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Fixed points and dynamic programming in complex-valued controlled metric spaces

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Abstract. In this paper, we introduce the concepts of $(\alpha - \Theta)$-contraction and Reich-type contraction within the framework of complex-valued controlled metric spaces (CVCMS). We also present related fixed point theorems for CVCMS, building on the works considered in the literature review for controlled metric type spaces. To demonstrate the practical implications and significance of our results, we provide several examples and an application in dynamic programming.

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1. Introduction and preliminaries

Fixed point theory has numerous applications across various fields of mathematics, engineering, and physics. It provides foundational tools for solving equations and modeling dynamic systems. One of the most significant contributions to this field was made by Stefan Banach [7], who introduced the Banach contraction theorem. This theorem not only established a method for proving the existence and uniqueness of fixed points in complete metric spaces but also laid the groundwork for further advancements in analysis and topology. Over the years, many researchers have extended Banach's theorem in diverse ways, exploring its implications in more complex and abstract settings, such as in non-linear analysis, differential equations, and optimization problems. These extensions have broadened the scope of fixed point theory, making it a versatile and powerful tool in both theoretical and applied mathematics.

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Azam et al. [6] gave the notion named complex-valued metric space (CV - metric space). Rao et al. [22] generalized CV - metric space as complex-valued b-metric space(CVBMS), which was later extended by Ullah et al. [23] by introducing a metric and naming this notion as complex valued-extended b-metric space (CVEb-metric space), further discussed by Belhenniche et al. [9]). In 2018 Mlaiki et al. [19] presented a controlled metric space that behaves as a different extension of b - metric space. In [20] Mlaiki et al. extended the results in [19]. Many mathematicians and researchers are working on this idea. Al-Mazrooei et al. [3] presented common fixed-point results for generalized contactions, Hussain et al. [15] discussed Fixed Point results For Nonlinear Contraction, Panday et al. [21] discussed Rational Type Contraction and Durdana Lateef presented Fisher [17] and Kannan [18] types contractions for controlled metric spaces.

Recently, the idea of the controlled metric space was further generalized by Aslam et al. [5], by introducing the new generalized space with the name of complex-valued controlled metric type space (CVCMS). They also presented different contraction theorems to illustrate the new concept. Ahmad et al. [2] presented rational type $(\alpha-\Theta)$ - contraction and Ahmad et al. [1] presented Reich type contraction for controlled metric spaces. In this new paper our aim is to define few type of contractions for the case of CVCMS. Moreover, several examples are provided, along with an application to dynamic programming.

Suppose we represent $\mathbb{C}$ for the complex numbers set with $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathbb{C}$. To compare two complex numbers we will use " $\precsim$ " symbol as the *partial order* on the set $\mathbb{C}$, which is called in related literature as *lexicographic order*,

"$\mathfrak{e}_1 \precsim \mathfrak{e}_2$" if and only if $Re(\mathfrak{e}_1) \leq Re(\mathfrak{e}_2)$ or $(Re(\mathfrak{e}_1) = Re(\mathfrak{e}_2)$ and $Im(\mathfrak{e}_1) \leq Im(\mathfrak{e}_2))$.

Taking under consideration the previous definition of partial order, we can say that "$\mathfrak{e}_1 \precsim \mathfrak{e}_2$" if from following conditions any one holds or satisfied:

(P_1) $Re(\mathfrak{e}_1) < Re(\mathfrak{e}_2)$ together with $Im(\mathfrak{e}_1) < Im(\mathfrak{e}_2)$;
(P_2) $Re(\mathfrak{e}_1) < Re(\mathfrak{e}_2)$ together with $Im(\mathfrak{e}_1) = Im(\mathfrak{e}_2)$;
(P_3) $Re(\mathfrak{e}_1) < Re(\mathfrak{e}_2)$ together with $Im(\mathfrak{e}_1) > Im(\mathfrak{e}_2)$;
(P_4) $Re(\mathfrak{e}_1) = Re(\mathfrak{e}_2)$ together with $Im(\mathfrak{e}_1) < Im(\mathfrak{e}_2)$.

From literature, we start from extension of b-metric and recall the concept of CVEb-metric space which was presented by N. Ullah et al. [23] in 2019.

Definition 1.1. [23] *Consider* $\mathbb{H}$ *as non empty set with a mapping* $\zeta : \mathbb{H} \times \mathbb{H} \to [1, \infty)$. *Then the functional defined as* $\mathrm{h} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ *is known to be CVEb-metric if all following described axioms holds good:*

(CEB_1) $0 \precsim \mathrm{h}(\mathfrak{e}, f)$ *and* $\mathrm{h}(\mathfrak{e}, f) = 0$ *if and only if* $\mathfrak{e} = f$,
(CEB_2) $\mathrm{h}(\mathfrak{e}, f) = \mathrm{h}(f, \mathfrak{e})$,
(CEB_3) $\mathrm{h}(\mathfrak{e}, g) \precsim \zeta(\mathfrak{e}, g)[\mathrm{h}(\mathfrak{e}, f) + \mathrm{h}(f, g)]$,
for all $\mathfrak{e}, f, g \in \mathbb{H}$. *A pair* $(\mathbb{H}, \mathrm{h})$ *is called a CVEb-metric space.*

Mlaiki et al.[19] gave the definition of the controlled metric type and there gave some fixed point results for this newly introduced type of metric.

Definition 1.2. *([19]) Consider* $\mathbb{H}$ *as non empty set with a mapping* $\zeta : \mathbb{H}\times\mathbb{H} \to [1,\infty)$*. The functional* $\mathrm{h} : \mathbb{H} \times \mathbb{H} \to [0,\infty)$ *is known to be controlled metric if the following conditions are satisfied:*

(CMT_1) $0 \leq \mathrm{h}(\mathfrak{e}, f)$ *and* $\mathrm{h}(\mathfrak{e}, f) = 0$ *if and only if* $\mathfrak{e} = f$,

(CMT_2) $\mathrm{h}(\mathfrak{e}, f) = \mathrm{h}(f, \mathfrak{e})$,

(CMT_3) $\mathrm{h}(\mathfrak{e}, g) \leq \zeta(\mathfrak{e}, f)\mathrm{h}(\mathfrak{e}, f) + \zeta(f, g)\mathrm{h}(f, g)$,

for all $\mathfrak{e}, f, g \in \mathbb{H}$*. A pair* $(\mathbb{H}, \mathrm{h})$ *is said to be a controlled metric type space .*

Recently Aslam et al. [5] introduced the concept of CVCMS defined as follows. As we observe, the CVC-metric is a symmetric norm, which is an important detail in order to obtain new fixed point results.

Definition 1.3. *Consider* $\mathbb{H}$ *as a non empty set with a mapping* $\zeta : \mathbb{H} \times \mathbb{H} \to [1,\infty)$*. The functional* $\mathrm{h} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ *is called as CVC- metric if following described axioms holds good:*

$(CCMT_1)$ $0 \precsim \mathrm{h}(\mathfrak{e}, f)$ *and* $\mathrm{h}(\mathfrak{e}, f) = 0$ *if and only if* $\mathfrak{e} = f$,

$(CCMT_2)$ $\mathrm{h}(\mathfrak{e}, f) = \mathrm{h}(f, \mathfrak{e})$,

$(CCMT_3)$ $\mathrm{h}(\mathfrak{e}, g) \precsim \zeta(\mathfrak{e}, f)\mathrm{h}(\mathfrak{e}, f) + \zeta(f, g)\mathrm{h}(f, g)$,

for all $\mathfrak{e}, f, g \in \mathbb{H}$*. A pair* $(\mathbb{H}, \mathrm{h})$ *is called a CVCMS.*

Example 1.4. [5] Let $\mathbb{H} = [0,\infty)$ and $\zeta : \mathbb{H} \times \mathbb{H} \to [1,\infty)$ be described as

$$\zeta(\mathfrak{e}, f) = \begin{cases} 1, & \text{if } \mathfrak{e}, f \in [0, 1], \\ 1 + \mathfrak{e} + f, & \text{otherwise.} \end{cases}$$

and $h : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ defined by the following

$$h(\mathfrak{e}, f) := \begin{cases} 0, & \mathfrak{e} = f \\ i, & \mathfrak{e} \neq f \end{cases}$$

Then $(\mathbb{H}, h)$ is a CVCMS.

