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On the existence of solutions to psi-Hilfer fractional neutral integro-differential equations with delay

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Abstract. This paper investigates a class of neutral-type fractional differential equations with finite delays, formulated through the generalized Ψ-Hilfer fractional derivative. This operator, being a broad framework that unifies various fractional derivatives, is highly effective in modeling dynamical processes with memory and hereditary characteristics. The primary objective is to establish sufficient conditions for the existence and uniqueness of solutions to such equations. The analysis employs fixed point theory-specifically Banach's contraction principle and Krasnoselskii's fixed point theorem—within an appropriately weighted function space. These tools ensure that the solutions are not only well-defined but also uniquely determined. Furthermore, two stability notions, namely Ulam–Hyers stability and its generalized form, are studied to verify that solutions remain close to the expected behavior under small perturbations in initial conditions or parameters. To demonstrate the applicability of the theoretical framework, an illustrative example with explicit functions and parameters is provided. The results strengthen the theoretical foundations of fractional calculus and open directions for further research on more generalized and complex delayed fractional systems.

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Abbreviations

- Fractional Differential Equation ($\mathcal{FDE}$)
- Fractional Integral Equation ($\mathcal{FIE}$)
- Fractional Calculus ($\mathcal{FC}$)
- Fractional Integral ($\mathcal{FI}$)
- Fractional Derivatives ($\mathcal{FD}$)
- Riemann-Liouville ($\mathcal{RL}$)
- Ψ-Hilfer Fractional Derivative ($\Psi - \mathcal{HFD}$)
- Ulam-Hyers Stability ($\mathcal{UHS}$)
- Ulam-Hyers Rassias Stability ($\mathcal{UHRS}$)
- Fractional Functional Integro-Differential Equation ($\mathcal{FIDE}$)

1. Introduction

$\mathcal{FC}$ has emerged as a powerful framework in modern mathematics and applied sciences, owing to its capability of describing processes influenced by memory and hereditary characteristics. Such systems frequently appear in diverse fields, including control systems, viscoelastic materials, biomedical engineering, signal transmission, and fluid mechanics. Classical integer-order models, fractional formulations provide greater flexibility and accuracy in capturing the underlying dynamics. Consequently, both theoretical investigations and practical applications of $\mathcal{FD}$ and $\mathcal{FI}$ have received considerable attention in recent decades [3, 5, 11].

To address different modeling needs, several operators have been introduced in the literature, such as $\mathcal{RL}$, Caputo, and Hadamard derivatives. A more generalized form is the Ψ-Hilfer operator, which integrates these classical definitions into a unified structure. Each operator offers unique interpretations, enhancing the scope of $\mathcal{FC}$ [3, 10, 11, 12].

Parallel to these developments, significant research has focused on the solvability of $\mathcal{FDE}$ and $\mathcal{FIE}$. Both linear and nonlinear models have been investigated using advanced mathematical tools, including fixed point theory, semigroup methods, and techniques from functional analysis [1, 2]. Moreover, stability analysis—particularly $\mathcal{UHS}$ and its extended form, $\mathcal{UHRS}$—plays a vital role in assessing the reliability of fractional models when subjected to perturbations. These approaches have recently been combined with Ψ-Hilfer derivatives to establish stability results for nonlinear equations [7, 8, 9].

Although a wide range of studies have considered fractional systems without delays or impulsive effects, fewer works address models involving both neutral terms and delay components within the Ψ-Hilfer framework. Foundational contributions can be found in [13, 14], yet further research is necessary to explore existence and stability under such conditions.

Motivated by this gap, the present paper investigates a class of $\mathcal{FIDE}$ with delay, governed by the Ψ-Hilfer fractional derivative. The general model under study is expressed as:

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\varsigma_1,\varsigma_2;\Psi}_{\mathfrak{a}_1+}\left[\mathfrak{f}(\iota)-\sum_{i=1}^{n}\mathcal{I}^{\xi_i}_{\mathfrak{a}_1+}\mathcal{Z}_i(\iota,\mathfrak{f}_i)\right] & =\mathcal{G}(\iota,\mathfrak{f}_\iota), \quad \iota\in\mathcal{J}=[\mathfrak{a}_1,\mathfrak{b}_1], \\ \mathfrak{f}(\iota) & =\Phi(\iota), \iota\in[\mathfrak{a}_1-\delta,\mathfrak{a}_1], \end{cases} \tag{1.1}$$

Here, ${}^{\mathcal{H}}\mathcal{D}^{\varsigma 1,\varsigma_2;\Psi}_{\mathfrak{a}_1^+}$ denotes the Ψ-$\mathcal{HFD}$ of order $\varsigma_1\in(0,1)$ and type $\varsigma_2\in[0,1]$, and $\mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}$ represents the Ψ-$\mathcal{RL}$ fractional integral. The functions $\mathcal{Z}_i$ and $\mathcal{G}$ are assumed to be continuous with respect to their arguments, and the initial condition Φ provides the required history for the system.

The remainder of this paper is organized as follows. Section 2 introduces the required background, notations, and fundamental tools from Ψ-Hilfer calculus and fixed point theory. Section 3 establishes existence, uniqueness, and $\mathcal{UHS}$. Section 4 provides a detailed illustrative example that supports the theoretical results.

