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Logarithmic Sobolev inequality in the variable exponent setting and its applications to hyperbolic differential equations with a logarithmic source term

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Abstract. We establish the generalized parametric logarithmic Sobolev inequalities in the Gagliardo-Nirenberg form for variable exponential space with log-Hölder exponential function. Employing the generalized parametric logarithmic Sobolev inequalities, we establish the existence of weak solutions to the boundary problem for the hyperbolic equation with logarithmic nonlinearity and involving variable exponents. Numerical examples and further applications will be addressed in a forthcoming paper.

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1. Introduction

In this work, we employ the logarithmic Sobolev inequality with variable exponent and investigate the solvability of the hyperbolic problem with variable exponent

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$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=div\left(|\nabla u|^{p(x)-2}\nabla u\right)+\theta\,|u|^{p(x)-2}\,u\ln|u|\,,$$
$$u\left(x,\ 0\right)=\phi_0\left(x\right),$$

$$\frac{\partial u\left(x,\ 0\right)}{\partial t}=\phi_1\left(x\right),$$
$$u\left(x,\ t\right)|_{\partial\Omega\times[0,\ T)}=\ 0,$$
$$\left.\frac{\partial u\left(x,\ t\right)}{\partial t}\right|_{\partial\Omega\times[0,\ T)}=\ 0,$$

where Ω is a bounded domain in R^l with a smooth boundary $\partial\Omega$, and $\phi_0,\ \phi_1$ are fixed initial functions; θ is a positive number. We define a function $p^*\ :\ \Omega\to R$ by $p^*\left(x\right)=\frac{lp(x)}{l-p(x)}$ if $p\left(x\right)<l$ and $p^*\left(x\right)=\infty$ if $p\left(x\right)\geq l$.

This type of equation models physical systems with spatially varying properties [1, 12, 27]. Materials science with non-uniform properties: The variable exponent allows the mathematical problem to model the dynamic behavior of materials where the elasticity, viscosity, or other material properties change from one location to another [24, 25]. Examples include elastoplastic materials, which can exhibit both elastic and plastic deformation. Geophysics and seismic wave propagation: The variable exponent can model different rock properties at various depths in the Earth's crust. It helps analyze how seismic waves travel through and are attenuated by these heterogeneous geological formations. In [27], S.D. Zeng, A.A. Khan, and S. Migorski used a framework of Oseen's type problems, which are fluid dynamics equations, and specifically addressed scenarios with non-smooth boundary conditions. The work establishes the existence of solutions for these inequalities and their applications in physics and engineering. The diffusive term in the equation is used in mathematical models for image denoising. In this application, the variable exponent can adapt the smoothing process to the specific local features of the image, such as preserving edges while removing noise [14]. In [14], the paper studied optimizing the algorithm for real-time applications, such as video surveillance. The optimizations included software pipeline, operation unit balancing, and other programming techniques to maximize processing efficiency.

Investigations of equations with logarithmic nonlinearity have a long history, see [11], and references therein. Variable exponents were studied in many papers, including [5, 7, 9, 15, 22, 23]. In [11], the authors analyze a hyperbolic equation with logarithmic non-linearity and a weak damping term, proving global existence of solutions using the potential well method and investigating growth and decay estimates. In [15], B.S. Wang, G.L. Hou, and B. Ge studied the existence and uniqueness of solutions for a quasilinear elliptic equation with a variable exponent and a convection term. In [19], the paper proved the existence of ground state sign-changing solutions for a class of second-order quasilinear elliptic equations. The equations are derived from models in nonlinear optics.

In the present article, we assume that a function $p\ :\ \Omega\rightarrow R$ satisfies the estimation

$$|p(x)-p(y)|\leq\frac{c}{\log\left(e+|x-y|^{-1}\right)}$$

and

$$|p(x)-p_{\infty}|\leq\frac{c_1}{\log(e+|x|)}$$

for all $x,\ y\in\Omega$, with some $c>0$, and some $c_1>0$, and real constant p_{∞}. Let $p,\ r\ :\ \Omega\rightarrow[0,\ \infty)$ and $p,\ r\in C(clos(\Omega))\cap L^{\infty}(\Omega)$. Let $p(x)<l$, $r(x)<\frac{lp(x)}{l-p(x)}$ for all $x\in clos(\Omega)$. Then, we establish that for each $\mu>0$, there exists a positive number $c(\mu)$ such that

$$\int_{\Omega}|u|^{p(x)}\ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right)dx\leq$$
$$\leq\mu\rho_{p(\cdot)}(\nabla u)+c(\mu)\rho_{p(\cdot)}(u)+c_1\frac{p_S}{p_m}\rho_{p(\cdot)}(u)\ln\left(\rho_{p(\cdot)}(u)\right)$$

for all $u\in L^{r(\cdot)}(\Omega)$ such that $|\nabla u|\in L^{p(\cdot)}(\Omega)$.

Also, we prove that assume $\phi_0\in W_{1,0}^{p(\cdot)}(\Omega)\setminus\{0\},\ \phi_0\in W_{\delta},\ 0<\delta<\ell,\ \phi_1\in L^2(\Omega)$, and $E_0<d$, then there exists a function $u\in L^{\infty}\left([0,\ \infty),\ W_{1,0}^{p(\cdot)}(\Omega)\setminus\{0\}\right)$, $\partial_t u\in L^{\infty}\left([0,\ \infty),\ L^2(\Omega)\right)$ such that

$$\int_{\Omega}\frac{\partial^2 u(x,\ t)}{\partial t^2}\varphi(x)\,dx+\int_{\Omega}|\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx$$
$$+\int_{\Omega}\frac{\partial u}{\partial t}\varphi dx=\theta\int_{\Omega}|u|^{p(x)-2}u\ln|u|\,\varphi dx$$

for all $\varphi\in W_{1,0}^{p(\cdot)}(\Omega)$. Thus, the hyperbolic problem (4.1) has a global weak solution $u\in L^{\infty}\left([0,\ \infty),\ W_{1,0}^{p(\cdot)}(\Omega)\setminus\{0\}\right)$, and the weak solution u grows as an exponential function as time approaches infinity (see Theorem 7).