Definition 1.5. *We consider* Θ *as the set of the functionals* $\Theta : (0,\infty) \to (1,\infty)$ *which satisfy the axioms described as under:*

(Θ_1) Θ *be a non-decreasing function.*

(Θ_2) *For each of the sequence* $\{l_n\} \subseteq l^+$, $\lim_{n\to\infty} \Theta(l_n) = 1 \iff \lim_{n\to\infty} l_n = 0^+$;

(Θ_3) $\exists$ *a constant* $0 < k < 1$ *and* $\vartheta \in (0,\infty]$*, so that,* $\lim_{l\to 0^+} \frac{\Theta(l)-1}{l^k} = \vartheta$.

Definition 1.6. *A self map* $\mathrm{T} : \mathbb{H} \longrightarrow \mathbb{H}$ *is known as* Θ*-contraction, if there exists a* Θ *which satisfy conditions* $(\Theta_1) - (\Theta_3)$ *and* $k \in (0, 1)$*, such that:*

$$\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f) \neq 0 \implies \Theta(\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f) \precsim [\Theta(\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f))]^k$$

for all $\mathfrak{e}, f \in \mathbb{H}$.

Considering the last section, we will present one application in dynamic programming, domain highly used in economics and financial phenomena. Taking into account the economics direction, during a simultaneous auction, each agent calculates their demand for a product at every possible price and puts it up to an auctioneer, which is known as a Walrasian auction. The price is then determined in such a way that the total demand of all agents is equal to the total amount of the property. The perfect match between supply and demand is achieved in a Walrasian auction.

The presence of a Walrasian auctioneer is the reason why most heterogeneous agent models in economics have an obvious symmetry. The only way to maintain symmetry is by agents acting in accordance with identical state values, which can only be maintained by them.

In this paper, we introduce the $(\alpha - \Theta)$ - contraction and Reich type contraction for complex-valued controlled metric spaces (CVCMS). Also related fixed point theorems for CVCMS are presented, as presented in [1] and [2] for controlled metric type space. To illustrate the results and their significance some examples and an application is also presented for the dynamic programming.

2. New fixed point results in CVCMS

In this section of the paper we give some new fixed point result for CVCMS with illustrative examples. First fixed point result is the following.

2.1. $(\alpha, \Theta)-$ type fixed point results in CVCMS

In this subsection, we will present some new fixed points results corresponding to the $(\alpha, \Theta)-$ contraction principle on the settings of the CVCMS. Further, in our theorems, we will consider $(\mathbb{H}, h)$ as a complete complex-valued controlled metric spaces and we denote it by CVCMS. By CVEb-metric space we will denote a complex-valued extended b-metric space.

Definition 2.1. *Assume* $(\mathbb{H}, \mathrm{h})$ *be CVCMS. A functional* $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ *is denoted as rational type* (α, Θ) *- contraction, if there exists* $\alpha : \mathbb{H} \times \mathbb{H} \to \mathbb{R}^+$*, where* $\kappa \in (0, 1)$ *with* $\Theta \in \Omega$*, so that*

$$\alpha(\mathfrak{e}, f)\Theta(\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f)) \precsim \Theta(M(\mathfrak{e}, f))^{\kappa} \tag{2.1}$$

here

$$M(\mathfrak{e}, f) = max\{\mathrm{h}(\mathfrak{e}, f), \mathrm{h}(\mathfrak{e}, \mathrm{T}\mathfrak{e}), \mathrm{h}(f, \mathrm{T}f), \frac{\mathrm{h}(\mathfrak{e}, \mathrm{T}\mathfrak{e})\mathrm{h}(f, \mathrm{T}f)}{1 + \mathrm{h}(\mathfrak{e}, f)}\} \tag{2.2}$$

for all $\mathfrak{e}, f \in \mathbb{H}$ *with* $\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f) \succsim 0$.

Theorem 2.2. *Consider* $(\mathbb{H}, \mathrm{h})$ *be a CVCMS. A function* $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ *be rational type* $(\alpha, \Theta)-$*contraction satisfying:*

1. T *is an* $\alpha-$*admissible;*
2. *there exist* $\mathfrak{e}_0 \in \mathbb{H}$*, so that the* $\alpha(\mathfrak{e}_0, \mathrm{T}\mathfrak{e}_0) \geq 0$*;*
3. T *is continuous;*
4. $\sup_{m \geq 1} \lim_{i \to \infty} \frac{\zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_{i+2})\zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_m)}{\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})} < 1.$

Assume also, for every $e \in \mathbb{H}$*, we have* $\lim_{n\to\infty} \zeta(\mathfrak{e}_n, \mathfrak{e})$ *and* $\lim_{n\to\infty} \zeta(\mathfrak{e}, \mathfrak{e}_n)$ *exist and are finite. Then there exist* $\mathfrak{e}^* \in \mathbb{H}$ *such that the* $\mathfrak{e}^* = \mathrm{T}\mathfrak{e}^*$.

Proof. Suppose $\mathfrak{e}_0 \in \mathbb{H}$ such that $\alpha(\mathfrak{e}_0, \mathrm{T}\mathfrak{e}_0) \geq 1$. Considering a sequence $\{\mathfrak{e}_n\}$ in $\mathbb{H}$ by $\mathfrak{e}_{n+1} = \mathrm{T}\mathfrak{e}_n$, for all $n \in N$. If there exists $n_0 \in N$ for which $\{\mathfrak{e}_{n_0+1} = \mathfrak{e}_{n_o}\}$, then $\mathrm{T}\mathfrak{e}_0 = \mathfrak{e}_0$ and hence the proof is completed. Thus, we assume that the sequence $\{\mathfrak{e}_{n+1} \neq \mathfrak{e}_n\}$, for all $n \in N$. By using (1) and (2), it's obvious

$$\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq 1,$$

for all $n \in N$. So, by (2.1), we obtain

$$1 \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) = \Theta(h(\mathrm{T}\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_n)) \precsim \alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1})\Theta(h(\mathrm{T}\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_n)).$$

Since T is a (α, Θ)-contraction, so for all $n \in N$, we have

$$\begin{aligned}
1 &\precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) \precsim \alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1})\Theta(h(\mathrm{T}\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_n)) \\
&\precsim \Theta(M(\mathfrak{e}_{n-1}, \mathfrak{e}_n))^\kappa \\
&= \Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_{n-1}), h(\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_n), \frac{h(\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_{n-1})h(\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_n)}{1 + h(\mathfrak{e}_{n-1}, \mathfrak{e}_n)}\})^\kappa \\
&= \Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_n), h(\mathfrak{e}_n, \mathfrak{e}_{n+1}), \frac{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n)h(\mathfrak{e}_n, \mathfrak{e}_{n+1})}{1 + h(\mathfrak{e}_{n-1}, \mathfrak{e}_n)}\})^\kappa \\
&\precsim \Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_n), h(\mathfrak{e}_n, \mathfrak{e}_{n+1}), h(\mathfrak{e}_n, \mathfrak{e}_{n+1})\})^\kappa \\
&= \Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_n, \mathfrak{e}_{n+1})\})^\kappa.
\end{aligned}$$

Thus

$$1 \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) \precsim \Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_n, \mathfrak{e}_{n+1})\})^\kappa. \tag{2.3}$$

If there exists n $\in N$ (a natural number), such that

$$\Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_n, \mathfrak{e}_{n+1})\})^\kappa = h(\mathfrak{e}_n, \mathfrak{e}_{n+1}),$$

then (2.3) becomes

$$1 \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1}))^\kappa \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})).$$

