

Notes on certain complex-type special functions in which the Gaussian function and its integral play essential roles

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Abstract. The primary aim of this scientific note is first to review the essential background on several special functions in which the Gaussian function in certain complex domains and its integral play fundamental roles, and subsequently to establish (or organize) a number of relevant results together with some of their potential implications.

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1. Introduction and motivation

In mathematics and several related scientific fields, we frequently encounter various types of academic investigations involving special functions, both in theoretical and applied contexts. Different forms of these functions – with complex (or real) variables (or parameters) – play an important role in a wide range of scientific studies in the written literature. In particular, they have undeniable significance in mathematical transformations and operator theory as is known.

This special research is closely related to various complex-type special functions. For example, one of them is the Faddeeva function of a complex variable (z) (or real variable (x)). It is a complex-valued function that is closely connected to several

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other well-known functions, such as the Fresnel integral, Dawson's integral, and the Voigt function. In the literature, as we know this special function is also referred to as the Kramp function, typically expressed in terms of the familiar complex variable (z).

In addition, the Faddeeva function plays a significant role in various physical problems, especially those concerning electromagnetic responses in complex media. At the same time, it appears in problems involving small-amplitude waves propagating through Maxwellian plasmas. In this connection, it is particularly notable in the expression for plasma permittivity, from which dispersion relations are also derived. Consequently, it is sometimes called the plasma dispersion function, although this name is often reserved for a certain rescaled version of the function.

Furthermore, the infrared permittivity of amorphous oxides exhibits phonon-induced resonances that can be too complicated to model with simple harmonic oscillators. The Brendel-Bormann oscillator model, which represents an infinite superposition of oscillators with slightly different frequencies following a Gaussian distribution, provides a more accurate description; its integrated response can be expressed in terms of the Faddeeva function.

Additionally, the Faddeeva function plays an essential role in analyzing electromagnetic waves used in AM radio. In particular, ground waves – vertically polarized waves that propagate over a lossy ground with finite resistivity and permittivity – can be characterized using this function. It also provides valuable information about changes in neutron cross sections of materials as temperature varies.

The primary aim of this special study is to examine the relevant function – which occupies a significant place in science and technology – in the complex plane, to present essential information about it, and to focus on certain key propositions accompanied by various illustrative examples.

At the same time, especially, this special investigation also seeks to explore specific aspects of the Gaussian function, which holds a prominent position in science and technology and plays a fundamental role among the special functions mentioned above. Subsequently, this study addresses the Faddeeva function (with complex variable z) within selected domains of the complex plane, with particular emphasis on fundamental propositions and several illustrative examples. Especially, we point out that the work [1] is only one of the fundamental references for the above material.

We can now move on to the next sections to introduce the necessary definitions and establish various theoretical concepts along with some related implications.

2. Necessary definitions and related information

In the mathematical and related scientific literature, the classical forms of the Gaussian function (with the variable ρ) and the Gaussian integral frequently appear in the expressions given by

$$\mathcal{G}(\rho) = \exp(-\rho^2) \quad (2.1)$$

and

$$\int_{-\infty}^{\infty} \exp(-\rho^2) d\rho, \tag{2.2}$$

respectively.

The relevant parameter (or variable) ρ , used in the mathematical expressions given in (2.1) and (2.2), may also appear as either a real or a complex variable in various mathematical documents. Since these situations play a big role in our specific results, we believe it is useful to highlight some pertinent information here. The first of these concerns the following expressions, which contain specific information directly related to the exponential function given in (2.1) and the generalized integral given in (2.2).

Remark 2.1. Let the notation \mathbb{R} denote the set of real numbers, and let the variable ρ be of the real variable x , serving as the independent variable of the function in (2.1). In this case, for the real-valued function

$$\mathcal{G}(x) = \exp(-x^2) \quad (x \in \mathbb{R})$$

and its associated improper integral being of the form in (2.2), each of the following assertions is supplied.