2. Preliminaries

In the following section, to introduce the essential preliminaries, including definitions, notations, and results from Ψ-Hilfer fractional calculus and nonlinear analysis, required for the later developments. These definitions are adapted to the weighted function space framework appropriate for our study.

Let $\mathcal{J}=[\mathfrak{a}_1,\mathfrak{b}_1]\subset\mathcal{R}$ be a compact interval , and let $\Psi\in\mathcal{C}^1(\mathcal{J},\mathcal{R})$ be an increasing function with $\Psi'(\tau)\neq 0 \quad \forall\tau\in\mathcal{J}$. This function Ψ defines the generalized fractional operator's kernel.

We introduce the following weighted spaces

$$\mathcal{C}^\eta_\Psi(\mathcal{J})=\{\mathfrak{f}:\mathcal{J}\to\mathcal{R}|(\Psi(\iota)-\Psi\ (\mathfrak{a}_1))^\eta\mathfrak{f}(\iota)\in\mathcal{C}(\mathcal{J})\}$$

with norm:

$$\|\mathfrak{f}\|_{\mathcal{C}^\eta_\Psi}=\sup_{\iota\in\mathcal{J}}|(\Psi(\iota)-\Psi\ (\mathfrak{a}_1))^\eta\mathfrak{f}(\iota)|.$$

$$\mathcal{C}^\eta_{\Psi,\delta}(\mathcal{J})=\{\mathfrak{f}:[\mathfrak{a}_1-\delta,\mathfrak{b}_1]\to\mathcal{R}|(\Psi(\iota)-\Psi\ (\mathfrak{a}_1))^\eta\mathfrak{f}(\iota)\in\mathcal{C}([\mathfrak{a}_1-\delta,\mathfrak{b}_1])\}$$

with norm:

$$\|\mathfrak{f}\|_{\mathcal{C}^\eta_\Psi}=\sup_{\iota\in[\mathfrak{a}_1-\delta,\mathfrak{b}_1]}|(\Psi(\iota)-\Psi\ (\mathfrak{a}_1))^\eta\mathfrak{f}(\iota)|.$$

For a fixed delay parameter $\delta>0$, we denote $\mathcal{C}_\delta=\mathcal{C}([-\delta,0],\mathcal{R})$ with norm:

$$\|\phi\|_{\mathcal{C}_\delta}=\sup_{\tau\in[-\delta,0]}|\phi(\tau)|.$$

For $\mathfrak{f}\in\mathcal{C}([\mathfrak{a}_1-\delta,\mathfrak{b}_1],\mathcal{R})$, the delay norm is given by

$$\|\mathfrak{f}_\tau\|_{\mathcal{C}_\delta}=\sup_{\rho\in[-\delta,0]}|\mathfrak{f}(\tau+\rho)|.$$

Let $\mathcal{C}_{\Psi}^{\eta} = \mathcal{C}([\mathfrak{a}_1 - \delta, \mathfrak{b}_1], \mathcal{R})$ denote the Weighted space of real-valued continuous function defined on the extended interval $[\mathfrak{a}_1 - \delta, \mathfrak{b}_1]$ with the norm:

$$\|\mathfrak{f}\|_{\mathcal{C}_{\Psi}^{\eta}} = \sup_{\tau \in [\mathfrak{a}_1 - \delta, \mathfrak{b}_1]} |\mathfrak{f}(\tau)|.$$

Definition 2.1. [13]*: Let* $\varsigma_1 > 0$*. The* Ψ*-*$\mathcal{RL} - \mathcal{FI}$ *of a function* f *on* $\mathcal{J}$ *is given by*

$$\mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi} \mathfrak{f}(\iota) = \frac{1}{\Gamma(\varsigma_1)} \int_{\mathfrak{a}_1}^{\tau} \Psi'(\mathfrak{s}) \big(\Psi(\tau) - \Psi(\mathfrak{s})\big)^{\varsigma_1 - 1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s}.$$

Definition 2.2. [13] *For* $0 < \varsigma_1 < 1$ *and* $0 \leq \varsigma_2 \leq 1$*, the* Ψ*-*$\mathcal{HFD}$ *of a function* f *is expressed as*

$$\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi} \mathfrak{f}(\iota) = \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota}\right)^{\mathfrak{n}} \big(\mathcal{I}_{\mathfrak{a}_1+}^{(n-\varsigma_1);\Psi}\big) \mathfrak{f}(\iota), \quad \mathfrak{n} = [\varsigma_1] + 1$$

Definition 2.3. [13] *For* $\mathfrak{n} - 1 < \varsigma_1 < \mathfrak{n} (\mathfrak{n} \in \mathcal{N})$ *and* $\mathfrak{f}, \Psi \in \mathcal{C}^{\mathfrak{n}}(\mathcal{J}, \mathcal{R})$*,then* Ψ*- Caputo fractional derivative of a function* $\mathfrak{f}$ *of order* ς_1 *is defined by*

$${}^{C}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi} \mathfrak{f}(\iota) = \mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi} \left(\mathfrak{f}(\iota) - \sum_{\mathfrak{i}=0}^{\mathfrak{n}-1} \frac{\mathfrak{f}_{\Psi}^{[\mathfrak{i}]}(\mathfrak{a}_1)}{k!} \big(\Psi(\iota) - \Psi(\mathfrak{s})\big)^{\mathfrak{i}} \right)$$

