambda-\left(s^2+st+t^2\right)\right]}, & \text{if } 0\leq |l(\mu)|<\frac{1}{2(1+2\alpha)\left[3\lambda-\left(s^2+st+t^2\right)\right]},\\ 2\beta(A-B)|l(\mu)|, & \text{if } |l(\mu)|\geq \frac{1}{2(1+2\alpha)\left[3\lambda-\left(s^2+st+t^2\right)\right]},\end{cases}$$

where

$$l(\mu)=\frac{\beta(1-\mu)}{\Delta_1},$$

and Δ_1 is defined in (3.9).

By substituting $m = 1, \eta = 0, \lambda = 1, s = 1, t = -1$ in Theorem 3.1, the following result is obvious:

Remark 3.4. If $f \in \mathcal{M}_{\Sigma}^{s}(\beta, \alpha; A, B)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta(A-B)}{2(1+2\alpha)}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{4(1+2\alpha)}, \\ 2\beta(A-B)|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{4(1+2\alpha)}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(1-\mu)}{4\beta(1+2\alpha) - 4(\beta-1)(1+\alpha)^2}.$$

For $\eta = 0, \lambda = 1, s = 1, t = 0, A = 1, B = -1, \alpha = 0$, Theorem 3.1 agrees with the following result:

Remark 3.5. If $f \in \mathcal{S}_{\Sigma_m}^{\beta}$, then

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \begin{cases} \frac{2\beta}{2m}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{4m}, \\ 4\beta|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{4m}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(\frac{m+1}{2} - \mu)}{\beta\left[2 + 2m + 2m^2\right] - m^2(\beta-1)}.$$

For $m = 1, \eta = 0, \lambda = 1, s = 1, t = 0, A = 1, B = -1, \alpha = 0$, Theorem 3.1 coincides with the following result:

Remark 3.6. If $f \in \mathcal{S}_{\Sigma}^{*}(\beta)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \beta, & \text{if } 0 \leq |l(\mu)| < \frac{1}{4}, \\ 4\beta|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{4}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(1-\mu)}{\beta+1}.$$

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