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Characterizations of pseudolinear and semistrictly quasilinear functions

Vsevolod I. Ivanov

Abstract. In this paper, we obtain several new complete characterizations of pseudolinear functions. Two of the results are of first-order and one is derivative free. The results are derived in terms of the Clarke-Rockafellar subdifferential. Additionally, we prove a characterization of the semistrictly quasilinear functions. It is similar to the derivative free characterization of the pseudolinear functions. We also find the conditions such that a semistrictly quasilinear function becomes pseudolinear.

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1. Introduction

The concepts of pseudolinearity and semistrict quasilinearity provide in a very natural way generalizations of the linearity. These classes include many useful functions. Among them are, for instance, the linear fractional functions. Several interesting properties of pseudolinear functions appeared in [6, 14, 15, 17, 21, 22]. More generalized convex functions have been studied for example in the books [5, 9, 18, 19, 20]. Quasilinear functions have been studied in [19] under the name quasimonotone functions.

In this paper, we obtain more characterizations of pseudolinear functions. In Theorem 2.4, we obtain necessary conditions for pseudolinearity, but they are not sufficient. The respective necessary and sufficient conditions are obtained in Theorem 2.5. Theorem 2.2 is well known and we prove it for reader's convenience. In fact, it is a generalization to non-differentiable functions of a theorem due to Chew and Choo

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[6]. It is interesting which is the largest class of functions such that the conditions of Theorem 2.4 hold. We prove that they become necessary and sufficient when the function is semistrictly quasilinear, which implies that the mentioned class is the exact one. Every pseudolinear function, which is subdifferentiable with respect to the Clarke-Rockafellar subdifferential, is semistrictly quasilinear. We also find the conditions that a semistrictly quasilinear function must satisfy to become pseudolinear.

In the sequel, we suppose that $\mathbb{E}$ is a Banach space. We denote by $\mathbb{E}^*$ its dual and the duality pairing between the vectors $a \in \mathbb{E}^*$ and $b \in \mathbb{E}$ by $\langle a, b\rangle$, by $\mathbb{R}$ the set of reals, by $\overline{\mathbb{R}}$ the union $\mathbb{R} \cup \{+\infty\}$, by $B(x, r)$ the closed ball of a center x with a radius r. Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be a proper extended real-valued function, whose domain is the set

$$\operatorname{dom}(f) := \{x \in \mathbb{E} \mid f(x) < +\infty\}.$$

Definition 1.1. *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be a proper extended real-valued function and $x \in \operatorname{dom}(f)$. The Clarke-Rockafellar generalized derivative of f at x in direction v is defined by*

$$f^{\uparrow}(x, v) = \sup_{\varepsilon>0} \limsup_{(y,\alpha)\downarrow_f x;t\downarrow 0} \inf_{u\in B(v,\varepsilon)} [f(y + tu) - \alpha]/t,$$

where $(y, \alpha) \downarrow_f x$ means that $y \to x$, $\alpha \to f(x)$, $\alpha \geq f(y)$ (see [23]*), and $y \to x$ implies that the norm $\|y - x\|$ approaches 0. If f happens to be lower semicontinuous at x the definition can be expressed in the slightly simpler form*

$$f^{\uparrow}(x, v) = \sup_{\varepsilon>0} \limsup_{y\downarrow_f x;t\downarrow 0} \inf_{u\in B(v,\varepsilon)} [f(y + tu) - f(y)]/t,$$

where $y \downarrow_f x$ means that $y \to x$, $f(y) \to f(x)$. When f is finite and locally Lipschitz, this derivative coincides with the Clarke generalized derivative [7]*, which is defined by*

$$f^0(x, v) = \limsup_{y\to x;t\downarrow 0} [f(y + tv) - f(y)]/t.$$

The Clarke-Rockafellar subdifferential of f at x is defined as follows:

$$\partial^{\uparrow} f(x) = \{x^* \in \mathbb{E}^* \mid \langle x^*, v\rangle \leq f^{\uparrow}(x, v), \quad \forall v \in \mathbb{E}\}$$

with the convention that $\partial^{\uparrow} f(x) = \emptyset$ if $x \notin \operatorname{dom}(f)$. The Clarke's subdifferential (or Clarke's generalized gradient) of f at x is defined as follows:

$$\partial f(x) = \{x^* \in \mathbb{E}^* \mid \langle x^*, v\rangle \leq f^0(x, v), \quad \forall v \in \mathbb{E}\}.$$