Where $\mathfrak{n} = [\varsigma_1] + 1$ *for* $\varsigma_1 \neq \mathcal{N}$*, and* $\mathfrak{f}_{\Psi}^{[\mathfrak{i}]}(\iota) = \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota}\right)^{\mathfrak{i}} \mathfrak{f}(\iota)$*.*

In particular, if $\mathfrak{n} = \varsigma_1$*, we have* ${}^{C}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi} \mathfrak{f}(\iota) = \mathfrak{f}_{\Psi}^{[\mathfrak{n}]}(\iota)$*.*

Definition 2.4. *([13]) For* $\mathfrak{n} - 1 < \varsigma_1 < \mathfrak{n} (\mathfrak{n} \in \mathcal{N})$ *and* $\mathfrak{f}, \Psi \in \mathcal{C}^{\mathfrak{n}}(\mathfrak{a}_1, \mathfrak{b}_1)$ *such that* Ψ *is an increasing with* $\Psi'(\iota) \neq 0$ *for all* $\iota \in [\mathfrak{a}_1, \mathfrak{b}_1]$*. Then the left-sided* Ψ*-Hilfer fractional derivative of* $\mathfrak{f}$ *of order* ς_1 *and type* $0 \leq \varsigma_2 \leq 1$ *is defined by*

$${}^{\mathcal{H}}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1,\varsigma_2;\Psi} \mathfrak{f}(\iota) = \mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_2(\mathfrak{n}-\varsigma_1);\Psi} \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota}\right)^{\mathfrak{n}} \big(\mathcal{I}_{\mathfrak{a}_1+}^{(1-\varsigma_2)(\mathfrak{n}-\varsigma_1);\Psi}\big) \mathfrak{f}(\iota)$$

One has

$${}^{\mathcal{H}}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1,\varsigma_2;\Psi} \mathfrak{f}(\iota) = \mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_2(n-\varsigma_1);\Psi} \mathcal{D}_{\mathfrak{a}_1+}^{\eta;\Psi} \mathfrak{f}(\iota)$$

where

$$\mathcal{D}_{\mathfrak{a}_1+}^{\eta;\Psi} \mathfrak{f}(\iota) = \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota}\right)^{\mathfrak{n}} \big(\mathcal{I}_{\mathfrak{a}_1+}^{(n-\eta);\Psi}\big) \mathfrak{f}(\iota), \quad \eta = \varsigma_1 + \varsigma_2(n - \varsigma_1)$$

Lemma 2.5. [13] *If* $\varsigma_1, \varsigma_2 > 0$*,* $\mathfrak{f} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ *then*

$$\big(\mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi}\big)\big(\mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_2;\Psi}\big) \mathfrak{f}(\iota) = \mathcal{I}_{\mathfrak{a}_1+}^{(\varsigma_1+\varsigma_2;\Psi)} \mathfrak{f}(\iota)$$

Theorem 2.6. [13] *If* $\varsigma_1, \sigma > 0$*, and* $0 \leq \varsigma_2 \leq 1$*. Then*

$$\mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi} \big[\Psi(\iota) - \Psi(\mathfrak{a}_1)\big]^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\varsigma_1 + \sigma)} \big(\Psi(\iota) - \Psi\ (\mathfrak{a}_1)\big)^{\varsigma_1+\sigma-1}$$

and

$${}^{\mathcal{H}}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1,\varsigma_2;\Psi} \big[\Psi(\iota) - \Psi(\mathfrak{a}_1)\big]^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma - \varsigma_1)} \big(\Psi(\iota) - \Psi(\mathfrak{a}_1)\big)^{\sigma-\varsigma_1-1}.$$

Theorem 2.7. [13] *Let* $\mathfrak{f} \in \mathcal{C}_{\eta[\mathfrak{a}_1,\mathfrak{b}_1]}, \mathfrak{o} < \eta < \varsigma_1 < 1$. *Then we have*

$$\mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi}\mathfrak{f}(\mathfrak{a}_1) = \lim_{\iota\to a+} \mathcal{I}_{\mathfrak{a}_1+}^{\varsigma_1;\Psi}\mathfrak{f}(\iota) = \mathfrak{o}.$$

Theorem 2.8. *(Banach fixed point theorem) Let* $(\mathcal{X}, d)$ *be a nonempty complete metric space, and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a contraction mapping. Then, there exists a unique point* $\mathfrak{x} \in \mathcal{X}$ *such that* $\mathcal{T}(\mathfrak{x}) = \mathfrak{x}$*; that is,* $\mathfrak{x}$ *is a fixed point of* $\mathcal{T}$.