Which is obviously a contradiction.
So $\Theta(max\{h(\mathfrak{e}_{n-1}, \mathfrak{e}_n), h(\mathfrak{e}_n, \mathfrak{e}_{n+1})\})^\kappa = h(\mathfrak{e}_{n-1}, \mathfrak{e}_n)$ for all natural numbers $n \in N$. Therefore by (2.3), we have

$$\begin{aligned}
1 &\precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1}))^\kappa \\
&\precsim \Theta(h(\mathfrak{e}_{n-1}, \mathfrak{e}_n))^{\kappa^2} \\
&\precsim \Theta(h(\mathfrak{e}_{n-2}, \mathfrak{e}_{n-1}))^{\kappa^3} \\
&\dots \\
&\precsim \Theta(h(\mathfrak{e}_0, \mathfrak{e}_1))^{\kappa^n}.
\end{aligned}$$

Thus by (2.3), we have

$$1 \precsim \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) \precsim \Theta(h(\mathfrak{e}_0, \mathfrak{e}_1))^{\kappa^n}. \tag{2.4}$$

Taking $n \to \infty$ in (2.4), we get

$$\lim_{n\to\infty} \Theta(h(\mathfrak{e}_n, \mathfrak{e}_{n+1})) = 1.$$

By (Θ_2) we get

$$\lim_{n\to\infty} h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) = 0.$$

By (Θ_3), there exists a $0 < \upsilon < 1$ with $\zeta \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1}{(h(\mathfrak{e}_n, \mathfrak{e}_{n+1}))^\upsilon} = \zeta.$$

Suppose that for $\zeta < \infty$. In this case, consider $\lambda = \frac{\zeta}{2} > 0$. So by definition, there exists $n_1 \in N$, such that

$$|\frac{\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1}{(h(\mathfrak{e}_n, \mathfrak{e}_{n+1}))^v} - \zeta| \leq \lambda, \text{ for all } n > n_1.$$

This implies that

$$\frac{\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1}{(h(\mathfrak{e}_n, \mathfrak{e}_{n+1}))^v} \geq \zeta - \lambda = \frac{\zeta}{2} = \lambda, \forall n > n_1.$$

Then

$$nh(\mathfrak{e}_n, \mathfrak{e}_{n+1})^h \precsim \mu n[\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1], \text{ for all } n > n_1,$$

with $\mu = \frac{1}{\lambda}$. Now we suppose that $\zeta = \infty$ with $\lambda > 0$. By the definition, there exists $n_1 \in N$, such that

$$\lambda \leq \frac{\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1}{h(\mathfrak{e}_n, \mathfrak{e}_{n+1})^v}, \text{ for all } n > n_1.$$

This implies

$$nh(\mathfrak{e}_n, \mathfrak{e}_{n+1})^v \precsim \mu n[\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1], \text{ for all } n > n_1,$$

where we have $\mu = \frac{1}{\lambda}$. Hence, in all type of situations, there exists $\mu > 0$ together with $n_1 \in N$ such that

$$nh(\mathfrak{e}_n, \mathfrak{e}_{n+1})^v \precsim \mu n[\Theta(h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1})) - 1], \tag{2.5}$$

for all $n > n_1$. So by (2.4) and (2.5), we get

$$nh(\mathfrak{e}_n, \mathfrak{e}_{n+1})^v \precsim \mu n([(\Theta(h(\mathfrak{e}_0, \mathfrak{e}_1)))]^{k^n} - 1).$$

Applying $n \to \infty$, we get the following

$$\lim_{n \to \infty} nh(\mathfrak{e}_n, \mathfrak{e}_{n+1})^v = 0.$$

So, there exists $n_2 \in N$, such that

$$h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \precsim \frac{1}{n^{1/v}}, \forall n > n_2.$$

Further, considering the triangular inequality for $p \geq 1$, we get

$$\begin{aligned}
h(\mathfrak{e}_n, \mathfrak{e}_{n+p}) &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p})h(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p}) \\
&\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2})h(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) \\
&\quad + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+p})h(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+p}) \\
&\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2})h(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) \\
&\quad + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3})h(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3}) \\
&\quad + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+p})\zeta(\mathfrak{e}_{n+3}, \mathfrak{e}_{n+p})h(\mathfrak{e}_{n+3}, \mathfrak{e}_{n+p})
\end{aligned}$$

...

$$\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{n+p-2}[\prod_{j=n+1}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p}))\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})h(\mathfrak{e}_i, \mathfrak{e}_{i+1})]$$
$$+\prod_{i=n+1}^{n+p-1}\zeta(\mathfrak{e}_i, \mathfrak{e}_{n+p})h(\mathfrak{e}_{n+p-1}, \mathfrak{e}_{n+p}),$$

which leads to the following

$$\begin{aligned}
h(\mathfrak{e}_n, \mathfrak{e}_{n+p}) &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{n+p-2}(\prod_{j=n+1}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p}))\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})h(\mathfrak{e}_i, \mathfrak{e}_{i+1}))\\
&+\prod_{i=n+1}^{n+p-1}\zeta(\mathfrak{e}_{n+p-1}\mathfrak{e}_{n+p})h(\mathfrak{e}_{n+p-1}, \mathfrak{e}_{n+p})\\
&= \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{n+p-1}[\prod_{j=n+1}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p}))\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})h(\mathfrak{e}_i, \mathfrak{e}_{i+1})]\\
&\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{n+p-1}[\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p}))\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})h(\mathfrak{e}_i, \mathfrak{e}_{i+1})]\\
&\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{n+p-1}[\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})\frac{1}{i^{1/k}}.
\end{aligned}$$

Thus,

$$h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{n+p-1}[\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})\frac{1}{i^{1/k}}. \tag{2.6}$$

Now, consider

$$\begin{aligned}
\sum_{i=n+1}^{n+p-1}[\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})\frac{1}{n^{1/k}} &= \sum_{i=n+1}^{n+p-1}\frac{1}{i^{1/k}}[\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})\\
&\precsim \sum_{i=n+1}^{\infty}\frac{1}{i^{1/k}}[\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1}) = \sum_{i=n+1}^{\infty}X_iY_i,
\end{aligned}$$

where

$$X_i = \frac{1}{i^{1/k}},$$

and

$$Y_i = [\prod_{j=0}^{i}\zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1}).$$

Since, $\frac{1}{k} > 0$, and $\sum\limits_{i=n+1}^{\infty} (\frac{1}{i^{1/k}})$ converges and here $Y_i = [\prod\limits_{j=0}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})$ is also increasing and bounded above, we have $\lim\limits_{i\to\infty} = \sup(Y_i)$, exists and non zero.

Hence, it is obvious that the product $[\prod\limits_{j=0}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})$ converges. Thus $\sum\limits_{i=n+1}^{\infty} X_i Y_i$ converges.

Further, let us consider the partial sum

$$R_p = \sum_{i=0}^{p}[\prod_{j=0}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_{n+p})]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})\frac{1}{i^{1/k}}.$$

Now from (2.6), we have

$$h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + [R_{n+p-1} - R_n]. \tag{2.7}$$

By applying the ratio test and then using the above mentioned condition (2.2), we have the guarantee of the existence of the $\lim\limits_{n\to\infty} R_n$. So it can be said that $\{R_n\}$ is Cauchy. Now apply the limit $n \to +\infty$ in (2.7), we get

$$\lim_{n\to\infty} h(\mathfrak{e}_n, \mathfrak{e}_{n+p}) = 0.$$

which shows that $\{\mathfrak{e}_n\}$ is the Cauchy sequence in $(\mathbb{H}, h)$, so $\{\mathfrak{e}_n\}$ is convergent and converges to $u \in \mathbb{H}$. Now we show that $u = \mathrm{T}u$.

Since $\mathfrak{e}_n \to u$ when $n \to \infty$ and also the mapping T is continuous, so we get $\mathrm{T}\mathfrak{e}_n \to \mathrm{T}u$ when we apply the limit $n \to \infty$.

So we have

$$h(u, \mathrm{T}u) = \lim_{n\to\infty} h(\mathfrak{e}_{n+1}, \mathrm{T}u) = \lim_{n\to\infty} h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}u) = 0$$

and hence it is clear that $u = \mathrm{T}u$. □

Further, we will give the above result for the cases of CVEb-metric space, complex-valued complete b-metric space (CVb-metric space), complex-valued complete metric space (CV-metric space). We will observe that our space is more general than the spaces enumerated here.