- (a) It is a decreasing function for all $x > 0$, and an increasing function for all $x < 0$.
- (b) For all $x \in \mathbb{R}$, $0 < \mathcal{G}(x) \leq 1$. Hence, it is a bounded function on \mathbb{R} .
- (c) It is an even function; namely, $\int_{-\infty}^0 \mathcal{G}(x) dx = \int_0^{\infty} \mathcal{G}(x) dx$.
- (d) It is a continuous and integrable function on \mathbb{R} .
- (e) $\int_{-\infty}^{\infty} \mathcal{G}(x) dx = 2 \int_{-\infty}^0 \mathcal{G}(x) dx = 2 \int_0^{\infty} \mathcal{G}(x) dx = \sqrt{\pi}$.

The second special information is some specific results directly related to the complex variable form of the Gaussian function given in (2.2).

Remark 2.2. Let the notation \mathbb{C} denote the set of complex numbers, and also let the mentioned variable ρ be the familiar complex variable z , which is the independent variable of the function given as in (2.1). In this case, for the complex-type function

$$\mathcal{G}(z) = \exp(-z^2) \quad (z = x + iy; i^2 = -1; x, y \in \mathbb{R}),$$

each of the following assertions is satisfied.

- (a) It is a continuous, differentiating and integrable function on \mathbb{C} . Hence, it is an entire function.
- (b) $|\mathcal{G}(z)| = \exp[-\Re e(z^2)] = \exp(y^2 - x^2)$.
- (c) $\Re e\{\mathcal{G}(z)\} = \exp[-\Re e(z^2)] \text{Cos}[\Re e(-z^2)] = e^{y^2 - x^2} \text{Cos}(2xy)$.
- (d) $\Im m\{\mathcal{G}(z)\} = \exp[-\Re e(z^2)] \text{Sin}[\Im m(-z^2)] = -e^{y^2 - x^2} \text{Sin}(2xy)$.
- (e) $\text{Arg}\{\mathcal{G}(z)\} = \Im m(-z^2) = -2xy \pmod{2\pi}$.

It is well known in the theory of complex functions that curves lying in certain regions of the complex plane, whether continuous or piecewise continuous, play a fundamental role in the evaluation of complex integrals. Accordingly, drawing on classical analysis and taking into account the detailed statements presented in Remarks 2.1 and 2.2, the following observations will significantly facilitate the analysis of the principal complex error functions along the relevant curves.

Remark 2.3. Let z be any parameter in the complex plane \mathbb{C} , and also let Ω be a complex curve defined by

$$\Omega : \Omega(\sigma) = z\sigma \quad (0 \leq \sigma \leq 1),$$

which connects the origin of the plane \mathbb{C} to a given complex point $z \in \mathbb{C}$, and also let the real-valued function

$$\kappa(\rho) = \exp(-\rho^2) \quad (\rho \in \mathbb{R}).$$

Then, when considering the curve Ω , each of the following statements is provided.

- (a) $|\exp(-(z\sigma)^2)| = \exp(-\Re[(z\sigma)^2])$.
- (b) The complex-type function $\kappa(\rho z)$ is continuous and its modulus is also bounded on the curve Ω .
- (c) $\int_0^z \kappa(\rho) d\rho = \int_{\Omega} \kappa(\rho) d\rho = z \int_0^1 \kappa(z\sigma) d\sigma$.
- (d) $\int_0^1 |\kappa(z\sigma)| d\sigma \leq \int_0^1 \exp(-\sigma^2 |z|^2) d\sigma$.
- (e) $|z| \int_0^1 \exp(-\sigma^2 |z|^2) d\sigma = \int_0^{|z|} \exp(-\tau^2) d\sigma$.
- (f) $|\int_0^z \kappa(\rho) d\rho| = |\int_{\Omega} \kappa(\rho) d\rho| = |z| \int_0^1 |\kappa(z\sigma)| d\sigma \leq \max\left\{|z|, \frac{\sqrt{\pi}}{2}\right\}$.
- (g) $\int_0^z \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_z^{\infty} \kappa(\rho) d\rho$.
- (h) $\int_z^{\infty} \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_0^z \kappa(\rho) d\rho$.
- (i) $\int_{|z|}^{\infty} \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_0^{|z|} \kappa(\rho) d\rho$.
- (j) $\int_0^{|z|} \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_{|z|}^{\infty} \kappa(\rho) d\rho$.
- (k) $\int_{|z|}^{\infty} \kappa(\rho) d\rho \leq \int_0^{|z|} \kappa(\rho) d\rho \leq \int_0^{\infty} \kappa(\rho) d\rho$.
- (l) $\left| \int_0^z \kappa(\rho) d\rho \right| \leq \int_0^{|z|} \kappa(\rho) d\rho$.
- (m) $\left| \int_z^{\infty} \kappa(\rho) d\rho \right| \leq \int_{|z|}^{\infty} \kappa(\rho) d\rho$.