Theorem 2.9. *Let* $\mathcal{X}$ *be a Banach space, and let* $\mathcal{S} \subset \mathcal{X}$ *be a nonempty, closed, bounded, and convex subset. Consider two mappings* $\mathcal{T}1, \mathcal{T}2 : \mathcal{S} \to \mathcal{X}$ *such that for all* $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}$, *the point* $\mathcal{T}1(\mathfrak{x}) + \mathcal{T}2(\mathfrak{y})$ *belongs to* $\mathcal{S}$. *If* $\mathcal{T}1$ *is a contraction and* $\mathcal{T}2$ *is completely continuous, then the equation* $\mathfrak{x} = \mathcal{T}_1\mathfrak{x} + \mathcal{T}_2\mathfrak{y}$ *admits at least one solution* $\mathfrak{x} \in \mathcal{S}$.

3. Main results

For convenience define the parameter:

$$\begin{aligned}\Lambda &= \Lambda_1 + \Lambda_2 \\ &= \frac{\left(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1)\right)^{\eta}}{\Gamma(\eta + \mathfrak{1})}\|\omega\| + \sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{n}} \frac{\left(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1)\right)^{\xi_i}}{\Gamma(\xi_i + \mathfrak{1})}\|\nu_i\| \end{aligned} \tag{3.1}$$

The assumptions listed below will be used in establishing the main results.

- (A_1) The function $\mathcal{G}, \mathcal{Z}_i : \mathcal{J} \times \mathcal{C}_\delta \to \mathcal{R}$ are continuous for each $i = 1, 2, ...n$.
- (A_2) There exist measurable function $\omega(\iota), \nu_i(\iota)$ such that for all $\mathfrak{f}, \bar{\mathfrak{f}} \in \mathcal{C}_\delta$ and $\iota \in \mathcal{J}$,

$$|\mathcal{G}(\iota, \mathfrak{f}) - \mathcal{G}(\iota, \bar{\mathfrak{f}})| \leq \omega(\iota)\|\mathfrak{f} - \bar{\mathfrak{f}}\|_{\mathcal{C}_\delta},$$
$$|\mathcal{Z}_i(\iota, \mathfrak{f}) - \mathcal{Z}_i(\iota, \bar{\mathfrak{f}})| \leq \nu_i(\iota)\|\mathfrak{f} - \bar{\mathfrak{f}}\|_{\mathcal{C}_\delta}$$

- (A_3) There exist constants $\Phi, \Phi_i \geq 0$ for all $(\iota, \mathfrak{f}) \in \mathcal{J} \times \mathcal{C}_\delta$,

$$|\mathcal{G}(\iota, \mathfrak{f})| \leq \Phi\|\mathfrak{f}\|_{\mathcal{C}_\delta}, |\mathcal{Z}_i(\iota, \mathfrak{f})| \leq \Phi_i.$$

Definition 3.1. *A function* $\mathfrak{f} \in \mathcal{C}^{\eta}_{\Psi,\delta}$ *is said to be a solution of (1) if it satisfies*

$$^{\mathcal{H}}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1,\varsigma_2;\Psi}\left[\mathfrak{f}(\iota) - \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}_{\mathfrak{a}_1+}^{\xi_i}\mathcal{Z}_i(\iota, \mathfrak{f}_i)\right] = \mathcal{G}(\iota, \mathfrak{f}_\iota), \quad \iota \in \mathcal{J} = [a, b],$$

and the initial condition $\mathfrak{f}(\iota) = \Phi(\iota), \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]$.

Lemma 3.2. *Let* $0 < \varsigma_1, \sigma < 1$, $0 \leq \varsigma_2 \leq 1$ *and* $\Psi'(\iota) \neq 0$ *on* $\mathcal{J} = [a, b]$. *If* $\mathfrak{h}, \mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ *with* $\mathfrak{h}(\mathfrak{a}_1) = 0$ *and* $\lambda(\iota) = 0$, *then the linear problem*

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{D}_{\mathfrak{a}_1+}^{\varsigma_1,\varsigma_2;\Psi}\left[\mathfrak{f}(\iota) - \mathfrak{h}(\iota)\right] &= \mathfrak{g}(\iota), \quad \iota \in \mathcal{J}, \\ \mathfrak{f}(\iota) &= \lambda(\iota), \quad \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1], \end{aligned}$$

has a unique solution given by

$$\mathfrak{f}(\iota) = \begin{cases} \mathfrak{h}(\iota) + \mathcal{I}_{\mathfrak{a}_1+}^{\eta;\Psi}\mathfrak{g}(\iota), & \iota \in \mathcal{J}, \\ \lambda(\iota), & \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1] \end{cases}$$

Theorem 3.3. *If assumption $(A_1)-(A_2)$ hold and*

$$\Lambda < 1 \tag{3.2}$$

then equation (1.1) admits a unique solution on $[\mathfrak{a}_1-\delta,\mathfrak{b}_1]$.