Theorem 2.3. *Let* $(\mathbb{H}, \mathrm{h})$ *be a CVEb-metric space with a rational type* $(\alpha, \Theta)-$*Contraction, as* $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ *such that,*

$$\Theta(\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f)) \precsim \Theta(M(\mathfrak{e}, f))^k, \tag{2.8}$$

where

$$M(\mathfrak{e}, f) = max\{\mathrm{h}(\mathfrak{e}, f), \mathrm{h}(\mathfrak{e}, \mathrm{T}\mathfrak{e}), \mathrm{h}(f, \mathrm{T}f), \frac{\mathrm{h}(\mathfrak{e}, \mathrm{T}\mathfrak{e})\mathrm{h}(f, cf)}{1 + \mathrm{h}(\mathfrak{e}, f)}\}, \text{ for all } \mathfrak{e}, f \in \mathbb{H}, \tag{2.9}$$

with $\mathrm{h}(\mathrm{T}\mathfrak{e}, \mathrm{T}f) \succsim 0$. *Assume also that the following assertions also hold*

1. T *is continuous.*
2. $\sup\limits_{m\geq 1} \lim\limits_{i\to\infty} \frac{\zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_{i+2})\zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_m)}{\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})} < 1.$

In addition to this assume also that, for every $l \in \mathbb{H}$, we get $\lim_{i\to\infty} \zeta(\mathfrak{e}_n, \mathfrak{e})$ and $\lim_{i\to\infty} \zeta(\mathfrak{e}, \mathfrak{e}_n)$ exist and are finite. Then, there exists $\mathfrak{e}^* \in \mathbb{H}$ such that $\mathfrak{e}^* = \mathrm{T}\mathfrak{e}^*$.

Corollary 2.4. *Let $(\mathbb{H}, \mathrm{h})$ be a CVEb-metric space and with a rational type $(\alpha, \Theta)-$Contraction, as $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ such that,*

1. T *is $\alpha-$admissible;*
2. *there exists $\mathfrak{e}_0 \in \mathbb{H}$ such that $\alpha(\mathfrak{e}_0, \mathrm{T}\mathfrak{e}_0) \geq 0$;*
3. T *is continuous;*
4. $sup_{m\geq 1} \lim_{i\to\infty} \frac{\zeta(\mathfrak{e}_{i+1},\mathfrak{e}_{i+2})\zeta(\mathfrak{e}_{i+1},\mathfrak{e}_m)}{\zeta(\mathfrak{e}_i,\mathfrak{e}_{i+1})} < 1$.

In addition to this assume also that, for every $l \in \mathbb{H}$, we get $\lim_{i\to\infty} \zeta(\mathfrak{e}_n, \mathfrak{e})$ and $\lim_{i\to\infty} \zeta(\mathfrak{e}, \mathfrak{e}_n)$ exist and are finite. Then, there exists $\mathfrak{e}^* \in \mathbb{H}$ such that $\mathfrak{e}^* = \mathrm{T}\mathfrak{e}^*$.

Proof. If we take $\zeta(\mathfrak{e}, f) = \zeta(f, g)$ in above theorem, Theorem (2.2), we get the conclusion. □

Corollary 2.5. *Let $(\mathbb{H}, \mathrm{h})$ be a CV-complete b-MS and with a rational type $(\alpha, \Theta)-$Contraction, as $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ such that,*

1. T *is $\alpha-$admissible;*
2. *there exists a $\mathfrak{e}_0 \in \mathbb{H}$ so that $\alpha(\mathfrak{e}_0, \mathrm{T}\mathfrak{e}_0) \geq 0$;*
3. T *is continuous;*

Then there exists $\mathfrak{e}^* \in \mathbb{H}$ such that $\mathrm{T}\mathfrak{e}^* = \mathfrak{e}^*$.

Proof. If we take $\zeta(\mathfrak{e}, f) = \zeta(f, g) = b \geq 1$ in the above theorem, Theorem (2.2). □

Corollary 2.6. *Let $(\mathbb{H}, \mathrm{h})$ be a CV-complete MS and $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ be a rational type $(\alpha, \Theta)-$Contraction such that:*

1. T *is $\alpha-$admissible;*
2. *there exists a $\mathfrak{e}_0 \in \mathbb{H}$ so that $\alpha(\mathfrak{e}_0, \mathrm{T}\mathfrak{e}_0) \geq 0$;*
3. T *is continuous;*

Then there exists $\mathfrak{e}^* \in \mathbb{H}$ such that $\mathrm{T}\mathfrak{e}^* = \mathfrak{e}^*$.

Proof. If we take $\zeta(\mathfrak{e}, f) = \zeta(f, g) = 1$ in the above theorem, Theorem (2.2). □

Example 2.7. Let $\mathbb{H} = \{0, 1, 2\}$. Define $\zeta : \mathbb{H} \times \mathbb{H} \to [1, \infty]$ and $h : \mathbb{H} \times \mathbb{H} \longrightarrow [1, \infty]$ as $\zeta(\mathfrak{e}, f) = 1 + \mathfrak{e} + f$ and

$$\begin{aligned} h(2,2) = {}& h(0,0) = h(1,1) = 0 \\ h(2,0) = {}& h(0,2) = 1 + 2i, \\ h(1,0) = {}& h(0,1) = 4 + 3i, \\ h(1,2) = {}& h(2,1) = 2 - 2i \end{aligned}$$

We define next $\mathrm{T} : \mathbb{H} \to \mathbb{H}$ by $\mathrm{T}(0) = 0; \mathrm{T}(1) = 2; \mathrm{T}(2) = 0$.

For $k = \frac{9}{10}$ and by defining $\Theta(\kappa) = \mathfrak{e}^{\sqrt{\kappa}}$, we checked multiple cases to sustain the assumptions of the main result.

Case 1**:** If $\mathfrak{e} = 0, f = 1$, we have

$$\begin{aligned}\Theta(\ h(\mathrm{T}\mathfrak{e}, \mathrm{T}f)) &= \Theta(\ h(\mathrm{T}0, \mathrm{T}1)) = \Theta(h(0,2)) = \mathfrak{e}^{\sqrt{\sqrt{5}}} \\ &\precsim (\mathfrak{e}^{\sqrt{5}})^{\frac{9}{10}} \\ &= [\Theta(max\{5, 0, \sqrt{8}, 0\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(0,1),\ h(0,\mathrm{T}0),\ h(1,\mathrm{T}1), \frac{h(0,\mathrm{T}0)\ h(1,\mathrm{T}1)}{1+\ h(0,1)}\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(\mathfrak{e},f),\ h(\mathfrak{e},\mathrm{T}\mathfrak{e}),\ h(f,\mathrm{T}f), \frac{h(\mathfrak{e},\mathrm{T}\mathfrak{e})\ h(f,\mathrm{T}f)}{1+\ h(\mathfrak{e},f)}\})]^{\frac{9}{10}}.\end{aligned}$$

Case 2**:** If $\mathfrak{e} = 1, f = 2$, we have

$$\begin{aligned}\Theta(\ h(\mathrm{T}\mathfrak{e}, \mathrm{T}f)) &= \Theta(\ h(\mathrm{T}1, \mathrm{T}2)) = \Theta(h(1,2)) = \mathfrak{e}^{\sqrt{\sqrt{5}}} \\ &\precsim (\mathfrak{e}^{\sqrt{\sqrt{8}}})^{\frac{9}{10}} \\ &= [\Theta(max\{\sqrt{8}, \sqrt{8}, \sqrt{5}, \frac{\sqrt{8}\sqrt{5}}{1+\sqrt{8}}\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(1,2),\ h(1,\mathrm{T}1),\ h(2,\mathrm{T}2), \frac{h(1,\mathrm{T}1)\ h(2,\mathrm{T}2)}{1+\ h(1,2)}\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(\mathfrak{e},f),\ h(\mathfrak{e},\mathrm{T}\mathfrak{e}),\ h(f,\mathrm{T}f), \frac{h(\mathfrak{e},\mathrm{T}\mathfrak{e})\ h(f,\mathrm{T}f)}{1+\ h(\mathfrak{e},f)}\})]^{\frac{9}{10}}.\end{aligned}$$