Definition 1.2. *A real-valued function $f : \mathbb{E} \to \mathbb{R}$ is called pseudoconvex (in terms of the Clarke directional derivative) iff the following implication is satisfied*

$$f(y) < f(x) \quad \Rightarrow \quad \langle x^*, y - x\rangle < 0, \quad \forall x^* \in \partial f(x). \tag{1.1}$$

A proper extended real function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is called pseudoconvex (in terms of the Clarke-Rockafellar subdifferential) iff the following implication is satisfied

$$f(y) < f(x) \quad \Rightarrow \quad \langle x^*, y - x\rangle < 0, \quad \forall x^* \in \partial^{\uparrow} f(x). \tag{1.2}$$

Recall that a real function f is said to be quasiconvex on a convex set X if,

$$f[x + t(y - x)] \leq \max\{f(x), f(y)\}, \quad \forall x \in X\ \forall y \in X,\ \forall t \in [0, 1].$$

The following result is due to Daniilidis, Hadjisavvas [8, Proposition 2.2].

Lemma 1.3. *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be a lower semicontinuous pseudoconvex function with a convex domain. Then f is quasiconvex.*

The following result is a particular case of Lemma 3 in the paper by Ivanov [10]:

Lemma 1.4. *Let f be a lower semicontinuous proper extended pseudoconvex real function, defined on some convex set S, included in an open set in a Banach space $\mathbb{E}$. Then the following implication holds*

$$x \in S,\ y \in S,\ f(y) \le f(x) \quad \Rightarrow \quad \langle x^*, y - x\rangle \le 0, \quad \forall x^* \in \partial^{\uparrow} f(x).$$

The next results were also derived by Ivanov [10]:

Lemma 1.5. *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be a lower semicontinuous and radially continuous proper extended real-valued function with a convex domain. Then, f is pseudoconvex if and only if there exists a positive function $p : \mathcal{E} \times \mathcal{E} \times \mathbb{E}^* \to (0, +\infty)$ with*

$$\begin{gathered} p(x, y, x^*)\ \langle x^*, y - x\rangle + p(y, x, y^*)\ \langle y^*, x - y\rangle \le 0, \\ \forall (x, y) \in \operatorname{dom}(f) \times \operatorname{dom}(f),\ \forall (x^*, y^*) \in \partial f(x) \times \partial f(y). \end{gathered} \tag{1.3}$$

2. Characterizations of pseudolinear functions

In this section, we apply the characterizations of pseudoconvex functions to obtain characterizations of pseudolinear ones.

Recall that a function f is said to be pseudoconcave iff $-f$ is pseudoconvex. A function f is said to be pseudolinear iff f is both pseudoconvex and pseudoconcave. In the characterizations of pseudoconvex functions, we suppose that f is a proper extended real-valued function, which implies that $f(x) > -\infty$ for every $x \in \mathbb{E}$. We want to apply these results to functions such that both f and $-f$ are proper. Therefore f should be a finite function.

Remark 2.1. We repeat once again that the Clarke generalized gradient is denoted by ∂f and the Clarke-Rockafellar subdifferential by $\partial^{\uparrow}$.

The next theorem is well known. Its proof appears in terms of various derivatives and subdifferential in [1, 6, 17, 22]. It was proved initially in [6]. Our proof is more general, but similar. We prove it, because the construction in it is used in the proof of another result.