It is well established that the essential expressions introduced in Remarks 2.1, 2.2, and 2.3 encode significant information commonly encountered in both the complex and Cartesian planes, as well as in a wide range of computations and definitions in function and integral theory. For further details, the comprehensive references cited in [1]-[8] may be consulted. In line with the objectives of the present work, we now

turn our attention to a collection of fundamental definitions and their associated calculations.

In the mathematical literature, the function of the complex variable z , commonly referred to as the Faddeeva (or Kramp) function and denoted by $\mathcal{W}(z)$, represents a scaled version of the complex complementary error function. As detailed in [9], it is defined by

$$\mathcal{W}(z) = \exp(-z^2) \operatorname{erfc}(-iz) \quad (i^2 = -1; z \in \mathbb{C}). \tag{2.3}$$

Here, the function $\operatorname{erfc}(z)$ denotes the complementary error function of the complex variable z , which is defined as

$$\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-\rho^2) d\rho \quad (z \in \mathbb{C}). \tag{2.4}$$

Taking into account the fundamental results of classical analysis and the computational result stated in Remark 2.1(e), the following identity can be readily obtained:

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \left(\int_0^z \exp(-\rho^2) d\rho + \int_z^\infty \exp(-\rho^2) d\rho \right) \\ = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-\rho^2) d\rho \\ = 1. \end{aligned} \tag{2.5}$$

Here, the complex-valued integral appearing above, namely,

$$\frac{2}{\sqrt{\pi}} \int_0^z \exp(-\rho^2) d\rho \quad (z \in \mathbb{C}), \tag{2.6}$$

is known as the fundamental definition of the error function with complex variable z , and it is commonly denoted by $\operatorname{erf}(z)$.

In particular, in view of the special information given in (2.4)-(2.6), it is straightforward to obtain

$$\operatorname{erf}(z) + \operatorname{erfc}(z) = 1 \quad (z \in \mathbb{C}), \tag{2.7}$$

that is, this identity is only one of the fundamental relationships between the error function and the complementary error function. For more relations between the main error functions and some of related applications, see [1, 4, 8, 9, 13, 15, 16].

We further emphasize that the principal error functions (of the complex variable z), namely the complex functions:

$$\operatorname{erf}(z) \quad \text{and} \quad \operatorname{erfc}(z) \quad (z \in \mathbb{C}) \tag{2.8}$$

are only two of the most important functions within the family of all error functions.

At the same time, as indicated in the first part of this special study, these complex functions also possess several broad applications (or special relationships) with many special functions arising in various fields of science and technology. As a representative example, the rescaled function of the complex variable z ,

$$\mathcal{Z}(z) = i\sqrt{\pi} \mathcal{W}(z) \quad (z \in \mathbb{C}) \tag{2.9}$$

is also defined in terms of the Faddeeva function defined by (2.3). When the complex parameter z is expressed in the familiar form $z = x + iy$, the Faddeeva function $\mathcal{W}(z)$ can be also written as

$$\mathcal{W}(z) \equiv \mathcal{W}(x + iy) = u(x, y) + iv(x, y) = u + iv, \quad (2.10)$$

where the real-valued functions $u(x, y)$ and $v(x, y)$ are known, respectively, as the real Voigt function and the imaginary Voigt function with real parameters x and y . These special functions represent the real and imaginary parts of the (complex) Faddeeva function. Moreover, for the rescaled function being of the form:

$$\mathcal{Z}(z) \equiv \mathcal{Z}(x + iy),$$

it is clear that

$$\Re(\mathcal{Z}(z)) = -\sqrt{\pi} v, \quad (2.11)$$

$$\Im(\mathcal{Z}(z)) = \sqrt{\pi} u, \quad (2.12)$$

$$|\mathcal{Z}(z)| = \sqrt{\pi} |\mathcal{W}(z)| = \sqrt{\pi(u^2 + v^2)} \quad (2.13)$$

and

$$\arg(\mathcal{Z}(z)) = -\arctan\left(\frac{v}{u}\right) + 2\ell\pi \quad (\ell \in \mathbb{Z}),$$

which immediately yields that

$$\text{Arg}(\mathcal{Z}(z)) = -\arctan\left(\frac{v}{u}\right) \pmod{2\pi}. \quad (2.14)$$

Thus, the complex function $\mathcal{Z}(z)$ is fully characterized by the above expressions once the relationship between u and v , determined by the complex functions defined in (2.3), is specified.