Case 3**:** If $\mathfrak{e} = 0, f = 2$, we have

$$\begin{aligned}\Theta(\ h(\mathrm{T}\mathfrak{e}, \mathrm{T}f)) &= \Theta(\ h(\mathrm{T}0, \mathrm{T}2)) = \Theta(h(0,0)) = \mathfrak{e}^{0} \\ &\precsim (\mathfrak{e}^{\sqrt{\sqrt{5}}})^{\frac{9}{10}} \\ &= [\Theta(max\{\sqrt{5}, 0, \sqrt{5}, 0\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(0,2),\ h(0,\mathrm{T}0),\ h(2,\mathrm{T}2), \frac{h(0,\mathrm{T}0)\ h(2,\mathrm{T}2)}{1+\ h(0,2)}\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(\mathfrak{e},f),\ h(\mathfrak{e},\mathrm{T}\mathfrak{e}),\ h(f,\mathrm{T}f), \frac{h(\mathfrak{e},\mathrm{T}\mathfrak{e})\ h(f,\mathrm{T}f)}{1+\ h(\mathfrak{e},f)}\})]^{\frac{9}{10}}.\end{aligned}$$

Case 4**:** If $\mathfrak{e} = f = 0, \mathfrak{e} = f = 1, \mathfrak{e} = f = 2$, we have

$$\begin{aligned}\Theta(\ h(\mathrm{T}\mathfrak{e}, \mathrm{T}f)) &= \mathfrak{e}^{0} \\ &\precsim [\Theta(max\{\ h(\mathfrak{e},f),\ h(\mathfrak{e},\mathrm{T}\mathfrak{e}),\ h(f,\mathrm{T}f), \frac{h(\mathfrak{e},\mathrm{T}\mathfrak{e})\ h(f,\mathrm{T}f)}{1+\ h(\mathfrak{e},f)}\})]^{\frac{9}{10}} \\ &= [\Theta(max\{\ h(\mathfrak{e},f),\ h(\mathfrak{e},\mathrm{T}\mathfrak{e}),\ h(f,\mathrm{T}f), \frac{h(\mathfrak{e},\mathrm{T}\mathfrak{e})\ h(f,\mathrm{T}f)}{1+\ h(\mathfrak{e},f)}\})]^{\kappa}.\end{aligned}$$

So, all assumptions of the above said theorem are satisfied. So, T has $\mathfrak{e} = 0$ as unique fixed point.

We can also satisfy many other results as a special cases of the main Theorem 2.2.

2.2. Reich type contractions on CVCMS

Theorem 2.8. *Suppose* $(\mathbb{H}, h)$ *be a CVCMS by the functions* $\zeta : \mathbb{H} \times \mathbb{H} \to [1, \infty)$. *Suppose also, the mapping* $\mathrm{T} : \mathbb{H} \times \mathbb{H}$ *be such that there are* $\alpha, \beta, \gamma \in (0, 1)$ *and* $k = \frac{\alpha+\beta}{1-\gamma} < 1$,

$$h(\mathrm{T}\mathfrak{e}, \mathrm{T}f) \precsim \alpha h(\mathfrak{e}, f) + \beta h(\mathfrak{e}, \mathrm{T}\mathfrak{e}) + \gamma h(f, \mathrm{T}f), \tag{2.10}$$

for all $\mathfrak{e}, f \in \mathbb{H}$. *For* $\mathfrak{e}_0 \in \mathbb{H}$, *take* $\mathfrak{e}_n = \mathrm{T}^n \mathfrak{e}_0$. *Assume that*

$$\sup_{m \geq 1} \lim_{\iota \to \infty} \frac{\zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_{i+2})}{\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})} \zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_m) < \frac{1}{k}. \tag{2.11}$$

Suppose that

$\lim_{n\to\infty} \zeta(\mathfrak{e}, \mathfrak{e}_n)$ *and* $\lim_{n\to\infty} \zeta(\mathfrak{e}_n, \mathfrak{e})$ *exist and are finite, and* $\gamma \lim_{n\to\infty} \zeta(\mathfrak{e}_n, \mathfrak{e}) < 1$ *for every* $e \in \mathbb{H}$.

Then, T *possess a unique fixed point.*

Proof. The assumed sequence $\{\mathfrak{e}_n\}$ verifies $\mathfrak{e}_{n+1} = \mathrm{T}\mathfrak{e}_n$ for all $n \in \mathbb{N}$. It is obvious that if there exists $n_0 \in \mathbb{N}$ for which $\mathfrak{e}_{n_0+1} = \mathrm{T}\mathfrak{e}_{n_0}$, then $\mathrm{T}\mathfrak{e}_{n_0} = \mathfrak{e}_{n_0}$, and the proof is trivially finished. So here we assume that $\{\mathfrak{e}_{n+1}\} \neq \{\mathfrak{e}_n\}$ for every $n \in \mathbb{N}$. So, by (2.10), we get

$$h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) = h(\mathrm{T}\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_n) \precsim \alpha h(\mathfrak{e}_{n-1}, \mathfrak{e}_n) + \beta h(\mathfrak{e}_{n-1}, \mathrm{T}\mathfrak{e}_{n-1}) + \gamma h(\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_n)$$

$$= \alpha h(\mathfrak{e}_{n-1}, \mathfrak{e}_n) + \beta h(\mathfrak{e}_{n-1}, \mathfrak{e}_n) + \gamma h(\mathfrak{e}_n, \mathfrak{e}_{n+1}), \tag{2.12}$$

which implies that

$$h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \precsim \frac{\alpha + \beta}{1 - \gamma} h(\mathfrak{e}_{n-1}, \mathfrak{e}_n) = k h(\mathfrak{e}_{n-1}, \mathfrak{e}_n). \tag{2.13}$$

Thus, we have

$$h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \precsim k h(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \precsim k^2 h(\mathfrak{e}_{n-2}, \mathfrak{e}_{n-1}) \precsim ... \precsim k^n h(\mathfrak{e}_0, \mathfrak{e}_1).$$

For all $n, m \in \mathbb{N}$ with $n < m$, we have,

$$\begin{aligned} h(\mathfrak{e}_n, \mathfrak{e}_m) &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_m)h(\mathfrak{e}_{n+1}, \mathfrak{e}_m) \\ &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2})h(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) \\ &\quad + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_m)h(\mathfrak{e}_{n+2}, \mathfrak{e}_m) \\ &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2})h(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) \\ &\quad + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3})h(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3}) \\ &\quad + \zeta(\mathfrak{e}_{n+1}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+2}, \mathfrak{e}_m)\zeta(\mathfrak{e}_{n+3}, \mathfrak{e}_m)h(\mathfrak{e}_{n+3}, \mathfrak{e}_m) \\ &\precsim \cdots \\ &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{m-2} [\prod_{j=n+1}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})h(\mathfrak{e}_i, \mathfrak{e}_{i+1}) \\ &\quad + \prod_{i=n+1}^{m-1} \zeta(\mathfrak{e}_i, \mathfrak{e}_m)h(\mathfrak{e}_{m-1}, \mathfrak{e}_m). \end{aligned}$$

Which implies that

$$\begin{aligned} h(\mathfrak{e}_n, \mathfrak{e}_m) &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})h(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + \sum_{i=n+1}^{m-2} [\prod_{j=n+1}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})h(\mathfrak{e}_i, \mathfrak{e}_{i+1}) \\ &\quad + [\prod_{i=n+1}^{m-1} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_{m-1}, \mathfrak{e}_m)h(\mathfrak{e}_{m-1}, \mathfrak{e}_m) \\ &\precsim \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})k^n h(\mathfrak{e}_0, \mathfrak{e}_1) + \sum_{i=n+1}^{m-2} [\prod_{j=n+1}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})k^i h(\mathfrak{e}_0, \mathfrak{e}_1) \\ &\quad + [\prod_{i=n+1}^{m-1} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_{m-1}, \mathfrak{e}_m)k^{m-1} h(\mathfrak{e}_0, \mathfrak{e}_1) \\ &= \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1})k^n h(\mathfrak{e}_0, \mathfrak{e}_1) + \sum_{i=n+1}^{m-1} [\prod_{j=n+1}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})k^i h(\mathfrak{e}_0, \mathfrak{e}_1). \end{aligned} \tag{2.14}$$

Let

$$R_g = \sum_{i=0}^{g} [\prod_{j=0}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})k^i h(\mathfrak{e}_0, \mathfrak{e}_1).$$

Consider

$$u_i = [\prod_{j=0}^{i} \zeta(\mathfrak{e}_j, \mathfrak{e}_m)]\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})k^i h(\mathfrak{e}_0, \mathfrak{e}_1).$$

We have

$$\frac{u_{i+1}}{u_i} = \zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_m)\frac{\zeta(\mathfrak{e}_{i+1}, \mathfrak{e}_{i+2})}{\zeta(\mathfrak{e}_i, \mathfrak{e}_{i+1})}k.$$

Considering the condition (2.11) and applying the ratio test, we can see that the series $\sum_i u_i$ converges. Thus, $\lim_{n\to\infty} R_n$ exists. So, we say the real sequence $\{R_n\}$ is a Cauchy sequence.