Proof. Let $\mathcal{T}:\mathcal{C}^{\eta}_{\Psi,\delta}\to\mathcal{C}^{\eta}_{\Psi,\delta}$ as

$$(\mathcal{T}\mathfrak{f})(\iota)=\begin{cases}\mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}[\mathcal{G}(\iota,\mathfrak{f}_\iota)]+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}[\mathcal{Z}_i(\iota,\mathfrak{f}_\iota)], & \iota\in\mathcal{J},\\ \lambda(\iota), \quad \iota\in[\mathfrak{a}_1-\delta,\mathfrak{a}_1]. & \end{cases}$$

Now we need to show that $\mathcal{T}$ is a contraction under the Λ-condition.
Using assumption (A_2), for $\iota\in\mathcal{J}$,

$$\begin{aligned}|\mathcal{T}_{\mathfrak{f}}(\iota)-\mathcal{T}_{\bar{\mathfrak{f}}}(\iota)| &\le \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}|\mathcal{G}(\iota,\mathfrak{f}_\iota)-\mathcal{G}(\iota,\bar{\mathfrak{f}}_\iota)|+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}|\mathcal{Z}_i(\iota,\mathfrak{f}_\iota)-\mathcal{Z}_i(\iota,\bar{\mathfrak{f}}_\iota)|\\ &\le \frac{(\Psi\ (\mathfrak{b}_1)-\Psi\ (\mathfrak{a}_1))^{\eta}}{\Gamma(\eta+1)}\|\omega\|\|\mathfrak{f}_\iota-\bar{\mathfrak{f}}_\iota\|_{\mathcal{C}_\delta}\mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}(1)(\iota)\\ &+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\frac{(\Psi\ (\mathfrak{b}_1)-\Psi\ (\mathfrak{a}_1))^{\xi_i}}{\Gamma(\xi_i+1)}\|\nu_i\|\|\mathfrak{f}_\iota-\bar{\mathfrak{f}}_\iota\|_{\mathcal{C}_\delta}\mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}(1)(\iota)\\ &\le\left[\frac{(\Psi\ (\mathfrak{b}_1)-\Psi\ (\mathfrak{a}_1))^{\eta}}{\Gamma(\eta+1)}\|\omega\|+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\frac{(\Psi\ (\mathfrak{b}_1)-\Psi\ (\mathfrak{a}_1))^{\xi_i}}{\Gamma(\xi_i+1)}\|\nu_i\|\right]\|\mathfrak{f}_\iota-\bar{\mathfrak{f}}_\iota\|_{\mathcal{C}^{\eta}_{\Psi,\delta}}\\ &\le(\Lambda_1+\Lambda_2)\|\mathfrak{f}_\iota-\bar{\mathfrak{f}}_\iota\|_{\mathcal{C}^{\eta}_{\Psi,\delta}}.\end{aligned}$$

Hence,

$$\|\mathcal{T}_{\mathfrak{f}}-\mathcal{T}_{\bar{\mathfrak{f}}}\|_{\mathcal{C}^{\eta}_{\Psi,\delta}}\le\Lambda\|\mathfrak{f}_\iota-\bar{\mathfrak{f}}_\iota\|_{\mathcal{C}^{\eta}_{\Psi,\delta}}.$$

Since $\Lambda<1$, $\mathcal{T}$ is a contraction and Theorem 2.8 guaranties a unique fixed point, which corresponds to the unique solution of (1.1) □

Theorem 3.4. *If assumptions $(A_1)-(A_3)$ are satisfied and*

$$\frac{\big(\Psi\ (\mathfrak{b}_1)-\Psi\ (\mathfrak{a}_1)\big)^{\eta}}{\Gamma(\eta+1)}\Phi<1. \tag{3.3}$$

then (1) has at least one solution on $[\mathfrak{a}_1-\delta,\mathfrak{b}_1]$

Proof. Suppose $\mathcal{C}^{\eta}_{\Psi,\delta}=\mathcal{C}([\mathfrak{a}_1-\delta,\mathfrak{b}_1],\mathcal{R})$ be the weighted space of continuous real-valued functions with the supremum norm, and define the closed ball:

$$\mathcal{B}_{\varrho}=\{\mathfrak{f}\in\mathcal{C}^{\eta}_{\Psi,\delta}:\|\mathfrak{f}\|_{\mathcal{C}^{\eta}_{\Psi,\delta}}\le\varrho\},$$

where

$$\varrho\ge\frac{\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\frac{(\Psi(\mathfrak{b}_1)-\Psi(\mathfrak{a}_1))^{\xi_i}}{\Gamma(\xi_i+1)}\Phi_i}{1-\frac{(\Psi(\mathfrak{b}_1)-\Psi(\mathfrak{a}_1))^{\eta}}{\Gamma(\eta+1)}\Phi}$$

Define the operator $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ where

$$\mathcal{T}_1(\mathfrak{f})(\iota) = \begin{cases} \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}\left[\mathcal{G}(\iota, \mathfrak{f}_\iota)\right], & \iota \in \mathcal{J}, \\ 0, & \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]. \end{cases}$$

$$\mathcal{T}_2(\mathfrak{f})(\iota) = \begin{cases} \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}\left[\mathcal{Z}_i(\iota, \mathfrak{f}_\iota)\right], & \iota \in \mathcal{J}, \\ \lambda, & \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]. \end{cases}$$

Step 1: $\mathcal{T}_1(\mathfrak{f}) + \mathcal{T}_2(\mathfrak{f}) \in \mathcal{B}_\varrho$.
For $\iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]$,

$$|\mathcal{T}_1(\mathfrak{f})(\iota) + \mathcal{T}_2(\mathfrak{f})(\iota)| \leq |\lambda(\iota)| \leq \|\lambda\|_{\mathcal{C}_\delta} \leq \varrho.$$