2. Preliminaries

We assume that an exponent p belongs to the class of log-Hölder continuous functions $P^{\log}(\Omega),\ \Omega\subseteq R^l$, and denote $p_m=\inf_{x\in\Omega}p(x)$ and $p_S=\sup_{x\in\Omega}p(x)$. For further information, see [5, 6]. We define a modular function by

$$\rho_{p(\cdot)}(u)=\int_{\Omega}|u(x)|^{p(x)}\,dx,$$

and the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ norm by

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0\ :\ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right)\leq 1\right\}.$$

In the case variable exponent $p(\cdot)$ is identical to the constant, we obtain the standard L^p spaces.

For each $a > 0$ such that $\rho\left(\frac{u}{a}\right) < \infty$, we have that $\lambda \mapsto \rho\left(\frac{u}{\lambda}\right)$ is a continuous function on $[a, \infty)$ such that $\lim_{\lambda\to\infty} \rho\left(\frac{u}{\lambda}\right) = 0$. For all $\lambda \geq 1$, we have

$$\lambda^{p_m} \rho(u) \leq \rho(\lambda u) \leq \lambda^{p_S} \rho(u).$$

For all $0 < \lambda < 1$, we obtain

$$\lambda^{p_S} \rho(u) \leq \rho(\lambda u) \leq \lambda^{p_m} \rho(u).$$

There are different modular functions, for example

$$\tilde{\rho}_{p(\cdot)}(u) = \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} dx,$$

which satisfies the following inequalities

$$\tilde{\rho}_{p(\cdot)}(u) \leq \rho_{p(\cdot)}(u) \leq \tilde{\rho}_{p(\cdot)}(2u)$$

and the induced norm satisfies the relation

$$\|u\|_{L^{p(\cdot)}} = \|u\|_{\tilde{\rho}_{p(\cdot)}} \leq \|u\|_{\rho_{p(\cdot)}} \leq 2\|u\|_{\tilde{\rho}_{p(\cdot)}}.$$

The Banach space $W^{p(\cdot)}_{1,\,0}(\Omega)$ is defined by

$$W^{p(\cdot)}_{1,\,0}(\Omega) = \left\{u \in W^1_{1,\,0}(\Omega) \ : \ u,\ \nabla u \in L^{p(\cdot)}(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{W^{p(\cdot)}_{1,\,0}} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u \in W^{p(\cdot)}_{1,\,0}(\Omega)$.

Proposition 2.1. *Let $p \in P^{\log}(\Omega)$, then for all $u \in W^{p(\cdot)}_{1,\,0}(\Omega)$ the estimate*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c\, diam(\Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds with a constant c depending only on $p(\cdot)$ and l.

Proposition 2.2. *Let $p,\ r\ :\ \Omega \to [0, \infty)$ and $p,\ r \in C(clos(\Omega)) \cap L^\infty(\Omega)$. Let $p(x) < l$, $r(x) < \frac{lp(x)}{l-p(x)}$ for all $x \in clos(\Omega)$. Then, there exists a continuous and compact embedding $W^{p(\cdot)}_1(\Omega) \to L^{r(\cdot)}(\Omega)$.*

The proof and additional information on the Sobolev and Poincare inequalities can be found in [5, 6, 15].

Similar to classical theory, we have that, in case $p_S < \infty$, the set $C_0^\infty(\Omega)$ of all smooth functions with compact support is dense in $L^{p(\cdot)}(\Omega)$ for every open domain Ω.

In the variable exponent setting, we reformulate the Holder theorem as follows: for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, the inequality

$$\int_\Omega |u(x)\, v(x)|\, dx \leq \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)}$$

holds with $q(x) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$.

For all $p_1,\ p_2 \in P^{\log}(\Omega)$ such that $p_1(x) \le p_2(x)$ almost everywhere in Ω, the embedding $L^{p_2(\cdot)}(\Omega) \to L^{p_1(\cdot)}(\Omega)$ is continuous.

Theorem 2.3. *Let $p \in P^{\log}(\Omega)$, $1 < p_m \le p_S < l$. Let the function $p^* : \Omega \to R$ be defined by $p^*(x) = \frac{lp(x)}{l-p(x)}$ if $p(x) < l$ and $p^*(x) = \infty$ if $p(x) \ge l$. Then, the inequality*

$$\|u\|_{L^{p^*(\cdot)}(\Omega)} \le c\,\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$ with a positive constant c depending only on the l and $p(\cdot)$.

For all $s(\cdot),\ p(\cdot) > 1$, the interpolation theory yields the inequality

$$\|u\|_{L^{s(\cdot)}(\Omega)} \le 4\,\|u\|^{\theta}_{L^{p(\cdot)}(\Omega)}\,\|u\|^{1-\theta}_{L^{r(\cdot)}(\Omega)},$$

where $\frac{1}{s(\cdot)} = \frac{\theta}{p(\cdot)} + \frac{1-\theta}{r(\cdot)}$ [5, Section 7].

Straightforwardly, from the variable exponent Sobolev-Poincare inequalities and the interpolation theory, we obtain the following interpolation theorem.

Theorem 2.4. *Let $p \in P^{\log}(\Omega)$, $1 < p_m \le p_S < \infty$, and $1 < p(\cdot) < l$, $1 < s(\cdot) < p(\cdot)$. Then, the inequality*

$$\|u\|_{L^{s(\cdot)}(\Omega)} \le c\,\|u\|^{\theta}_{L^{p(\cdot)}(\Omega)}\,\|u\|^{1-\theta}_{L^{r(\cdot)}(\Omega)}$$

holds for all $u \in L^{r(\cdot)}(\Omega) \cap W_1^{p(\cdot)}(\Omega)$ with a positive constant c depending only on the l and $p(\cdot),\ s(\cdot),\ r(\cdot)$, where $\frac{1}{s(\cdot)} = \theta\left(\frac{1}{p(\cdot)} - \frac{1}{l}\right) + \frac{1-\theta}{r(\cdot)}$, $\theta \in [0,\ 1)$.