For discussions of the relationships and applications between complex error functions and other special functions, the reader is referred to the earlier studies listed in [9]-[16]. Moreover, because the present study focuses on complex functions of a single variable and their applications, the works cited in [17]-[19] constitute essential references for researchers in this area.

The next section is devoted to presenting several propositions involving complex error functions defined in (2.4) and (2.6), as well as their associated complex functions.

3. Various propositions, relations and some of possible implications

In this main section, several special results concerning the Faddeeva function of the complex variable z are presented, including comprehensive statements, properties, and their potential implications and applications. Selected results are subsequently proved.

Proposition 3.1. *Let z be any member of the set \mathbb{C} . Then, each of the following propositions dealing with the Faddeeva function of the complex variable z holds:*

$$\mathcal{W}(-z) = \overline{\mathcal{W}(\bar{z})}, \tag{3.1}$$

$$\mathcal{W}(z) + \mathcal{W}(-z) = 2 \exp(-z^2), \tag{3.2}$$

$$\mathcal{W}(z) = \exp(-z^2) [1 - \operatorname{erf}(-iz)], \tag{3.3}$$

$$\mathcal{W}(iz) = \exp(z^2) [1 - \operatorname{erf}(z)] \tag{3.4}$$

and

$$\sqrt{\pi} \mathcal{W}'(z) + 2\sqrt{\pi} z \mathcal{W}(z) - 2i = 0. \tag{3.5}$$

Proof. For the relevant proofs, it suffices to make appropriate use of the main results presented in (2.3)-(2.8). Detailed proofs of Propositions (3.2) and (3.4) are provided, whereas the proofs of Propositions (3.1), (3.3), and (3.5) are left to the interested reader.

For the proof of the statement given in (3.2), the complex function defined by (2.3) immediately yields the assertion:

$$\mathcal{W}(-z) = \exp(-z^2) \operatorname{erfc}(iz) \quad (z \in \mathbb{C}), \tag{3.6}$$

thus, from (2.3) and (3.6), the desired result follows immediately.

For the proof of the statement given in (3.4), using the definition in (2.3), we obtain

$$\begin{aligned} \mathcal{W}(iz) &= \exp\left[-(-iz)^2\right] \operatorname{erfc}[-i(iz)] \\ &= \exp(z^2) \operatorname{erfc}(z) \quad (z \in \mathbb{C}), \end{aligned} \tag{3.7}$$

and by using the main relationship given in (2.7) together with (3.7), the proof can be readily completed based on the principal complex functions described in (2.8). \square

As additional results of potential interest regarding the complex function in (2.3), and given that the complex functions in (2.8) are analytic, several propositions naturally arise concerning series expansions with real or complex parameters, which are particularly suitable for approximate calculations. We now proceed to establish some of these results.

Proposition 3.2. *Let z belong to the set \mathbb{C} . Then, each of the following statements is true.*

$$\exp(-z^2) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{2j}, \tag{3.8}$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \tag{3.9}$$

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \tag{3.10}$$

$$\operatorname{erf}(-iz) = -\frac{2i}{\sqrt{\pi}} \int_0^z \exp(\xi^2) d\xi \tag{3.11}$$

$$= -\frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \tag{3.12}$$

$$\operatorname{erfc}(-iz) = 1 + \frac{2i}{\sqrt{\pi}} \int_0^z \exp(\xi^2) d\xi \tag{3.13}$$

$$= 1 + \frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1} \tag{3.14}$$

and

$$\mathcal{W}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{2j} + \frac{2i}{\sqrt{\pi}} z \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^k \frac{(-1)^k}{(k-\ell)! [2(k-\ell)+1]} z^{2k} \right). \tag{3.15}$$