For $\iota \in \mathcal{J}$, using assumption (A_3), we estimate

$$\begin{aligned} |\mathcal{T}_1(\mathfrak{f})(\iota) + \mathcal{T}_2(\mathfrak{f})(\iota)| &\leq \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}|\mathcal{G}(\iota, \mathfrak{f}_\iota)| + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}|\mathcal{Z}_i(\iota, \mathfrak{f}_\iota)| \\ &\leq \Phi\|\mathfrak{f}\|_{\mathcal{C}^\eta_{\Psi,\delta}} \frac{(\Psi(\iota) - \Psi\ (\mathfrak{a}_1))^\eta}{\Gamma(\eta + 1)} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \frac{\left(\Psi(\iota) - \Psi\ (\mathfrak{a}_1)\right)^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i \\ &\leq \frac{(\Psi(\mathfrak{b}_1) - \Psi(\mathfrak{a}_1))^\eta}{\Gamma(\eta + 1)} \Phi\varrho + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \frac{\left(\Psi(\mathfrak{b}_1) - \Psi(\mathfrak{a}_1)\right)^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i \\ &\leq \varrho. \end{aligned}$$

So, $\mathcal{T}_1(\mathfrak{f}) + \mathcal{T}_2(\mathfrak{f}) \in \mathcal{B}_\varrho$.
Step 2: Since $\mathcal{T}$ is a contraction as shown in Theorem 3.3, it follows that $\mathcal{T}_1$ also satisfies the contraction property.
Step 3: $\mathcal{T}_2$ is completely continuous on $\mathcal{B}_\varrho$.
Due to the continuity of $\mathcal{Z}_i(., \mathfrak{f}(.))$ and $\lambda(.)$, it follows that $\mathcal{T}_2$ is continuous.
Moreover

$$\|\mathcal{T}_2\mathfrak{f}\|_{\mathcal{C}_\delta} = \sup_{\iota \in [-\delta, 0]} |\mathcal{T}_2\mathfrak{f}(\iota)| = \sup_{\iota \in [-\delta, 0]} |\lambda(\iota)| = \|\lambda\|_{\mathcal{C}_\delta} \leq \|\lambda\|_{\mathcal{C}^\eta_{\Psi,\delta}} \leq \varrho, \quad \mathfrak{f} \in \mathcal{B}_\varrho.$$

and

$$\|\mathcal{T}_2\mathfrak{f}\|_{\mathcal{C}} = \sup_{\iota \in \mathcal{J}} |\mathcal{Z}_2\mathfrak{f}(\iota)| \leq \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \frac{\left(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1)\right)^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i = \mathcal{L}, \quad \mathfrak{f} \in \mathcal{B}_\varrho.$$

Thus, combining these estimates

$$\|\mathcal{T}_2\mathfrak{f}\|_{\mathcal{C}^\eta_{\Psi,\delta}} \leq \varrho + \mathcal{L}$$

This confirms that $\mathcal{T}_2$ is uniformly bounded on the set $\mathcal{B}_\varrho$.
It remains to establish the equicontinuity of $\mathcal{T}_2$ on $\mathcal{B}_\varrho$.

The operator derivative can be estimate as follows,

$$
\begin{aligned}
|(\mathcal{T}_2)^{(1)}_{\Psi}(\iota| &\le \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i-1;\Psi}_{\mathfrak{a}_1+}|\mathcal{Z}_i(\iota,\mathfrak{f}_\iota)| \\
&\le \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \Phi_i \mathcal{I}^{\xi_i-1;\Psi}_{\mathfrak{a}_1+}(1)(\iota) \\
&\le \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \frac{\left(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1)\right)^{\xi_i - 1}}{\Gamma(\xi_i)} \Phi_i = l
\end{aligned}
$$

Now, for any $\iota_1, \iota_2 \in \mathcal{J}$,

$$
|(\mathcal{T}_2\mathfrak{f})(\iota_2) - (\mathcal{T}_2\mathfrak{f})(\iota_1)| = \int_{\iota_1}^{\iota_2} |(\mathcal{T}_2)(s)| ds \le l(\iota_2 - \iota_1).
$$

As $\iota_2 - \iota_1 \to 0$, this expression tends to zero uniformly, proving equicontinuity. By the Arzela- Ascoli theorem, since $\mathcal{T}_2$ is uniformly bounded and equicontinuous on $\mathcal{B}_\varrho$, it is relatively compact in $\mathcal{C}^{\eta}_{\Psi,\delta}$. Together with continuity we conclude that$\mathcal{T}_2$ is compact on $\mathcal{B}_\varrho$. □