Now, using (2.14), we get

$$h(\mathfrak{e}_n, \mathfrak{e}_m) \precsim h(\mathfrak{e}_0, \mathfrak{e}_1)[k^n \zeta(\mathfrak{e}_n, \mathfrak{e}_{n+1}) + (R_{m-1}, R_n)]. \tag{2.15}$$

Above, we used $\zeta(\mathfrak{e}, f) \geq 1$. Assuming $n, m \to \infty$ in (2.15), we obtain $\lim_{n,m\to\infty} h(\mathfrak{e}_n, \mathfrak{e}_m) = 0$.

Thus, the sequence $\{\mathfrak{e}_n\}$ is Cauchy in the CVCMS $(\mathbb{H}, h)$. So, there is some $\mathfrak{e}^* \in \mathbb{H}$ such that

$$\lim_{n\to\infty} h(\mathfrak{e}_n, \mathfrak{e}^*) = 0.$$

that is, $\mathfrak{e}_n \to \mathfrak{e}^*$ as $n \to \infty$.

In the following we will show that $\mathfrak{e}^*$ is a fixed point of T. By (2.10) and by condition (3), we have

$$\begin{aligned}
h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*) &\precsim \zeta(\mathfrak{e}^*, \mathfrak{e}_{n+1}) h(\mathfrak{e}^*, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*) h(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*)\\
&= \zeta(\mathfrak{e}^*, \mathfrak{e}_{n+1}) h(\mathfrak{e}^*, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*) h(\mathrm{T}\mathfrak{e}_n, \mathrm{T}\mathfrak{e}^*)\\
&\precsim \zeta(\mathfrak{e}^*, \mathfrak{e}_{n+1}) h(\mathfrak{e}^*, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*)[\alpha h(\mathfrak{e}_n, \mathfrak{e}^*) + \beta h(\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_n) + \gamma h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*)]\\
&= \zeta(\mathfrak{e}^*, \mathfrak{e}_{n+1}) h(\mathfrak{e}^*, \mathfrak{e}_{n+1}) + \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*)[\alpha h(\mathfrak{e}_n, \mathfrak{e}^*) + \beta h(\mathfrak{e}_n, \mathrm{T}\mathfrak{e}_{n+1}) + \gamma h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*)].
\end{aligned}$$

Taking the limit as $n \to \infty$ and using (2.12), (2.13) and the fact that $\lim_{n\to\infty} h(\mathfrak{e}_n, \mathfrak{e})$ and $\lim_{n\to\infty} h(\mathfrak{e}, \mathfrak{e}_n)$ exist, and are finite, we obtain that

$$h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*) \precsim [\gamma \lim_{n\to\infty} \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*)] h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*).$$

Suppose that $\mathfrak{e}^* \neq \mathrm{T}\mathfrak{e}^*$, keeping in the mind that $\gamma \lim_{n\to\infty} \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*) < 1$, so

$$0 < h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*) \precsim [\gamma \lim_{n\to\infty} \zeta(\mathfrak{e}_{n+1}, \mathrm{T}\mathfrak{e}^*)] h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*) < h(\mathfrak{e}^*, \mathrm{T}\mathfrak{e}^*).$$

It is a clear contradiction which yields the result that $\mathfrak{e}^* = \mathrm{T}\mathfrak{e}^*$. So easily the uniqueness of the fixed point follows and it also completes the proof. □

Example 2.9. Assume that $\mathbb{H} = \{0, 1, 2\}$. And define $\zeta : \mathbb{H} \times \mathbb{H} \to [1, \infty]$ and $h : \mathbb{H} \times \mathbb{H} \to [1, \infty]$ as $\zeta(\mathfrak{e}, f) = 1 + \mathfrak{e} + f$ and

$$\begin{aligned}
h(2,2) = {}& h(0,0) = h(1,1) = 0\\
h(2,0) = {}& h(0,2) = 1 + 2i,\\
h(1,0) = {}& h(0,1) = 4 + 3i,\\
h(1,2) = {}& h(2,1) = 2 - 2i
\end{aligned}$$

We define the mappingT $: \mathbb{H} \to \mathbb{H}$ by $\mathrm{T}(0) = 1, \mathrm{T}(1) = 1, \mathrm{T}(2) = 1$. For $\alpha = \frac{1}{11}, \beta = \frac{3}{11}, \gamma = \frac{3}{11}$ and setting $\mathfrak{e}_0 = 0, \mathfrak{e}_1 = 2$ and $\mathfrak{e}_n = 1$, for all $n \geq 2$, condition (2.11) is satisfied. For (2.10) we check different cases to sustain the assumptions of

the main result.
Case 01**:** If $\mathfrak{e} = 0, f = 1$, we have

$$\begin{aligned} h(\mathrm{T}\mathfrak{e}, \mathrm{T}f) =& h(\mathrm{T}0, \mathrm{T}1) = 0 \\ & \precsim \alpha h(p, \mathfrak{e}) + \beta h(p, \mathrm{T}p) + \gamma h(\mathfrak{e}, \mathrm{T}\mathfrak{e}). \end{aligned}$$

Case 02**:** If $\mathfrak{e} = 1, f = 2$, we have

$$\begin{aligned} h(\mathrm{T}\mathfrak{e}, \mathrm{T}f) =& h(\mathrm{T}1, \mathrm{T}2) = 0 \\ & \precsim \alpha h(p, \mathfrak{e}) + \beta h(p, \mathrm{T}p) + \gamma h(\mathfrak{e}, \mathrm{T}\mathfrak{e}). \end{aligned}$$

Case 03**:** If $\mathfrak{e} = 0, f = 2$, we have

$$\begin{aligned} h(\mathrm{T}\mathfrak{e}, \mathrm{T}f) =& h(\mathrm{T}0, \mathrm{T}2) = 0 \\ & \precsim \alpha h(p, \mathfrak{e}) + \beta h(p, \mathrm{T}p) + \gamma h(\mathfrak{e}, \mathrm{T}\mathfrak{e}). \end{aligned}$$

Case 04**:** If $\mathfrak{e} = f = 0, \mathfrak{e} = f = 1, \mathfrak{e} = f = 2$, we have

$$\begin{aligned} h(\mathrm{T}\mathfrak{e}, \mathrm{T}f) &= 0 \\ & \precsim \alpha h(p, \mathfrak{e}) + \beta h(p, \mathrm{T}p) + \gamma h(\mathfrak{e}, \mathrm{T}\mathfrak{e}). \end{aligned}$$

So, all assumptions of the above said theorem (2.8) are accomplished. So, T has $\mathfrak{e} = 1$ as a unique fixed point. We can also satisfy many results as special case of main Theorem (2.8).

3. Application to dynamic programming

The idea of *Dynamic programming* was given by Richard Bellman (see [10]). This idea was used for solving a large decision issues by splitting them into simpler fitted sub-issues which are solved iteratively over time. Dynamic programming techniques were highly used in optimization, control issues, remarkably, impulsive control issues.