Theorem 2.2. *Let f be a lower semicontinuous proper extended function with a convex domain S. Then f is pseudolinear on S with respect to the Clarke-Rockafellar subdifferential if and only if there exists a positive function $p : S \times S \times \mathbb{E}^* \to (0, +\infty)$ with*

$$f(y) - f(x) = p(x, y, x^*)\, \langle x^*, y - x\rangle, \quad \forall\, x \in S\ \forall y \in S,\ \forall x^* \in \partial^{\uparrow} f(x) \tag{2.1}$$

such that $\partial^{\uparrow} f(x) \ne \emptyset$.

Proof. Let f be pseudolinear. We prove that there exists a function p satisfying (2.1). Consider the function, defined as follows:

$$p(x,y,x^*) = \begin{cases} \frac{f(y)-f(x)}{\langle x^*, y-x\rangle}, & \text{if } \langle x^*, y-x\rangle \neq 0 \\ 1, & \text{if } \langle x^*, y-x\rangle = 0. \end{cases} \tag{2.2}$$

We prove that it is positive. Let $\langle x^*, y-x\rangle > 0$. It follows from pseudoconvexity that $f(y) \geq f(x)$. Suppose that it is possible that $f(y) = f(x)$. Then by Lemma 1.4 we obtain that $\langle x^*, y-x\rangle \leq 0$, which is a contradiction. Let $\langle x^*, y-x\rangle < 0$. Since $\partial^\uparrow(sf)(x) = s\partial^\uparrow f(x)$ for every $s \in \mathbb{R}$, then $-x^* \in \partial^\uparrow(-f)(x)$. We conclude from here that $f(y) \leq f(x)$. By Lemma 1.4 we obtain that the case $f(y) = f(x)$ is impossible. Therefore $p > 0$.

We prove that the function p satisfies (2.1). It is enough to show that $\langle x^*, y-x\rangle = 0$ implies that $f(y) = f(x)$. Indeed, assume the contrary. If $f(y) < f(x)$, then by pseudoconvexity we obtain that $\langle x^*, y-x\rangle < 0$, a contradiction. If $f(y) > f(x)$, then by pseudoconcavity we again get a contradiction.

Let $x \in S$, $y \in S$, $x^* \in \partial^\uparrow f(x)$ and equation (2.1) is satisfied. Obviously (2.1) implies that f is pseudolinear. ☐

Theorem 2.3. *Let f be a locally Lipschitz real-valued function, defined on some convex set S included in an open set Γ in a Banach space $\mathbb{E}$. Then, f is pseudolinear on S if and only if there exists a positive function $p : S \times S \times \mathbb{E}^* \to (0, +\infty)$ with*

$$\begin{gathered} p(x,y,x^*)\ \langle x^*, y-x\rangle + p(y,x,y^*)\ \langle y^*, x-y\rangle = 0, \\ \forall (x,y) \in S \times S,\ \forall (x^*, y^*) \in \partial f(x) \times \partial f(y). \end{gathered} \tag{2.3}$$

Proof. Let f be pseudolinear. We prove that inequality (2.3) holds. Choose arbitrary $x \in S$, $y \in S$. It follows from Theorem 2.2 that there exists a function $p : S \times S \times \mathbb{E}^* \to (0, +\infty)$ with

$$f(y) - f(x) = p(x,y,x^*)\ \langle x^*, y-x\rangle, \quad \forall x^* \in \partial f(x) \tag{2.4}$$

and

$$f(x) - f(y) = p(y,x,y^*)\ \langle y^*, x-y\rangle, \quad \forall y^* \in \partial f(y). \tag{2.5}$$

If we add (2.4) and (2.5), then we obtain (2.3).

The converse claim follows from Lemma 1.5, applied both to f and $-f$. ☐

Equation (2.3) is a type of monotonicity of the subdifferential of a pseudolinear function. More properties of such maps are studied in [4]. More types monotone maps have been defined in [12, 13].