Proposition 2.5. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < r(\cdot) < \infty$, then the inequality*

$$\int_\Omega \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right) dx \le \frac{1}{1-\frac{p_S}{r_m}} \ln\left(\frac{\left(\rho_{r(\cdot)}(u)\right)^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)}\right)$$

holds for all $L^{p(\cdot)}(\Omega) \cap L^{r(\cdot)}(\Omega)$.

Proof. We obtain that the derivative of the function of t on $(0,\ \infty)$

$$f(t) = t \ln\left(\int_\Omega |u|^{\frac{1}{t}}\, dx\right) \quad (\text{for } u \ne 0 \text{ a.e.})$$

in the following form

$$f'(t) = t \ln\left(\int_\Omega |u|^{\frac{1}{t}}\, dx\right) - \frac{1}{t}\left(\int_\Omega |u|^{\frac{1}{t}}\, dx\right)^{-1} \int_\Omega |u|^{\frac{1}{t}} \ln(|u|)\, dx$$

for all $t \in (0,\ \infty)$. Employing the convexity argument, for all$0 < \tilde{t} < t < \infty$, we have

$$f'(t) \ge \frac{f(\tilde{t}) - f(t)}{\tilde{t} - t}$$

thus, we obtain

$$\ln\left(\int_\Omega |u|^{p(x)}\, dx\right) - \int_\Omega p(x) \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln(|u|)\, dx \ge \frac{1}{\frac{p_S}{r_m} - 1} \ln\left(\frac{\left(\rho_{r(\cdot)}(u)\right)^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)}\right)$$

so

$$-\ln\left(\int_\Omega |u|^{p(x)}\,dx\right)+\int_\Omega p(x)\,\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\ln(|u|)\,dx\le\frac{1}{1-\frac{p_S}{r_m}}\ln\left(\frac{\left(\rho_{r(\cdot)}(u)\right)^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)}\right).$$

□

3. Logarithmic Sobolev inequalities in the variable spaces

We consider the analog of logarithmic Sobolev inequalities.

Theorem 3.1. *Let $p\in P^{\log}(\Omega)$, $1<p(\cdot)<\infty$, then there exists a constant $c_1>0$ depending only on l and $p(\cdot)$ such that the inequality*

$$\int_\Omega\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right)dx\le c_1\ln\left(c(l,\ p(\cdot)\ )\,\frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{p(\cdot)}(u)}\right)$$

holds for all $u\in L^{p(\cdot)}(\Omega)$ such that $|\nabla u|\in L^{p(\cdot)}(\Omega)$.

Proof. We assume that $p\in P^{\log}(\Omega)$, $p(x)<l$, then we obtain that

$$\rho_{r(\cdot)}(u)\le\tilde{c}(l,\ p(\cdot)\ )\,\rho_{p(\cdot)}(\nabla u)$$

for all $u\in W^{p(\cdot)}_{1,0}(\Omega)$ for $r(x)\le p^*(x)$. We have

$$\rho_{r(\cdot)}(u)\le\left(\tilde{c}(l,\ p(\cdot)\ )\,\rho_{p(\cdot)}(\nabla u)\right)^{\theta}\rho_{p(\cdot)}(u)^{1-\theta}.$$

□

We can assume $r(x)=\frac{lp(x)}{l-1}$ and $\theta=\frac{1}{p_S}$, then we have

$$\int_\Omega\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right)dx\le$$

$$\le\frac{r_m}{r_m-p_S}\ln\left(\frac{\left(\left(\tilde{c}(l,\ p(\cdot)\ )\,\rho_{p(\cdot)}(\nabla u)\right)^{\theta}\rho_{p(\cdot)}(u)^{1-\theta}\right)^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)}\right)$$

and

$$\int_\Omega\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right)dx\le\frac{r_m}{r_m-p_S}\ln\left(c(l,\ p(\cdot)\ )\,\frac{\rho_{p(\cdot)}(\nabla u)^{\frac{1}{r_m}}}{\rho_{p(\cdot)}(u)^{1-\frac{p_S-1}{r_m}}}\right),$$

$$\int_\Omega \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right) dx \le$$
$$\le \frac{r_m}{r_m - p_S} \ln\left(c\left(l,\ p(\cdot)\ \right)\right)$$
$$+\frac{1}{r_m}\frac{r_m}{r_m - p_S} \ln\left(\rho_{p(\cdot)}(\nabla u)\right)$$
$$-\left(1-\frac{p_S - 1}{r_m}\right)\frac{r_m}{r_m - p_S} \ln\left(\rho_{p(\cdot)}(u)\right) \le$$
$$\le c_1 \ln\left(c\left(l,\ p(\cdot)\ \right)\frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{p(\cdot)}(u)}\right)$$

for all $u \in L^{p(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Theorem 3.2. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < \infty$, and $r(x) \le p^*(x)$, then the inequality*

$$\int_\Omega \frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \ln\left(\frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)}\right) dx \le c_1 \ln\left(c\left(l,\ p(\cdot),\ r(\cdot)\ \right)\frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{r(\cdot)}(u)}\right)$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$

Proof. Let $p \in P^{\log}(\Omega)$, $p(x) < l$. Then, we have

$$\rho_{s(\cdot)}(u) \le \left(\tilde{c}\left(l,\ p(\cdot)\ \right)\rho_{p(\cdot)}(\nabla u)\right)^{\theta} \rho_{r(\cdot)}(u)^{1-\theta}.$$

□

We obtain

$$\int_\Omega \frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \ln\left(\frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)}\right) dx \le \frac{1}{1-\frac{r_S}{s_m}} \ln\left(\frac{\left(\rho_{s(\cdot)}(u)\right)^{\frac{r_S}{s_m}}}{\rho_{r(\cdot)}(u)}\right),$$

so, we have

$$\int_\Omega \frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \ln\left(\frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)}\right) dx \le$$
$$\le \frac{1}{1-\frac{r_S}{s_m}} \ln\left(\frac{\left(\left(\tilde{c}\left(l,\ p(\cdot)\ \right)\rho_{p(\cdot)}(\nabla u)\right)^{\theta}\rho_{r(\cdot)}(u)^{1-\theta}\right)^{\frac{r_S}{s_m}}}{\rho_{r(\cdot)}(u)}\right) \le$$
$$\le \frac{1}{1-\frac{r_S}{s_m}} \ln\left(c\left(l,\ p(\cdot)\ \right)\frac{\left(\left(\rho_{p(\cdot)}(\nabla u)\right)^{\frac{1}{r_S}}\rho_{r(\cdot)}(u)^{\frac{r_S-1}{r_S}}\right)^{\frac{r_S}{s_m}}}{\rho_{r(\cdot)}(u)}\right) \le$$
$$\le c_1 \ln\left(c\left(l,\ p(\cdot),\ r(\cdot)\ \right)\frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{r(\cdot)}(u)}\right)$$

with $\theta = \frac{1}{r_S}$ for $r(x) \le p^*(x)$.