Definition 3.5. *Problem (1.1) is $\mathcal{UHS}$ if there exists $\mathfrak{c} \in \mathcal{R}^+$ such that, for every $\epsilon > 0$ and $\bar{\mathfrak{f}}$ satisfying*

$$
\begin{cases}
\left| {}^{\mathcal{H}}\mathcal{D}^{\varsigma_1,\varsigma_2;\Psi}_{\mathfrak{a}_1+}\left[\mathfrak{f}(\iota) - \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}\mathcal{Z}_i(\iota,\mathfrak{f}_i)\right] - \mathcal{G}(\iota,\mathfrak{f}_\iota)\right| \le \epsilon, & \iota \in \mathcal{J} = [\mathfrak{a}_1, \mathfrak{b}_1], \\
|\bar{\mathfrak{f}}(\iota) - \lambda(\iota)| \le \epsilon, \quad \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1],
\end{cases}
\tag{3.4}
$$

there exists a unique solution $\mathfrak{f} \in \mathcal{C}_\delta$ with

$$
\|\bar{\mathfrak{f}} - \mathfrak{f}\|_{\mathcal{C}_\delta} \le \mathfrak{c}\epsilon.
$$

Definition 3.6. *Problem (1.1) is generalized $\mathcal{UHS}$ if there exists $\sigma \in \mathcal{C}(\mathcal{R}^+, \mathcal{R}^+)$,with $\sigma(0) = 0$,such that, for every $\epsilon > 0$, $\bar{\mathfrak{f}} \in \mathcal{C}^{\eta}_{\Psi,\delta}$ satisfying the inequality in (3.4), there exist a unique solution $\mathfrak{f} \in \mathcal{C}_\delta$ of (1.1) such that,*

$$
\|\bar{\mathfrak{f}} - \mathfrak{f}\|_{\mathcal{C}_\delta} \le \sigma(\epsilon).
$$

Remark 3.7. $\bar{\mathfrak{f}} \in \mathcal{C}_\delta$ is a solution of the inequality (3.4) if and only if there exist $\mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ such that

- $|\mathfrak{g}(\iota)| \le \epsilon, \quad \forall \iota \in \mathcal{J}$ and
- ${}^{\mathcal{H}}\mathcal{D}^{\varsigma_1,\varsigma_2;\Psi}_{\mathfrak{a}_1+}\left[\bar{\mathfrak{f}}(\iota) - \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}\mathcal{Z}_i(\iota,\bar{\mathfrak{f}}_i)\right] = \mathcal{G}(\iota,\bar{\mathfrak{f}}_\iota) + \mathfrak{g}(\iota), \quad \forall \iota \in \mathcal{J}.$

Theorem 3.8. *If the condition (A_2) and (3.2) are satisfied. then the solution of (1.1) is $\mathcal{UHS}$ and generalized $\mathcal{UHS}$.*

Proof. Suppose $\epsilon \in \mathcal{R}^+$, let $\bar{\mathfrak{f}} \in \mathcal{C}^{\eta}_{\Psi,\delta}$ be any solution of the inequality. By Remark 3.7, there exists $\mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ with $|\mathfrak{g}(\iota)| \le \epsilon, \quad \iota \in \mathcal{J}$, such that the perturbed problem is given by

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\varsigma_1,\varsigma_2;\Psi}_{\mathfrak{a}_1+}\left[\bar{\mathfrak{f}}(\iota) - \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}\mathcal{Z}_i(\iota,\bar{\mathfrak{f}}_i)\right] = \mathcal{G}(\iota,\bar{\mathfrak{f}}_\iota) + \mathfrak{g}(\iota), \quad \forall \iota \in \mathcal{J}. \\ \bar{\mathfrak{f}}(\iota) = \lambda(\iota), \quad \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]. \end{cases} \tag{3.5}$$

By Lemma 3.2 , the solution $\bar{\mathfrak{f}}(\iota)$ to the perturbed problem is given by

$$\bar{\mathfrak{f}}(\iota) = \begin{cases} \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}\left[\mathcal{G}(\iota,\bar{\mathfrak{f}}_\iota) + \mathfrak{g}(\iota)\right] + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}\left[\mathcal{Z}_i(\iota,\bar{\mathfrak{f}}_\iota)\right], \quad \iota \in \mathcal{J}, \\ \lambda(\iota), \quad \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]. \end{cases}$$

Let $\mathfrak{f} \in \mathcal{C}_\delta$ be the solution of (1.1) which satisfies:

$$\mathfrak{f}(\iota) = \begin{cases} \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}\left[\mathcal{G}(\iota,\mathfrak{f}_\iota)\right] + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}\left[\mathcal{Z}_i(\iota,\mathfrak{f}_\iota)\right], \quad \iota \in \mathcal{J}, \\ \lambda(\iota), \quad \iota \in [\mathfrak{a}_1 - \delta, \mathfrak{a}_1]. \end{cases}$$

Now, estimate the difference $|\bar{\mathfrak{f}} - \mathfrak{f}|$ for $\iota \in \mathcal{J}$:

$$|\bar{\mathfrak{f}}(\iota) - \mathfrak{f}(\iota)| \leq \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}|\mathcal{G}(\iota,\bar{\mathfrak{f}}_\iota) - \mathcal{G}(\iota,\mathfrak{f}_\iota)| + \mathcal{I}^{\eta;\Psi}_{\mathfrak{a}_1+}|\mathfrak{g}(\iota)| + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \mathcal{I}^{\xi_i;\Psi}_{\mathfrak{a}_1+}|\mathcal{Z}_i(\iota,\bar{\mathfrak{f}}_\iota) - \mathcal{Z}_i(\iota,\mathfrak{f}_\iota)|$$