Theorem 2.4. *Let S be a convex set, included in some open set Γ in a Banach space $\mathbb{E}$. Suppose that f is a locally Lipschitz function, which is pseudolinear on S with respect to the Clarke's derivative. Then for all $x \in S$, $y \in S$, and $\lambda \in [0,1]$ there exists a number $b > 0$, which depends on x, y, λ such that the following conditions are satisfied:*

$$f[x + \lambda(y-x)] = \lambda\, b\, f(y) + (1 - \lambda\, b) f(x), \tag{2.6}$$

$$0 < b \leq 1/\lambda, \quad \forall \lambda \in (0,1]. \tag{2.7}$$

Proof. Choose arbitrary points $x \in S$, $y \in S$ and a number $\lambda \in (0,1)$. Denote $z(\lambda) = x + \lambda(y - x)$. We have $\partial f(z(\lambda)) \neq \emptyset$. Take arbitrary $\xi \in \partial f(z(\lambda))$. It follows from Theorem 2.2 that there exists a positive function $q : S \times S \times \mathbb{E}^* \to (0, +\infty)$ such that

$$q(z(\lambda), x, \xi)[f(x) - f(z(\lambda))] = \langle \xi, x - z(\lambda) \rangle = \lambda \langle \xi, x - y \rangle \tag{2.8}$$

and

$$q(z(\lambda), y, \xi)[f(y) - f(z(\lambda))] = \langle \xi, y - z(\lambda) \rangle = (1 - \lambda) \langle \xi, y - x \rangle \tag{2.9}$$

where $q = 1/p$. Let us multiply (2.8) by $(1 - \lambda)$, (2.9) by λ, and add the obtained inequalities. Then we obtain that (2.6) holds where

$$b = q(z(\lambda), y, \xi)/[\lambda\, q(z(\lambda), y, \xi) + (1 - \lambda)\, q(z(\lambda), x, \xi)]. \tag{2.10}$$

It follows from (2.10) that $0 < \lambda\, b < 1$ if $0 < \lambda < 1$ and $x \neq y$.

Equality (2.6) is satisfied with $b = 1$ if $\lambda = 1$. It also holds if $\lambda = 0$.

We prove that b does not depend on ξ. Equation (2.6) implies that if $f(y) \neq f(x)$, then

$$b = \frac{f[x + \lambda(y - x)] - f(x)}{\lambda[f(y) - f(x)]}.$$

Suppose that $f(y) = f(x)$. Since the function f is both pseudoconvex and pseudoconcave, then by Lemma 1.3, f is both quasiconvex and quasiconcave. Therefore,

$$f(x) = \min\{f(x), f(y)\} \leq f[x+\lambda(y-x)] \leq \max\{f(x), f(y)\} = f(x) \quad \text{for all } \lambda \in [0,1].$$

We conclude from here that $f[x + \lambda(y - x)] = f(x)$ for all $\lambda \in [0,1]$.

It is seen that really b does not depend on ξ. □

The next result gives us a derivative-free complete characterization of pseudolinear functions.

Proposition 2.5. *Let S be a convex set in a Banach space $\mathbb{E}$. Suppose that f is a continuously differentiable function, defined on some open convex set, which contains S. Then the following claims are equivalent:*

(a) *f is pseudolinear on S;*

(b) *there is a function $b : S \times S \times [0,1] \to (0, +\infty)$ such that for all $x \in S$, $y \in S$ there exists the limit*

$$q(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda), \tag{2.11}$$

$q(x, y)$ is strictly positive, and for each $\lambda \in [0,1]$ equation (2.6) *and inequality* (2.7) *are satisfied.*

Proof. We prove implication (a) ⇒ (b). Let f be pseudolinear on S. It follows from Theorem 2.4, taking into account that for continuously differentiable functions the Clarke directional derivative coincides with the usual Frechet derivative, that there exists a function $b(x, y, \lambda)$ which satisfy (2.6) and (2.7). We prove that there exists the limit $\lim_{\lambda \downarrow 0} b(x, y, \lambda)$, and it is strictly positive. Take arbitrary points x, $y \in S$. Let us consider two cases:

1) $f(x) \neq f(y)$. The function

$$b(x, y, \lambda) = \frac{f[x + \lambda(y - x)] - f(x)}{\lambda[f(y) - f(x)]}$$

satisfies (2.6) and (2.7). The limit

$$q(x,y) = \lim_{\lambda\downarrow 0} b(x,y,\lambda) = \frac{\nabla f(x)(y-x)}{f(y)-f(x)}.$$

If $f(y) < f(x)$, then, by pseudoconvexity $\nabla f(x)(y-x) < 0$ and $q(x,y) > 0$. If $f(y) > f(x)$, then by pseudoconcavity $\nabla f(x)(y-x) > 0$ and again $q(x,y) > 0$.

2) $f(x) = f(y)$. In this case we can choose $b(x,y,\lambda) = 1$. It is obvious that $q(x,y) = \lim_{\lambda\downarrow 0} b(x,y,\lambda) = 1 > 0$.

We prove implication (b) ⇒ (a). If $\lambda > 0$, then (2.6) can be written as

$$\frac{f[x+\lambda(y-x)]-f(x)}{\lambda} = b(x,y,\lambda)[f(y)-f(x)].$$

Taking the limit as $\lambda \downarrow 0$, we obtain that

$$\nabla f(x)(y-x) = q(x,y)[f(y)-f(x)].$$

It follows from here that f is both pseudoconvex and pseudoconcave. □

Example 2.6. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x) = x_2/x_1$, where

$$S = \{x = (x_1,x_2) \mid x_1 > 0\}.$$

The function f is pseudolinear over S. This function satisfies (2.6) and (2.7). We have

$$(f(x+\lambda(y-x)) - f(x))/\lambda = (x_1y_2 - x_2y_1)/(x_1(x_1+\lambda(y_1-x_1)))$$

and

$$b(x,y,\lambda) = (f(x+\lambda(y-x)) - f(x))/(\lambda(f(y)-f(x))) = y_1/(x_1+\lambda(y_1-x_1)).$$

Therefore $0 < \lambda b \le 1$ for all $x \in S$, $y \in S$, $\lambda \in (0,1]$.

3. On semistrictly quasilinear and pseudolinear functions

It is interesting which is the class of functions such that the necessary conditions from Theorem 2.4 become both necessary and sufficient. In this section, we show that this property can be generalized and it becomes necessary and sufficient when the function is semistrictly quasilinear.

Definition 3.1. *A function f, defined on a convex set S is called semistrictly quasiconvex iff for all x, $y \in S$, $\lambda \in (0,1)$ the following implication holds:*

$$f(y) < f(x) \quad\Rightarrow\quad f[x+\lambda(y-x)] < f(x).$$

If the function $-f$ is semistrictly quasiconvex, then f is called semistrictly quasiconcave.

Definition 3.2. *A function f, defined on a convex set, is called semistrictly quasilinear iff it is both semistrictly quasiconvex and semistrictly quasiconcave.*

Proposition 3.3 ([11]). *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a radially lower semicontinuous semistrictly quasiconvex function. Then f is quasiconvex.*

Remark 3.4. It is well known that every differentiable pseudoconvex function, defined on a convex set, is semistrictly quasiconvex on this set; see, for example, [2, Theorem 3.5.11]. A generalization of this claim to non-differentiable functions in terms of Clarke-Rockafellar derivative was obtained in Ref. [10].

Proposition 3.5. *Let S be a convex set, included in some open convex set Γ in a Banach space $\mathbb{E}$. Suppose that f is a continuous function, which is pseudolinear on S. Then f is both semistrictly quasiconvex and semistrictly quasiconcave.*

Proof. The claim follows directly from the known theorem that every pseudoconvex function is semistrictly quasiconvex. ☐

Lemma 3.6. *A function f defined on a Banach space $\mathbb{E}$ is both semistrictly quasiconvex and semistrictly quasiconcave if and only if the following implication holds*

$$x \in \operatorname{dom} f,\ y \in \operatorname{dom} f,\ f(y) < f(x),\ \lambda \in (0,1) \quad \Rightarrow \quad f(y) < f[x+\lambda(y-x)] < f(x).$$

Proof. The proof follows immediately from the definitions of semistrict quasiconvexity and semistrict quasiconcavity. ☐

It is interesting which is the widest class of functions, which satisfy the conditions (2.6) and $0 < \lambda b(x,y,\lambda) < 1$.