The results above can be presented in the parametric form.

Theorem 3.3. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < \infty$, and $r(x) \leq p^*(x)$. For each $\mu > 0$, there exists a positive number $c(\mu)$ such that*

$$\int_\Omega |u|^{p(x)} \ln\left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)}\right) dx \leq$$
$$\leq \mu\rho_{p(\cdot)}(\nabla u) + c(\mu)\rho_{p(\cdot)}(u) + c_1 \frac{p_S}{p_m}\rho_{p(\cdot)}(u) \ln\left(\rho_{p(\cdot)}(u)\right)$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$. Remark: The condition $r(x) \leq p^*(x)$ ensures the continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$.

4. Existence of a weak solution to the hyperbolic equation with logarithmic nonlinearity

In this section, we consider the hyperbolic equation involving variable exponents in the following form

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - div\left(|\nabla u|^{p(x)-2}\nabla u\right) + \frac{\partial u}{\partial t} = \theta |u|^{p(x)-2} u \ln|u|, \\ &u(x,\, 0) = \phi_0(x), \\ &\frac{\partial u(x,\, 0)}{\partial t} = \phi_1(x), \\ &u(x,\, t)|_{\partial\Omega\times[0,\, T)} = 0, \\ &\left.\frac{\partial u(x,\, t)}{\partial t}\right|_{\partial\Omega\times[0,\, T)} = 0, \end{aligned} \tag{4.1}$$

where Ω is a bounded domain in R^l with a smooth boundary $\partial\Omega$, and ϕ_0, ϕ_1 are fixed initial functions. Where a function $p^* : \Omega \to R$ is defined by $p^*(x) = \frac{lp(x)}{l-p(x)}$ if $p(x) < l$ and $p^*(x) = \infty$ if $p(x) \geq l$. We assume that θ is a positive number.

Definition 4.1. *The energy functional is defined by*

$$E_t(u) = \frac{1}{2}\int_\Omega |\partial_t u|^2 dx + \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \tag{4.2}$$
$$-\theta \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \ln|u|\, dx + \theta \int_\Omega \frac{1}{p(x)^2} |u|^{p(x)} dx. \tag{4.3}$$

We have the following energy estimate.

Lemma 4.2. *The energy functional E_t is a non-increasing function of $t \geq 0$ such that*

$$\frac{dE_t(u)}{dt} = -\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx \leq 0 \tag{4.4}$$

for all $t \geq 0$.

Proof. Multiplying the hyperbolic equation by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} dx - \int_\Omega div\left(|\nabla u|^{p(x)-2}\nabla u\right)\frac{\partial u}{\partial t} dx$$
$$+ \int_\Omega \frac{\partial u}{\partial t}\frac{\partial u}{\partial t} dx = \theta \int_\Omega |u|^{p(x)-2} u \ln|u| \frac{\partial u}{\partial t} dx.$$

Thus, we deduce

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2 dx+\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right.$$
$$\left.-\theta\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\ln|u|\,dx+\theta\int_\Omega\frac{1}{p(x)^2}|u|^{p(x)}dx\right)=-\int_\Omega\frac{\partial u}{\partial t}\frac{\partial u}{\partial t}dx\le 0.$$

□

The energy estimate is proven. By a density argument, this estimate remains valid for weak solutions.

Now, we are ready to prove the existence of a weak solution.

Theorem 4.3. *For each given* $\phi_0 \in W_{1,0}^{p(\cdot)}(\Omega)$ *and* $\phi_1 \in L^2(\Omega)$, *there exists a weak solution* u *to the boundary problem* (4.1), *in the following sense:* $u \in C\left([0,\ T],\ W_{1,0}^{p(\cdot)}(\Omega)\right)$, $\partial_t u \in C\left([0,\ T],\ L^2(\Omega)\right)$ *and* u *satisfies the identity*

$$\int_\Omega\frac{\partial^2 u(x,\ t)}{\partial t^2}\varphi(x)\,dx+\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx+$$
$$+\int_\Omega\frac{\partial u}{\partial t}\varphi dx=\theta\int_\Omega|u|^{p(x)-2}u\ln|u|\,\varphi dx \tag{4.5}$$

for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$.

Proof. We are going to employ the Galerkin approximation method. Let $\{\psi_k,\ k\in N\}$ be a basis in $W_{1,0}^{p(\cdot)}(\Omega)$, orthogonal in $L^2(\Omega)$. Let Λ_j be a linear span *span* $\{\psi_1,\ \psi_2,\ \dots,\ \psi_j\}$. We define the projections of the initial conditions on the finite-dimensional subspaces of Λ_j by

$$\phi_{0,j}(x)=\sum_{k=1,\ \dots,\ j}a_k\psi_k\xrightarrow[j\to\infty]{W_{1,0}^{p(\cdot)}(\Omega)}\phi_0(x),$$

$$\phi_{1,j}(x)=\sum_{k=1,\ \dots,\ j}b_k\psi_k\xrightarrow[j\to\infty]{W_{1,0}^{p(\cdot)}(\Omega)}\phi_1(x).$$