In [13] Bertsekas and Tsitsikls gave a new class of RL algorithms for example Value Iteration (VI) and Policy Iteration (PI) process. Then, in [12], Bertsekas and Ioffe gave a new iteration scheme, the Temporal Differences policy iteration $TD(\lambda)$ scheme, and they proven that the $TD(\lambda)$ scheme can be resumed to a PI scheme, appointed by λ-PIR. Some details about are given in the following studies [4, 8, 11, 14, 16].

This section is dedicated to analyse the application of Reich fixed-point theorem in CVCMS in order to prove the results of existence and uniqueness for the solution of the *dynamic programming Bellman's equation.*

Let H and Q be two CVCMS with $\tilde{S} \subseteq H$ be the state space with $\tilde{D} \subseteq Q$ be the decision space. Let $\pi : \tilde{S} \times \tilde{D} \to \tilde{S}$ be the process transformation and $\eth : \tilde{S} \times \tilde{D} \to \mathbb{C}$ with $\mathrm{G} : \tilde{S} \times \tilde{D} \times \mathbb{C} \to \mathbb{C}$ be some given functional, where $\mathbb{C}$ is the set of complex numbers. The optimal return map $F : \tilde{S} \to \mathbb{C}$ of the process of continuous decision is represented by the following functional equation of the form

$$F(\mathfrak{e}) = \sup_{f \in \tilde{D}} \{\eth(\mathfrak{e}, f) + \mathrm{G}(\mathfrak{e}, f, F(\pi(\mathfrak{e}, f)))\}, \text{ with } \mathfrak{e}, f \in \tilde{S}. \tag{3.1}$$

Let $\mathfrak{B}(\tilde{S})$ be collection of all complex-valued mappings, bounded on $\tilde{S}$.

Theorem 3.1. *Consider the previous instances true and we suppose further the following*

1. G *and* $\mathfrak{d}$ *are two bounded mappings;*
2. $|\mathrm{G}(\mathfrak{e}, f, \mathfrak{h}z) - \mathrm{G}(\mathfrak{e}, f, \mathfrak{k}z)| \precsim \alpha|\mathfrak{h}z - \mathfrak{k}z| + \beta|\mathfrak{h}z - \mathrm{T}\mathfrak{k}z| + \gamma|\mathfrak{k}z - \mathrm{T}\mathfrak{k}z|$,

for all $\mathfrak{h}, \mathfrak{k} \in \mathfrak{B}(\tilde{S})$ *and* $(\mathfrak{e}, f, \mathfrak{h}z) \in \tilde{S} \times \tilde{D} \times \mathbb{C}$, *where* $z \in \tilde{S}$ *and* $\alpha, \beta, \gamma \in (0, 1)$ *together with* $\alpha + \beta + \gamma < 1$ *and* $\kappa = \frac{\alpha+\beta}{1-\gamma} < 1$.

Then the functional equation (3.1) has a unique bounded solution on $\tilde{S}$.

Proof. We take $\mathfrak{B}(\tilde{S})$ the set of all complex-valued mappings, bounded on $\tilde{S}$, endowed with the norm $h : \mathfrak{B}(\tilde{S}) \times \mathfrak{B}(\tilde{S}) \to \mathbb{C}$ defined as $h(\mathfrak{e}, f) = \|\mathfrak{e} - f\|_\infty = |\mathfrak{e} - f|\mathfrak{e}^{-i\theta}$, where $|p| = \sqrt{s^2 + t^2}$ with $p = s + it$, for $s, t \in \mathbb{R}$, $\theta > 0$ and $i = \sqrt{-1} \in \mathbb{C}$.

Let $\varsigma : \mathfrak{B}(\tilde{S}) \times \mathfrak{B}(\tilde{S}) \to [1, \infty)$ be a functional defined as

$$\varsigma(p, \mathfrak{e}) = \begin{cases} 1, & \text{if } \mathfrak{e}, f \in (0, 1], \\ \frac{1+\max\{\mathfrak{e}, f\}}{\min\{\mathfrak{e}, f\}}, & \text{otherwise.} \end{cases}$$

We remark that $(\mathfrak{B}(\tilde{S}), h)$ is a complete CVCMS. We consider the operator $\mathrm{T} : \mathfrak{B}(\tilde{S}) \to \mathfrak{B}(\tilde{S})$ defined by $\mathrm{T}\mathfrak{h} = \varphi$, where

$$\varphi(\mathfrak{e}) = \sup_{y \in \tilde{D}} \{\mathfrak{d}(\mathfrak{e}, f) + \mathrm{G}(\mathfrak{e}, f, \mathfrak{h}(\pi(\mathfrak{e}, f)))\}.$$

We know that G and g are bounded, then $\varphi \in \mathfrak{B}(\tilde{S})$. Then, to find a bounded solution of the functional equation (3.1) is the same with getting a fixed point of T.

Let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{B}(\tilde{S})$ and $\mathrm{T}\mathfrak{h}_1 = \varphi_1$, $\mathrm{T}\mathfrak{h}_2 = \varphi_2$. Then we have

$$\varphi_1(p) = \sup_{y \in \tilde{D}} \{\mathfrak{d}(\mathfrak{e}, f) + \mathrm{G}(\mathfrak{e}, f, \mathfrak{h}_1(\pi(\mathfrak{e}, f)))\}.$$

$$\varphi_2(p) = \sup_{y \in \tilde{D}} \{\mathfrak{d}(\mathfrak{e}, f) + \mathrm{G}(\mathfrak{e}, f, \mathfrak{h}_2(\pi(\mathfrak{e}, f)))\}.$$

For $e \in \tilde{S}$ and $\mathfrak{e}_1, \mathfrak{e}_2 \in Q$ and $\epsilon > 0$ we get

$$\mathrm{T}\mathfrak{h}_1 e \prec \mathfrak{d}(\mathfrak{e}, f_1) + \mathrm{G}(\mathfrak{e}, f_1, \mathfrak{h}_1(\pi(\mathfrak{e}, f_1))) + \epsilon. \tag{3.2}$$

$$\mathrm{T}\mathfrak{h}_2 e \prec \mathfrak{d}(\mathfrak{e}, f_2) + \mathrm{G}(\mathfrak{e}, f_2, \mathfrak{h}_2(\pi(\mathfrak{e}, f_2))) + \epsilon. \tag{3.3}$$

Also, we have

$$\mathrm{T}\mathfrak{h}_1 e \succsim \mathfrak{d}(\mathfrak{e}, f_2) + \mathrm{G}(\mathfrak{e}, f_2, \mathfrak{h}_1(\pi(\mathfrak{e}, f_2))). \tag{3.4}$$

$$\mathrm{T}\mathfrak{h}_2 e \succsim \mathfrak{d}(\mathfrak{e}, f_1) + \mathrm{G}(\mathfrak{e}, f_1, \mathfrak{h}_2(\pi(\mathfrak{e}, f_1))). \tag{3.5}$$

From (3.2) and (3.5) we get

$$\begin{aligned} |\mathrm{T}\mathfrak{h}_1\mathfrak{e} - \mathrm{T}\mathfrak{h}_2\mathfrak{e}| &\precsim |\mathrm{G}(\mathfrak{e}, f, \mathfrak{h}_1(\pi(\mathfrak{e}, f_1))) - \mathrm{G}(\mathfrak{e}, f, \mathfrak{h}_2(\pi(\mathfrak{e}, f_2)))| + \epsilon \\ &\precsim \alpha|\mathfrak{h}_1\mathfrak{e} - \mathfrak{h}_2\mathfrak{e}| + \beta|\mathfrak{h}_1\mathfrak{e} - \mathrm{T}\mathfrak{h}_1\mathfrak{e}| + \gamma|\mathfrak{h}_2\mathfrak{e} - \mathrm{T}\mathfrak{h}_2\mathfrak{e}|. \end{aligned} \tag{3.6}$$

From (3.3) and (3.4) we get

$$\begin{aligned} |\mathrm{T}\mathfrak{h}_2\mathfrak{e} - \mathrm{T}\mathfrak{h}_1\mathfrak{e}| &\precsim |\mathrm{G}(\mathfrak{e}, f, \mathfrak{h}_2(\pi(\mathfrak{e}, f_2))) - \mathrm{G}(\mathfrak{e}, f, \mathfrak{h}_1(\pi(\mathfrak{e}, f_1)))| + \epsilon \\ &\precsim \alpha|\mathfrak{h}_2\mathfrak{e} - \mathfrak{h}_1\mathfrak{e}| + \beta|\mathrm{T}\mathfrak{h}_1\mathfrak{e} - \mathfrak{h}_1\mathfrak{e}| + \gamma|\mathrm{T}\mathfrak{h}_2\mathfrak{e} - \mathfrak{h}_2\mathfrak{e}|. \end{aligned} \tag{3.7}$$