Theorem 3.7. *Let f be a continuous function defined on some convex set in a Banach space $\mathbb{E}$. Then f is both semistrictly quasiconvex and semistrictly quasiconcave if and only if for all $x \in S$, $y \in S$ and $\lambda \in (0,1)$ there exists a number $b > 0$, which depend on x, y, λ such that $0 < \lambda b(x,y,\lambda) < 1$ and Condition (2.6) is satisfied.*

Proof. Let f be both semistrictly quasiconvex and semistrictly quasiconcave. Consider the function $b(x,y,\lambda)$ defined by

$$b = \frac{f[x+\lambda(y-x)] - f(x)}{\lambda[f(y)-f(x)]}.$$

Let $f(y) < f(x)$ and $0 < \lambda < 1$. We prove that

$$0 < f[x+\lambda(y-x)] - f(x)/[f(y)-f(x)] < 1. \tag{3.1}$$

It follows from the definition of semistrict quasiconvexity that $f[x+\lambda(y-x)] < f(x)$. Therefore

$$f[x+\lambda(y-x)] - f(x)/[f(y)-f(x)] > 0.$$

It follows from Lemma 3.6 that

$$f(y) < f[x+\lambda(y-x)] < f(x).$$

Hence (3.1) is satisfied

The case $f(y) > f(x)$ is similar. It follows from semistrict quasiconvexity that $f[x+\lambda(y-x)] < f(y)$. Therefore $f[x+\lambda(y-x)] - f(x) < f(y) - f(x)$, which implies that (3.1) is also satisfied. Therefore, Condition (2.6) holds and $0 < \lambda b < 1$.

Let $x \in S$, $y \in S$, $f(x) = f(y)$. By Proposition 3.3 f is both quasiconvex and quasiconcave. Therefore,

$$f[x+\lambda(y-x)] \le \max\{f(x), f(y)\} = f(x) \quad \text{for all } \lambda \in [0,1],$$

$$f[x+\lambda(y-x)] \geq \min\{f(x), f(y)\} = f(x) \quad \text{for all } \lambda \in [0,1].$$

We conclude from here that $f[x+\lambda(y-x)] = f(x)$ for all $\lambda \in [0,1]$. Hence, (2.6) is satisfied with $b = 1$ for every $\lambda \in (0,1)$ and $0 < \lambda b < 1$.

Conversely, suppose that $x \in S$, $y \in S$, $f(y) < f(x)$, $0 < \lambda < 1$, and $0 < \lambda b(x,y,\lambda) < 1$. We prove that f is both semistrictly quasiconvex and semistrictly quasiconcave. Since $0 < \lambda b(x,y,\lambda) < 1$, then we have

$$\lambda\, b\, f(y) + (1-\lambda\, b)\, f(x) < \lambda\, b\, f(x) + (1-\lambda\, b)\, f(x) = f(x),$$

and

$$\lambda\, b\, f(y) + (1-\lambda\, b)\, f(x) > \lambda\, b\, f(y) + (1-\lambda\, b)\, f(y) = f(y).$$

It follows from (2.6) that $f(y) < f[x+\lambda(y-x)] < f(x)$. By Lemma 3.6 f is both semistrictly quasiconvex and semistrictly quasiconcave. □

The following claim is well known [15].

Proposition 3.8. *Let S be an open convex set in a finite-dimensional space E, f be a Fréchet differentiable function, defined on S. Then f is pseudolinear on S if and only if the following sets are equal for all $x \in S$: $\{y \in S : \ \nabla f(x)(y-x) = 0\}$ and $\{y \in S : \ f(y) = f(x)\}$.*

The next theorem is similar, but different from Proposition 3.8.