The Galerkin approximate solution will be assumed to have the form

$$u_j(x,\ t)=\sum_{k=1,\ \dots,\ j}\xi_k(t)\,\psi_k(x),$$

with coefficients $\xi_k(t)=\langle u(x,\ t),\ \psi_k(x)\rangle$ that satisfy the approximate problems in Λ_j

$$\int_\Omega\frac{\partial^2 u_j}{\partial t^2}\varphi dx+\int_\Omega|\nabla u_j|^{p(x)-2}\nabla u_j\nabla\varphi dx+$$
$$+\int_\Omega\frac{\partial u_j}{\partial t}\varphi dx=\theta\int_\Omega|u_j|^{p(x)-2}u_j\ln|u_j|\,\varphi dx,$$

$$\begin{aligned} u_j(x,\ 0) &= \phi_{0,j} = \sum_{k=1,\ \ldots,\ j} \langle \phi_0,\ \psi_k \rangle \psi_k, \\ \frac{\partial u_j(x,\ 0)}{\partial t} &= \phi_{1,j} = \sum_{k=1,\ \ldots,\ j} \langle \phi_1,\ \psi_k \rangle \psi_k, \\ u_j(x,\ t)|_{\partial\Omega\times[0,\ T)} &= 0, \\ \left.\frac{\partial u_j(x,\ t)}{\partial t}\right|_{\partial\Omega\times[0,\ T)} &= 0. \end{aligned}$$

□

It is a system of j ordinary differential equations of the second order for the coefficients

$$\xi_k(t) = \langle u(x,\ t),\ \psi_k(x) \rangle,\ k = 1,\ \ldots,\ j$$

on $[0,\ t_j),\ 0 < t_j \le T$ for each index j. Therefore, the coefficients $\xi_k(t)$, $k = 1,\ \ldots,\ j$ satisfy the integral identity

$$\int_\Omega \left(\frac{\partial^2 u_j}{\partial t^2} - div\left(|\nabla u_j|^{p(x)-2} \nabla u_j \right) + \right. \\ \left. + \frac{\partial u_j}{\partial t} - \theta |u_j|^{p(x)-2} u_j \ln|u_j| \right) \psi_k(x)\, dx = 0, \quad k = 1,\ \ldots,\ j.$$

Next, we must prove that we can take $t_j = T$ for each j. Let $\frac{\partial u_j}{\partial t}$ be a test function, then we have

$$\int_\Omega \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_j}{\partial t} dx - \int_\Omega div\left(|\nabla u_j|^{p(x)-2} \nabla u_j \right) \frac{\partial u_j}{\partial t} dx \\ + \int_\Omega \frac{\partial u_j}{\partial t} \frac{\partial u_j}{\partial t} dx = \theta \int_\Omega |u_j|^{p(x)-2} u_j \ln|u_j| \frac{\partial u_j}{\partial t} dx,$$

integration by parts gives the following identity

$$\frac{d}{dt} E_t(u_j) = \frac{d}{dt} \left(\frac{1}{2} \int_\Omega \left(\frac{\partial u_j}{\partial t} \right)^2 dx + \int_\Omega \frac{1}{p(x)} |\nabla u_j|^{p(x)} dx \right. \\ \left. -\theta \int_\Omega \frac{1}{p(x)} |u_j|^{p(x)} \ln|u_j|\, dx + \theta \int_\Omega \frac{1}{p(x)^2} |u_j|^{p(x)} dx \right) = \\ = -\int_\Omega \left(\frac{\partial u_j}{\partial t} \right)^2 dx \le 0.$$

Integration of the last inequality with respect to time from 0 to t gives

$$E_t(u_j) \le E_0(u_j).$$

We obtain

$$E_t\left(u_j\right)=\frac{1}{2}\int_\Omega\left(\frac{\partial u_j}{\partial t}\right)^2dx+\int_\Omega\frac{1}{p\left(x\right)}\left|\nabla u_j\right|^{p(x)}dx-$$
$$-\theta\int_\Omega\frac{1}{p\left(x\right)}\left|u_j\right|^{p(x)}\ln\left|u_j\right|dx+\theta\int_\Omega\frac{1}{p\left(x\right)^2}\left|u_j\right|^{p(x)}dx.$$

By the parametric logarithmic inequality, we have

$$\int_\Omega\frac{1}{p\left(x\right)}\left|u_j\right|^{p(x)}\ln\left|u_j\right|dx\leq$$
$$\leq\frac{1}{p_m}\mu\int_\Omega\left|\nabla u_j\right|^{p(x)}dx+\frac{1}{p_m}c\left(\mu\right)\int_\Omega\left|u_j\right|^{p(x)}dx$$
$$+\frac{1}{p_m}\left(c_1\frac{p_S}{p_m}+1\right)\ln\left(\int_\Omega\left|u_j\right|^{p(x)}dx\right)\int_\Omega\left|u_j\right|^{p(x)}dx.$$

So,

$$E_t\left(u_j\right)\geq\frac{1}{2}\int_\Omega\left(\frac{\partial u_j}{\partial t}\right)^2dx$$
$$+\left(\frac{1}{p_S}-\frac{\theta}{p_m}\mu\right)\int_\Omega\left|\nabla u_j\right|^{p(x)}dx+\theta\left(\frac{1}{p_S^2}-\frac{1}{p_m}c\left(\mu\right)\right)\int_\Omega\left|u_j\right|^{p(x)}dx$$
$$-\frac{1}{p_m}\left(c_1\frac{p_S}{p_m}+1\right)\ln\left(\int_\Omega\left|u_j\right|^{p(x)}dx\right)\int_\Omega\left|u_j\right|^{p(x)}dx.$$

Assume $c=2E_0\left(u_j\right)$, we have

$$\int_\Omega\left(\frac{\partial u_j}{\partial t}\right)^2dx+2\left(\frac{1}{p_S}-\frac{\theta}{p_m}\mu\right)\int_\Omega\left|\nabla u_j\right|^{p(x)}dx$$
$$+\theta\left(\frac{1}{p_S^2}-\frac{1}{p_m}c\left(\mu\right)\right)\int_\Omega\left|u_j\right|^{p(x)}dx\leq$$
$$\leq c+2\frac{1}{p_m}\left(c_1\frac{p_S}{p_m}+1\right)\ln\left(\int_\Omega\left|u_j\right|^{p(x)}dx\right)\int_\Omega\left|u_j\right|^{p(x)}dx.$$