Applying assumption (A_2) and estimate $|\mathfrak{g}(\iota)| \leq \epsilon$, we have

$$\begin{aligned} |\bar{\mathfrak{f}}(\iota) - \mathfrak{f}(\iota)| \leq & \left(\frac{(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1))^\eta}{\Gamma(\eta+1)}\|\omega\| + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \frac{(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1))^{\xi_i}}{\Gamma(\xi_i+1)}\|\nu_i\|\right)\|\bar{\mathfrak{f}}_\iota - \mathfrak{f}_\iota\|_{\mathcal{C}_\delta} \\ & + \frac{(\Psi\ (\mathfrak{b}_1) - \Psi\ (\mathfrak{a}_1))^\eta}{\Gamma(\eta+1)}\epsilon. \end{aligned}$$

That is,

$$\|\bar{\mathfrak{f}} - \mathfrak{f}\|_{\mathcal{C}^\eta_{\Psi,\delta}} \leq \Lambda\|\bar{\mathfrak{f}} - \mathfrak{f}\|_{\mathcal{C}^\eta_{\Psi,\delta}} + \mathfrak{p}\epsilon$$

where

$$\mathfrak{p} = \frac{(\Psi(\mathfrak{b}_1) - \Psi(\mathfrak{a}_1))^\eta}{\Gamma(\eta+1)}$$

Therefore

$$\begin{aligned} (1-\Lambda)\|\bar{\mathfrak{f}}_\iota - \mathfrak{f}_\iota\|_{\mathcal{C}^\eta_{\Psi,\delta}} &\leq \mathfrak{p}\epsilon \\ \|\bar{\mathfrak{f}}_\iota - \mathfrak{f}_\iota\|_{\mathcal{C}^\eta_{\Psi,\delta}} &\leq \frac{\mathfrak{p}}{(1-\Lambda)}\epsilon. \end{aligned}$$

Let $\mathfrak{c} = \frac{\mathfrak{p}}{(1-\Lambda)} > 0$. Then,

$$\|\bar{\mathfrak{f}}_\iota - \mathfrak{f}_\iota\|_{\mathcal{C}^\eta_{\Psi,\delta}} \leq \mathfrak{c}\epsilon$$

Which proves the problem (1.1) is $\mathcal{UHS}$.
To show generalized $\mathcal{UHS}$, defined $\sigma(\epsilon) = \frac{\mathfrak{p}}{(1-\Lambda)}\epsilon \in \mathcal{C}(\mathcal{R}^+,\mathcal{R}^+)$ with $\sigma(0) = 0$.
Thus

$$\|\bar{\mathfrak{f}}_\iota - \mathfrak{f}_\iota\|_{\mathcal{C}^\eta_{\Psi,\delta}} \leq \sigma(\epsilon)$$

and the solution of (1.1) is generalized $\mathcal{UHS}$. □

4. An example

To demonstrate the applicability of the theoretical framework, consider the specific form of equation (1.1) on the interval $\mathcal{J} = [1, 2]$.
Let the function $\psi(\iota) = \iota^2$, with parameters:
$a = 1, b = 2, \varsigma_1 = \frac{1}{2}, \varsigma_2 = \frac{1}{2}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{2}, \delta > 0$.
Define the nonlinear terms as

$$\mathcal{Z}_i(\iota, \mathfrak{r}) = \frac{\iota^2}{(10+\iota)} \tan(\mathfrak{r}), \quad \mathcal{G}(\iota, \mathfrak{r}) = \frac{1}{[5(1+\iota^2)]} \cos(\mathfrak{r}) + \frac{\iota}{(20+\iota^2)} \tag{4.1}$$

Using the expressions, the constant are evaluated as
$\Lambda_1 = 0.3261$ and $\Lambda_2 = 0.1452$, $\Lambda = \Lambda_1 + \Lambda_2 = 0.4713 < 1$.
Hence, Theorem 3.3 is satisfied, guaranteeing uniqueness of the solution.
Moreover, with

$$\frac{(\Psi(\mathfrak{b}_1) - \Psi(\mathfrak{a}_1))^{\eta}}{\Gamma(\eta+1)} \Phi = 0.1355 < 1$$

The hypotheses of both Theorem 3.3 and Theorem 3.4 are satisfied. Therefore, we conclude that the problem (1.1) with $\mathcal{G}(\iota, \mathfrak{r})$ and $\mathcal{Z}_i(\iota, \mathfrak{r})$ defined in (4.1), admits a unique solution on the interval $[1, 2]$. Moreover, existence and uniqueness are guaranteed.

5. Conclusion

This paper examined a class of fractional neutral integro-differential equations with delay involving the Ψ-$\mathcal{HFD}$. By applying Banach's contraction principle and Krasnoselskii's fixed point theorem, conditions were established to guarantee the existence and uniqueness of solutions. In addition, Ulam–Hyers stability and its generalized form were demonstrated, confirming that the solutions remain stable under small perturbations.An illustrative example verified the theoretical results and highlighted the role of fractional orders, delay effects, and nonlinear terms in ensuring solvability. The study reinforces the utility of the Ψ-Hilfer operator as a versatile tool for analyzing systems with memory and hereditary properties.

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