Theorem 3.9. *Let S be an convex set in a Banach space E, f be a Fréchet differentiable semistrictly quasilinear function, defined on some open set Γ, containing S. Then f is pseudolinear on S if and only if the following implication holds:*

$$x \in S,\ y \in S,\ \nabla f(x)(y-x) = 0 \quad \Rightarrow \quad f(y) = f(x). \tag{3.2}$$

Proof. Let implication (3.2) be satisfied. We prove that f is pseudolinear. Take arbitrary $x \in S$, $y \in S$ such that $f(y) < f(x)$. By semistrict quasiconvexity we have $f[x+\lambda(y-x)] < f(x)$ for every $\lambda \in (0,1)$. Therefore $\nabla f(x)(y-x) \leq 0$. It follows from (3.2) that the case $\nabla f(x)(y-x) = 0$ is impossible, because $f(y) < f(x)$. Hence f is pseudoconvex. Using similar arguments we can prove that f is pseudoconcave. Both pseudoconvexity and pseudoconcavity imply that f is pseudolinear.

Suppose that f is pseudolinear. We prove that implication (3.2) holds. Let $x \in S$, $y \in S$, $\nabla f(x)(y-x) = 0$, but $f(x) \neq f(y)$. If $f(y) < f(x)$, by pseudoconvexity we have $\nabla f(x)(y-x) < 0$, which is a contradiction. If $f(y) > f(x)$, by pseudoconcavity we have $\nabla f(x)(y-x) > 0$, which is also a contradiction. Therefore (3.2) holds. □

We prove a generalization of this result.

Theorem 3.10. *Let S be a convex subset of an open set Γ in a Banach space $\mathbb{E}$. Suppose that $f : \Gamma \to \mathbb{R}$ is a continuous function, which is both semistrictly quasiconvex and semistrictly quasiconcave on S and $\partial^{\uparrow} f(x) \neq 0$, $\partial^{\uparrow}(-f)(x) \neq 0$ for all $x \in S$. Then f is pseudolinear with respect to the Clarke-Rockafellar subdifferential if and only if both implications hold:*

$$x \in S,\ y \in S,\ \xi \in \partial^{\uparrow} f(x),\ \langle \xi, y-x \rangle = 0 \quad \Rightarrow \quad f(y) \geq f(x). \tag{3.3}$$

and

$$x \in S,\ y \in S,\ \eta \in \partial^{\uparrow}(-f)(x),\ \langle \eta, y-x \rangle = 0 \quad \Rightarrow \quad f(y) \le f(x). \tag{3.4}$$

Proof. Let f be pseudolinear. We prove implication (3.3). Take arbitrary $x \in S$, $y \in S$, $\xi \in \partial^{\uparrow} f(x)$ such that $\langle \xi, y-x \rangle = 0$. If $f(y) < f(x)$, then by pseudoconvexity we have $\langle \xi, y-x \rangle < 0$, which contradicts $\langle \xi, y-x \rangle = 0$. The proof of implication (3.4) is similar. Take arbitrary $x \in S$, $y \in S$, $\eta \in \partial^{\uparrow}(-f)(x)$ such that $\langle \eta, y-x \rangle = 0$. If $f(y) > f(x)$, then by pseudoconcavity we have $\langle \eta, y-x \rangle > 0$, which is also a contradiction.

Conversely, suppose that implications (3.3) and (3.4) are fulfiled. We prove that f is pseudoconvex. Let x and y be arbitrary points from S. We prove that

$$\langle \xi, y-x \rangle > 0,\ \xi \in \partial^{\uparrow} f(x) \quad \text{implies} \quad f(y) \ge f(x).$$

Indeed, it follows from $\langle \xi, y-x \rangle > 0$ that $f^{\uparrow}(x, y-x) > 0$. By the definition of the Clarke-Rockafellar derivative, there exist $\varepsilon > 0$ and sequences $\{x_i\}_{i=1}^{\infty}$, $x_i \in \Gamma$, $\{t_i\}_{i=1}^{\infty}$, $t_i > 0$ such that $x_i \to x$, $t_i \downarrow 0$ and

$$\inf_{u \in B(y-x,\varepsilon)} [f(x_i + t_i u) - f(x_i)]/t_i > 0, \quad \forall i.$$