Now, we select μ such that both inequalities $\frac{p_m}{\theta p_S}>\mu$ and $\frac{p_m}{p_S^2}>c\left(\mu\right)$ hold together, then we obtain

$$\int_\Omega\left(\frac{\partial u_j}{\partial t}\right)^2dx+\int_\Omega\left|\nabla u_j\right|^{p(x)}dx+\int_\Omega\left|u_j\right|^{p(x)}dx\leq$$
$$\leq c\left(1+\frac{1}{p_m}\ln\left(\int_\Omega\left|u_j\right|^{p(x)}dx\right)\int_\Omega\left|u_j\right|^{p(x)}dx\right).$$

We deduce

$$u_j\left(\cdot,\ t\right)=u_j\left(\cdot,\ 0\right)+\int_{[0,\ t]}\frac{\partial u_j\left(\cdot,\ \tau\right)}{\partial t}d\tau.$$

Therefore, we obtain

$$\int_{\Omega}\left(\frac{\partial u_j(t)}{\partial t}\right)^2 dx=\int_{\Omega}\left(u_j(\cdot,\ 0)+\int_{[0,\ t]}\frac{\partial u_j(\cdot,\ \tau)}{\partial t}d\tau\right)^2 dx\leq$$

$$\leq 2\int_{\Omega}u_j(0)^2\,dx+2\int_{\Omega}\left(\int_{[0,\ t]}\frac{\partial u_j(\cdot,\ \tau)}{\partial t}d\tau\right)^2 dx\leq$$

$$\leq 2\int_{\Omega}u_j(0)^2\,dx+2T\int_{[0,\ t]}\int_{\Omega}\left(\frac{\partial u_j(\cdot,\ \tau)}{\partial t}\right)^2 dxd\tau.$$

Thus, we conclude

$$\int_{\Omega}\left(\frac{\partial u_j(t)}{\partial t}\right)^2 dx\leq 2\int_{\Omega}u_j(0)^2\,dx+2Tc_2\left(1+\frac{1}{p_m}\ln\left(\int_{\Omega}|u_j|^{p(x)}\,dx\right)\int_{\Omega}|u_j|^{p(x)}\,dx\right).$$

We take $c_3=\max\left(2\int_{\Omega}u_j(0)^2\,dx,\ 2Tc_2\right)$, so we have

$$\int_{[0,\ T]}\int_{\Omega}\left(\frac{\partial u_j(t)}{\partial t}\right)^2 dxdt\leq$$

$$2c_3\left(1+\int_{[0,\ t]}\frac{1}{p_m}\ln\left(\int_{\Omega}|u_j|^{p(x)}\,dx\right)\int_{\Omega}|u_j|^{p(x)}\,dxd\tau\right).$$

Thus, we have

$$\int_{[0,\ T]}\int_{\Omega}\left(\frac{\partial u_j(t)}{\partial t}\right)^2 dxdt\leq$$

$$\leq 2c_2\left(1+\int_{[0,\ t]}\left(c_3+\int_{\Omega}|u_j|^{p(x)}\,dx\right)\ln\left(c_3+\left(\int_{\Omega}|u_j|^{p(x)}\,dx\right)^{\frac{1}{p_m}}\right)d\tau\right)$$

and

$$\int_{[0,\ T]}\int_{\Omega}\left(\frac{\partial u_j(t)}{\partial t}\right)^2 dxdt\leq 2c_3\exp\left(2c_3t\right)=const.$$

We conclude

$$\int_{\Omega}\left(\frac{\partial u_j}{\partial t}\right)^2 dx+\int_{\Omega}|\nabla u_j|^{p(x)}\,dx+\int_{\Omega}|u_j|^{p(x)}\,dx\leq c$$

where a positive constant c does not depend on $t\in[0,\ T]$ and j. We obtain

$$\max_{t\in[0,\ t_j]}\int_{\Omega}\left(\frac{\partial u_j}{\partial t}\right)^2 dx+\max_{t\in[0,\ t_j]}\int_{\Omega}|\nabla u_j|^{p(x)}\,dx+\max_{t\in[0,\ t_j]}\int_{\Omega}|u_j|^{p(x)}\,dx\leq\tilde{c},$$

where a positive constant c does not depend on $t\in[0,\ T]$ and j.We take the limit as t_j tends to T. Thus, we have:

1) set $\{u_j\}$ is uniformly bounded in $L^2\left([0,\ T],\ W_{1,0}^{p(\cdot)}(\Omega)\right)$,

and

2) set $\left\{\frac{\partial u_j}{\partial t}\right\}$ is uniformly bounded in $L^{\infty}\left([0,\ T],\ L^2(\Omega)\right)$.

There exists a subsequence of $\{u_j,\ j\in N\}$, which will still be denoted by $\{u_j,\ j\in N\}$, such that:

1. Sequence $\{u_j\}$ converges to u *-weakly in $L^{\infty}\left([0,\ T],\ W_{1,0}^{p(\cdot)}(\Omega)\right)$ as j tends to infinity;
2. Sequence $\left\{\frac{\partial u_j}{\partial t}\right\}$ converges to $\frac{\partial u}{\partial t}$ *-weakly in $L^{\infty}\left([0,\ T],\ L^2(\Omega)\right)$ as j tends to infinity;
3. Sequence $\{u_j\}$ converges to u weakly in $L^2\left([0,\ T],\ W_{1,0}^{p(\cdot)}(\Omega)\right)$ as j tends to infinity;
4. Sequence $\left\{\frac{\partial u_j}{\partial t}\right\}$ converges to $\frac{\partial u}{\partial t}$ weakly in $L^2\left([0,\ T],\ L^2(\Omega)\right)$ as j tends to infinity.

By the Aubin–Lions–Simon lemma, we obtain that sequence $\{u_j\}$ converges to u in $L^2\left([0,\ T],\ L^2(\Omega)\right)$ as j tends to infinity, and $\{u_j\}$ converges to u almost everywhere in $\Omega\times[0,\ T]$ as j tends to infinity. By continuity of $r\to r^{p(x)-1}\ln r^{\theta}$, we obtain that sequence $\left\{|u_j|^{p(x)-2}u_j\ln|u_j|^{\theta}\right\}$ converges to $|u|^{p(x)-2}u\ln|u|^{\theta}$ almost everywhere in $\Omega\times[0,\ T]$ as j tends to infinity.