Proof. In light of the detailed information provided in the special studies [1], [4], [9], [15], and [16], since the main complex error function $\operatorname{erf}(z)$, defined as in (2.6), is an entire function, it has no singularities except at infinity, and its Taylor-Maclaurin expansion always converges as we know. The integral mentioned in (2.6) cannot be evaluated in closed form in terms of elementary complex functions, but by expanding the integrand of the analytic function, in the complex form, namely

$$\mathcal{K}(z) := \exp(z) \quad (z \in \mathbb{C}), \tag{3.16}$$

into its series expansion and then integrating term by term, one can easily obtain its Maclaurin series. However, for convenience, we also want to share two pieces of necessary-special information (just below).

The first is the series expansion for the necessary proof of the exponential function, *which* plays an important role for each of the propositions from (3.8) to (3.15), *which* is

$$\exp(z) = \sum_{j=0}^{\infty} \frac{1}{j!} z^j \quad (z \in \mathbb{C}). \tag{3.17}$$

The second is the revised new form with the help of the definition in (2.6), *which* relates to the error function $\operatorname{erf}(z)$. For it, by changing of the variable in the form $\rho = -i\xi$ for the integral presented by (2.6), the assertion given by

$$\begin{aligned} \operatorname{erf}(-iz) &= \frac{2}{\sqrt{\pi}} \int_0^{-iz} \exp(\rho^2) d\rho \\ &= -\frac{2i}{\sqrt{\pi}} \int_0^z \exp(\xi^2) d\xi \end{aligned} \tag{3.18}$$

is then obtained, *which* also implies that the power series being of the equivalent forms given by

$$\begin{aligned} \operatorname{erf}(-iz) &= -\frac{2i}{\sqrt{\pi}} \int_0^z \left(\sum_{j=0}^{\infty} \frac{1}{j!} \xi^{2j} \right) d\xi \\ &= -\frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \left(\frac{1}{j!} \int_0^z \xi^{2j} d\xi \right) \end{aligned}$$

$$= -\frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1} \quad (z \in \mathbb{C}). \tag{3.19}$$

□

Specifically, with regard to the relevant research and the remaining pending proofs, we would like to offer the following guidance:

- By setting $z := -z^2$ in the mentioned function given in (3.16) and then considering the series expansion in (3.17), the elementary result constituted as in (3.8) is first obtained.

- Through the instrumentality of the assertions given in (3.16) and (3.17), the essential equalities given in (3.9) and (3.10) are easily seen, respectively.

- By combining the assertions presented in (2.7) and (3.17), the essential results given in (3.12) and (3.14) are also obtained.

- For the proof of the final proposition, namely the assertion in (3.15), it suffices to consider the well-known Cauchy product of the associated power series of the complex functions erf and $erfc$.

- Furthermore, since the expressions given in (3.9) and (3.11), as well as those in (3.10) and (3.12), are directly related, the detailed series expansions corresponding to (3.9) and (3.11) have been omitted.

In terms of various possible implications and suggestions, the mentioned propositions, namely Proposition 3.1 and Proposition 3.2, which contain comprehensive information, will aid both us and related researchers. Now, we want to focus on some specific information in this direction, highlighting some specific examples. We also leave the determination of other implications to the dedicated efforts of interested researchers.

For some of those special implications, let us consider the information given in (2.9) and (3.2) (or (3.3)). With the help of the (complex) Faddeeva function defined in (2.3), we can easily arrive at the following special results:

$$\mathcal{W}(0) = 1, \tag{3.20}$$

$$\begin{aligned} \mathcal{W}(i) &= e \cdot erf c(1) \\ &= e [1 - erf(1)] \\ &= e \left(1 - \frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} \right), \end{aligned} \tag{3.21}$$

$$\mathcal{Z}(0) = i\sqrt{\pi}, \tag{3.22}$$

$$\mathcal{Z}(i) = ie\sqrt{\pi} erf c(1) \tag{3.23}$$

$$= ie\sqrt{\pi} \left(1 - \frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} \right), \tag{3.24}$$