Taking the number i sufficiently large we ensure that $x_i \in B(x,\varepsilon)$. Therefore, we have $y - x_i \in B(y-x,\varepsilon)$ and $f[x_i + t_i(y-x_i)] > f(x_i)$. Using that f is lower semicontinuous and semistrictly quasiconvex we conclude from Proposition 3.3 that it is quasiconvex on S. Therefore,

$$f(x_i) < f[x_i + t_i(y-x_i)] \le f(y).$$

Hence, $f(x) \le \liminf_{i\to\infty} f(x_i) \le f(y)$. It follows from the converse implication that

$$x \in S,\ y \in S,\ f(y) < f(x) \quad \text{imply} \quad \langle \xi, y-x \rangle \le 0, \quad \forall\, \xi \in \partial^{\uparrow} f(x).$$

Therefore, according to implication (3.3), we obtain that f is pseudoconvex.

We prove that $-f$ is pseudoconvex. Let x and y be arbitrary points from S. We prove that

$$\langle \eta, y-x \rangle > 0,\ \eta \in \partial^{\uparrow}(-f)(x) \quad \text{imply} \quad f(y) \le f(x).$$

Indeed, it follows from $\langle \eta, y-x \rangle > 0$ that $(-f)^{\uparrow}(x, y-x) > 0$. By the definition of the Clarke-Rockafellar derivative, there exist $\varepsilon > 0$ and sequences $\{x_i\}_{i=1}^{\infty}$, $x_i \in \Gamma$, $\{t_i\}_{i=1}^{\infty}$, $t_i > 0$ such that $x_i \to x$, $t_i \downarrow 0$ and

$$\inf_{u \in B(y-x,\varepsilon)} [-f(x_i + t_i u) + f(x_i)]/t_i > 0, \quad \forall i.$$

Taking the number i sufficiently large we ensure that $x_i \in B(x,\varepsilon)$. Therefore, we have $y - x_i \in B(y-x,\varepsilon)$ and $f[x_i + t_i(y-x_i)] < f(x_i)$. Using that f is upper semicontinuous and semistrictly quasiconcave, we conclude from Proposition 3.3 that it is quasiconcave. Therefore,

$$f(x_i) > f[x_i + t_i(y-x_i)] \ge f(y).$$

Hence, $f(x) \ge \limsup_{i\to\infty} f(x_i) \ge f(y)$. It follows from the converse implication that

$$x \in S,\ y \in S,\ f(y) > f(x) \quad \text{imply} \quad \langle \eta, y-x \rangle \le 0, \quad \forall\, \eta \in \partial^{\uparrow}(-f)(x).$$

Therefore, by implication (3.4), we obtain that $-f$ is pseudoconvex, which implies that the function f is pseudolinear.

□

Corollary 3.11. *Let S be an open convex set in a Banach space $\mathbb{E}$. Suppose that $f : S \to \mathbb{R}$ is a locally Lipschitz semistrictly quasilinear on S function. Then, f is pseudolinear with respect to the Clarke generalized directional derivative if and only if the following implication holds:*

$$x \in S,\ y \in S,\ \xi \in \partial f(x),\ \langle \xi, y - x\rangle = 0 \quad \Rightarrow \quad f(y) = f(x).$$

Remark 3.12. Theorem 3.9 is not a consequence of Corollary 3.11, because the Clarke subdifferential $\partial f(x)$ does not coincides with the gradient $\nabla f(x)$ when the function is Fréchet differentiable, but it is not necessarily continuously differentiable.

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Vsevolod I. Ivanov
Department of Mathematics, Technical University of Varna,
9010 Varna, Bulgaria,
e-mail: `vsevolodivanov@yahoo.com`