For all $\varphi\in\Lambda_j$, we obtain the identity

$$\int_{\Omega}\frac{\partial u_j}{\partial t}\varphi dx-\int_{\Omega}\phi_{1,j}\varphi dx+\int_{[0,\ t]}\int_{\Omega}|\nabla u_j|^{p(x)-2}\nabla u_j\nabla\varphi dxd\tau$$
$$+\int_{[0,\ t]}\int_{\Omega}\frac{\partial u_j}{\partial t}\varphi dxd\tau=\theta\int_{[0,\ t]}\int_{\Omega}|u_j|^{p(x)-2}u_j\ln|u_j|\varphi dxd\tau.$$

We pass to the limit as j goes to infinity, and deduce

$$\int_{\Omega}\frac{\partial u}{\partial t}\varphi dx=\int_{\Omega}\phi_1\varphi dx-\int_{[0,\ t]}\int_{\Omega}|\nabla u|^{p(x)-2}\nabla u\nabla\varphi dxd\tau-$$
$$-\int_{[0,\ t]}\int_{\Omega}\frac{\partial u}{\partial t}\varphi dxd\tau+\theta\int_{[0,\ t]}\int_{\Omega}|u|^{p(x)-2}u\ln|u|\varphi dxd\tau.$$

The right-hand side of the last identity is an absolutely continuous function. We differentiate at t, thus the identity

$$\theta\int_{\Omega}|u(x,\ t)|^{p(x)-2}u(x,\ t)\ln|u(x,\ t)|\varphi(x)dx=$$
$$=\int_{\Omega}\frac{\partial^2u(x,\ t)}{\partial t^2}\varphi(x)dx+\int_{\Omega}\frac{\partial u(x,\ t)}{\partial t}\varphi(x)dx$$
$$+\int_{\Omega}|\nabla u(x,\ t)|^{p(x)-2}\nabla u(x,\ t)\nabla\varphi(x)dx$$

holds for all $\varphi\in W_{1,0}^{p(\cdot)}(\Omega)$. Employing the initial conditions, we deduce:

1) sequence $\{u_j\}$ converges to u weakly in $L^2\left([0,\ T],\ W_{1,0}^{p(\cdot)}(\Omega)\right)$ as j tends to infinity;

2) sequence $\left\{\frac{\partial u_j}{\partial t}\right\}$ converges to $\frac{\partial u}{\partial t}$ weakly in $L^2\left([0,\ T],\ L^2(\Omega)\right)$ as j tends to infinity;

3) sequence $\{u_j\}$ converges to u in $C\left([0,\ T],\ L^2(\Omega)\right)$ as j tends to infinity; and,

4) sequence $\{u_j(x,\ 0)=\phi_{0,j}\}$ converges to $u(x,\ 0)=\phi_0(x)$ in $W_{1,0}^{p(\cdot)}(\Omega)\cap H_0^1(\Omega)$ as j tends to infinity.

Let $\eta\in C_0^\infty([0,\ T])$. Then, we have

$$-\int_\Omega \frac{\partial u_j}{\partial t}\eta'(t)\,\varphi dx=\int_{[0,\ t]}\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j\nabla\varphi\eta(\tau)\,dxd\tau-$$
$$-\int_{[0,\ t]}\int_\Omega \frac{\partial u_j}{\partial t}\varphi\eta(\tau)\,dxd\tau+\theta\int_{[0,\ t]}\int_\Omega |u_j|^{p(x)-2}u_j\ln|u_j|\,\varphi\eta(\tau)\,dxd\tau$$

for all $\varphi\in W_{1,0}^{p(\cdot)}(\Omega)$. We pass to the limit as j tends to infinity, and deduce

$$-\int_\Omega \frac{\partial u}{\partial t}\eta'(t)\,\varphi dx=\int_{[0,\ t]}\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\eta(\tau)\,dxd\tau-$$
$$-\int_{[0,\ t]}\int_\Omega \frac{\partial u}{\partial t}\varphi\eta(\tau)\,dxd\tau+\theta\int_{[0,\ t]}\int_\Omega |u|^{p(x)-2}u\ln|u|\,\varphi\eta(\tau)\,dxd\tau.$$

Therefore, we obtain that sequence $\left\{\frac{\partial u_j(x,\ 0)}{\partial t}=\phi_{1,j}\right\}$ converges to $\frac{\partial u(x,\ 0)}{\partial t}=\phi_1(x)$ in $L^2(\Omega)$ as index j tends to infinity. Thus, the existence of a weak solution has been proven.

5. New potential wells technique for variable exponent spaces

For small positive numbers δ, we define the variable exponent functionals by

$$J_\delta(u)=\delta\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx+\theta\int_\Omega \frac{1}{p(x)^2}|u|^{p(x)}dx-\theta\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\ln|u|\,dx,$$

and

$$I_\delta(u)=\delta\int_\Omega |\nabla u|^{p(x)}dx-\theta\int_\Omega |u|^{p(x)}\ln|u|\,dx.$$

For all $u\in W_{1,0}^{p(\cdot)}(\Omega)\setminus\{0\}$, for $\lambda>0$, we consider the mapping $\lambda\mapsto\Psi_\delta(\lambda)=J_\delta(\lambda u)$ defined by

$$\Psi_\delta(\lambda)=\delta\int_\Omega \frac{\lambda^{p(x)}}{p(x)}|\nabla u|^{p(x)}dx+$$

$$+\theta\int_\Omega \frac{\lambda^{p(x)}}{p(x)^2}|u|^{p(x)}dx-\theta\int_\Omega \frac{\lambda^{p(x)}}{p(x)}|u|^{p(x)}\ln|\lambda u|\,dx.$$