Further, from (3.6) and (3.7), and multiplying on both sides with $\mathfrak{e}^{-i\theta}$ (exponential function) we get

$$|T\mathfrak{h}_1\mathfrak{e} - T\mathfrak{h}_2\mathfrak{e}|\mathfrak{e}^{-i\theta} \precsim \alpha|\mathfrak{h}_1\mathfrak{e} - \mathfrak{h}_2\mathfrak{e}|\mathfrak{e}^{-i\theta} + \beta|\mathfrak{h}_1\mathfrak{e} - T\mathfrak{h}_1\mathfrak{e}|\mathfrak{e}^{-i\theta} + \gamma|\mathfrak{h}_2\mathfrak{e} - T\mathfrak{h}_2\mathfrak{e}|\mathfrak{e}^{-i\theta}. \quad (3.8)$$

Then, we have

$$\begin{aligned} h(T\mathfrak{h}_1\mathfrak{e}, T\mathfrak{h}_2\mathfrak{e}) = \|T\mathfrak{h}_1\mathfrak{e} - T\mathfrak{h}_2 e\|_\infty &\precsim \alpha\|\mathfrak{h}_1\mathfrak{e} - \mathfrak{h}_2 e\|_\infty \\ &+ \beta\|\mathfrak{h}_1\mathfrak{e} - T\mathfrak{h}_1 e\|_\infty + \gamma\|\mathfrak{h}_2\mathfrak{e} - T\mathfrak{h}_2 e\|_\infty \\ &= \alpha h(\mathfrak{h}_1\mathfrak{e}, \mathfrak{h}_2\mathfrak{e}) + \beta h(\mathfrak{h}_1\mathfrak{e}, T\mathfrak{h}_1\mathfrak{e}) + \gamma h(\mathfrak{h}_2\mathfrak{e}, T\mathfrak{h}_2\mathfrak{e}). \end{aligned} \quad (3.9)$$

It is easy to see , that for both cases of the functional $\varsigma(\mathfrak{e}, f)$, for $\mathfrak{e}, f \in [0,1]$ and otherwise. Moreover, the condition (2.10) and the related limits are verified. Then, for $\alpha, \beta, \gamma \in (0,1)$, with $\kappa = \frac{\alpha+\beta}{1-\gamma} < 1$, all the conditions of the Theorem 2.8 are satisfied.

Then, we have proved the existence and the uniqueness of a bounded solution of the equation (3.1) in $\mathfrak{B}(\tilde{S})$.

□

4. Conclusions

Fixed point theory it is a main tool in topology and nonlinear analysis, with the aim to obtain existence, uniqueness results and solution approximation schemes. Moreover, has a lot application in different fields as, dynamic optimization, optimal control, partial or common differential equations, differential inclusions and fractional equations.

Our paper is fully dedicated for the study of the existence and uniqueness of a fixed point for the case of a new type of space, CVCMS. Contractions presented by [1] and [2] for controlled metric spaces are now defined for CVCMS. Also, with the help of examples sustainability of results is verified. Moreover, we discussed the existence and the uniqueness of a bounded solution of the Bellman equation from dynamic programming field, applying our fixed point results.

The claim for these results is particularly evident in the fields of engineering, biological systems, management, economy, finance and information technology.

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References

[1] Ahmad, J., Al-Mazrooei, A.E., Aydi, H., De la Sen, M. *On Fixed Point Results in Controlled Metric Spaces,* J. Funct., 2020, 2108167.

[2] Ahmad, J., Lateef, D., *Fixed point theorems for rational type ($\alpha - \Theta$)-contractions in controlled metric spaces,* J. Nonlinear Sci. Appl., **13**(2020), 163–170.

[3] Al-Mazrooei, A.E., Ahmad, J., *Fixed Point Results in Controlled Metric Spaces with Applications,* Mathematics, **10(3)**(2022), 490 pp. https://doi.org/10.3390/math10030490

[4] Arutyunov, A., Jaćimović, V., Pereira, F., *Second order necessary conditions for optimal impulsive control problems*, Journal of Dynamical and Control Systems, **9(1)**(2022), 131-153.

[5] Aslam, M.S., Chowdhury, M.S.R., Guran, L., Alqudah, M. A., Abdeljawad, T. *Fixed point theory in complex valued controlled metric spaces with an application,* AIMS Mathematics, **7(7)**(2022), 11879-11904.

[6] Azam, A., Ahmad, J., Kumam, P., *Common fixed point theorems for multi-valued mappings in complex valued metric spaces,* Journal of Inequalities and Applications, **1**(2013), 578 pp.

[7] Banach, S., *Sur les opérations dans les ensembles abstraits et applications aux equations integrals,* Fund. Math., 1922, 133-181.

[8] Belhenniche A., Benahmed S., Pereira F.L., *Extension of λ-PIR for weakly contractive operators via fixed point theory,* Fixed Point Theory, **22(2)**(2021), 511-526. doi:10.24193/fpt-ro.2021.2.34.

[9] Belhenniche, A., Benahmed, S., Guran, L., *Existence of a solution for integral Urysohn type equations system via fixed points technique in complex valued extended b-metric spaces,* Journal of Prime Research in Mathematics, **16(2)**(2020), 109-122.

[10] Bellman, R. *The theory of dynamic programming,* Rand Corporation Report Santa Monica, CA, 1954.

[11] Bertsekas, D.P., *Dynamic Programming and Optimal Control,* Athena Scientific, 2005.

[12] Bertsekas, D.P., Ioffe, S., *Temporal differences-based policy iteration and applications in neuro-dynamic programming*, Lab. for Info. and Decision Systems Report, 1996, 2349.

[13] Bertsekas, D.P., Tsitsiklis, J.N., *Neuro-dynamic programming,* Athena Scientific, 1996.

[14] Fraga, S., Pereira, F., *Hamilton-Jacobi-Bellman equation and feedback synthesis for impulsive control,* IEEE Transactions on Automatic Control, **57(1)**(2021), 244-249.

[15] Hussain, S., *Fixed Point Theorems For Nonlinear Contraction In Controlled Metric Type Space,* Applied Mathematics E-Notes, **21**(2021), 53-61.

[16] Karamzin, D., De Oliveira, V., Pereira, F., Silva, G., *On some extension of optimal control theory,* European Journal of Control, **20(6)**(2014), 284-291.

[17] Lateef, D., *Fisher type fixed point results in controlled metric spaces,*Journal of Mathematics and Computer Science, **20**(2020), 234-240.

[18] Lateef, D., *Kannan fixed point theorem in C-metric spaces,* Journal of Mathematical Analysis, **10(5)**(2019), 34-40.

[19] Mlaiki, N., Aydi, H., Souayah, N., *Abdeljawad, T. Controlled metric type spaces and related contraction principle,* Mathematics, **6(10)**(2018), 1-6.

[20] Mlaiki, N., Aydi, H., Souayah, N., *Abdeljawad, T. An Improvement of Recent Results in Controlled Metric Type Spaces,* Filomat, **34(6)**(2020), 1853–1862.

[21] Panday, B., Pandey, A. K., Ughade, M., *Rational Type Contraction In Controlled Metric Spaces,* J. Math. Comput. Sci., **11(4)**(2021), 4631-4639.

[22] Rao, K. P.R., Swamy, P.R., Prasad, J.R., *A common fixed point theorem in complex valued b-metric spaces,* Bulletin of Mathematics and Statistics Research, **1(1)**(2013), 1-8.

[23] Ullah, N., Shagari, M. S., Azam, A., *Fixed Point Theorems in complex valued extended b metric spaces.* Moroccan J. of Pure and Appl. Anal. **5**(2019), no. 2, 140-163.

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