and

$$\mathcal{W}(z) = \frac{1}{i\sqrt{\pi}} \mathcal{Z}(z) \implies \mathcal{W}(-z) = \frac{1}{i\sqrt{\pi}} \mathcal{Z}(-z) \quad (z \in \mathbb{C}), \quad (3.25)$$

which immediately yields the logical implications given by

$$\begin{aligned} 2 \exp(-z^2) &= \mathcal{W}(z) + \mathcal{W}(-z) \\ &= \frac{1}{i\sqrt{\pi}} [\mathcal{Z}(z) + \mathcal{Z}(-z)] \\ \implies \mathcal{Z}(z) + \mathcal{Z}(-z) &= 2i\sqrt{\pi} \exp(-z^2) \quad (z \in \mathbb{C}). \end{aligned} \quad (3.26)$$

For an additional implication, let us focus on the complex-type differential equation related to the Faddeeva function and its first derivative, which is given in (3.5). For this, consider the following initial value problem:

$$\begin{cases} \sqrt{\pi} \mathcal{W}'(z) - 2z\sqrt{\pi} \mathcal{W}(z) - 2i = 0 \\ \mathcal{W}(0) = 1 \end{cases} \quad (3.27)$$

can easily be obtained by the help of the information given by (3.5) and (3.6), where $z \in \mathbb{C}$.

In fact, new higher-order initial value problems can also be created by taking various order derivatives of the differential equation mentioned above. We leave such new determinations to interested researchers.

4. Conclusion and recommendations

In the first section, the extensive role of complex error functions in science and technology, as well as some motivating factors, was emphasized.

The second section presented special definitions related to certain mathematical forms (integrals and functions), the complex error functions, and specific associated functions, organized as several detailed remarks. An illustrative example covering applications between (2.9) and (2.14) was also discussed.

In the third section, several essential statements concerning the aforementioned complex-valued functions are formulated, and some of them are proven. Possible implications of these results are presented in (3.1)-(3.15) as examples for interested researchers. Furthermore, the variety of complex-valued transformations – including numerous elementary computations that can be derived from (3.20)-(3.24) as well as the special computations highlighted in (2.11)-(2.14) – can, of course, be further expanded.

By using the Faddeeva function and related error functions derived from the main error functions indicated in (2.8), numerous propositions and their corresponding implications can be formulated. For instance, considering the complex imaginary error function

$$erfi(z) = -i \operatorname{erf}(iz) \quad (i^2 = -1; z \in \mathbb{C}), \quad (4.1)$$

many analogous results can be explored. For these, it suffices to refer to the detailed results in (2.3)-(4.2).

Additionally, in light of the specific information highlighted in (3.25)-(3.27), the special function given in (2.7) can be applied to the differential equation in (3.6), which takes the differential forms:

$$\mathcal{Z}'(z) + 2z\mathcal{Z}(z) + 2 = 0 \quad \text{or} \quad \begin{cases} \mathcal{Z}'(z) + 2z\mathcal{Z}(z) + 2 = 0 \\ \mathcal{Z}(0) = i\sqrt{\pi} \end{cases}, \quad (4.2)$$

and can be readily solved, where $z \in \mathbb{C}$. Of course, this equation can be readily solved. Moreover, the special-form differential equations in both (3.5) and (4.2) may be extended to other classes of equations, including higher-order differential equations.

In specially, building on the extensive information in Remarks 2.1-2.3, one can also determine lower and/or upper bounds for the modulus of complex-type functions $w(z)$, $\operatorname{erfc}(z)$, $\operatorname{erf}(z)$, $w(z)$, $w(iz)$, $\operatorname{erf}(iz)$ and $\operatorname{erfi}(z)$, as defined in (2.3), (2.4), (2.6), (3.6), (2.7), (3.18), and (4.2), respectively. For each of the studies mentioned here that involve complex functions, it is important to emphasize the role of constructing and analyzing the complex curve Ω in different ways, as highlighted in Remark 2.3. Furthermore, approximate evaluations of function series – arising as consequences of reasonable constraints on the relevant z (or $|z|$) parameters – can introduce additional perspectives on the limitations associated with the specified moduli.

As final words, for further details and applications of complex error functions derived from the main functions indicated in (2.8), interested researchers may consult the studies in [2], [3], [6], [8], [10], [13], and [19]-[24].

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