We estimate the logarithm term, and have

$$\int_{\Omega}|u|^{p(x)}\ln|u|\,dx\le$$
$$\le\int_{\{|u|\le 1\}}|u|^{p(x)}\ln|u|\,dx+\int_{\{|u|\ge 1\}}|u|^{p(x)}\ln|u|\,dx\le$$
$$\le c\int_{\{|u|\ge 1\}}|u|^{p(x)+\gamma}\,dx\le c\int_{\Omega}|u|^{p(x)+\gamma}\,dx$$

for all $\gamma>0$. By application of the Young inequality, we obtain that for each $\varepsilon>0$ there exists some constant $c(\varepsilon)>0$ such that

$$\int_{\Omega}|u|^{p(x)}\ln|u|\,dx\le c\int_{\Omega}|u|^{p(x)+\gamma}\,dx\le$$

$$\le c\tilde{\varepsilon}\rho_{p(\cdot)}(|\nabla u|)+cc(\tilde{\varepsilon})\,\rho_{p(\cdot)}(u)\le$$

$$\le\varepsilon\rho_{p(\cdot)}(|\nabla u|)+c(\varepsilon)\left(\rho_{p(\cdot)}(u)\right)^{\beta},$$

where we select $0<\gamma<\frac{p_m^2}{l}$ and

$$1<\beta=p_m\frac{\left(1-\frac{l\gamma}{p_m(p_m+\gamma)}\right)(p_m+\gamma)}{p_m^2-l\gamma}.$$

Lemma 5.1. *Let $u\in W_{1,0}^{p(\cdot)}(\Omega)\setminus\{0\}$. Then, we have $\lim\limits_{\lambda\to+0}\Psi_\delta(\lambda)=0$ and $\lim\limits_{\lambda\to\infty}\Psi_\delta(\lambda)=-\infty$.*

Proof. Straightforwardly, we have

$$\Psi_\delta(\lambda)=J_\delta(\lambda u)=\delta\int_{\Omega}\frac{1}{p(x)}|\lambda\nabla u|^{p(x)}\,dx-$$
$$-\theta\int_{\Omega}\frac{1}{p(x)}|\lambda u|^{p(x)}\ln|\lambda u|\,dx+\theta\int_{\Omega}\frac{1}{p(x)^2}|\lambda u|^{p(x)}\,dx\overset{\lambda\to 0}{\longrightarrow}0$$

and

$$\Psi_\delta(\lambda)\le$$
$$\le\int_{\Omega}\frac{1}{p(x)}\left(\left(\delta|\nabla u|^{p(x)}+\theta\frac{1}{p(x)}|u|^{p(x)}\right)\lambda^{p(x)}-\theta\lambda^{p(x)}|u|^{p(x)}\ln|\lambda u|\right)dx=$$
$$=\int_{\Omega}\frac{1}{p(x)}\left(\left(\delta|\nabla u|^{p(x)}+\theta\frac{1}{p(x)}|u|^{p(x)}-\theta|u|^{p(x)}\ln|\lambda|\right)\lambda^{p(x)}\right)dx$$
$$-\theta\int_{\Omega}\frac{1}{p(x)}\lambda^{p(x)}|u|^{p(x)}\ln|u|\,dx\overset{\lambda\to\infty}{\longrightarrow}-\infty.$$

□

Lemma 5.2. *Let* $u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$. *Then, there exists a solution* λ *to the equation* $\Psi'_\delta(\lambda) = 0$ *such that*

$$\min\{D_1, D_1\} \le \lambda \le \max\{D_1,\ D_1\}$$

where,

$$D_1 = \exp\left(\frac{1}{\theta B(u)}\left(\delta\frac{\min\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}{\max\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}A(u) - \theta C(u)\right)\right),$$

$$D_2 = \exp\left(\frac{1}{\theta B(u)}\left(\delta\frac{\max\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}{\min\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}A(u) - \theta C(u)\right)\right),$$

and

$$A(u) = \int_\Omega |\nabla u|^{p(x)}\,dx, \quad B(u) = \int_\Omega |u|^{p(x)}\,dx, \quad C(u) = \int_\Omega |u|^{p(x)}\ln|u|\,dx.$$

Proof. We have

$$\begin{aligned}\Psi'_\delta(\lambda) &= \delta\int_\Omega \lambda^{p(x)-1}|\nabla u|^{p(x)}dx + \theta\int_\Omega \frac{\lambda^{p(x)-1}}{p(x)}|u|^{p(x)}dx \\ &\quad - \theta\int_\Omega\left(\lambda^{p(x)-1}|u|^{p(x)}\ln|\lambda u| + \frac{\lambda^{p(x)-1}}{p(x)}|u|^{p(x)}\right)dx \\ &= \delta\int_\Omega \lambda^{p(x)-1}|\nabla u|^{p(x)}dx - \theta\int_\Omega \lambda^{p(x)-1}|u|^{p(x)}\ln|\lambda|dx \\ &\quad - \theta\int_\Omega \lambda^{p(x)-1}|u|^{p(x)}\ln|u|dx.\end{aligned}$$

Thus, we have

$$\begin{aligned}\Psi'_\delta(\lambda) &\ge \delta\min\left(\lambda^{p_S-1},\lambda^{p_m-1}\right)A(u) \\ &\quad - \theta\max\left(\lambda^{p_S-1},\lambda^{p_m-1}\right)(\ln|\lambda|B(u)+C(u)) = f_1(\lambda)\end{aligned}$$

and

$$\begin{aligned}\Psi'_\delta(\lambda) &\le \delta\max\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)A(u) \\ &\quad -\theta\min\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)(\ln|\lambda|B(u)+C(u)) = f_2(\lambda).\end{aligned}$$

Thus, we have that

$$\min\{D_1, D_1\} \le \lambda \le \max\{D_1,\ D_1\},$$

where, we denote

$$D_1 = \exp\left(\frac{1}{\theta B(u)}\left(\delta\frac{\min\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}{\max\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}A(u) - \theta C(u)\right)\right)$$

$$D_2 = \exp\left(\frac{1}{\theta B(u)}\left(\delta\frac{\max\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}{\min\left(\lambda^{p_S-1},\ \lambda^{p_m-1}\right)}A(u) - \theta C(u)\right)\right),$$

and

$$A(u) = \int_\Omega |\nabla u|^{p(x)}\, dx,$$
$$B(u) = \int_\Omega |u|^{p(x)}\, dx,$$
$$C(u) = \int_\Omega |u|^{p(x)} \ln |u|\, dx.$$

□

Lemma 5.3. *Let $u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$. Then, there exists a number $\lambda^* = \lambda^*(u) > 0$ such that the functional Ψ_δ attains its maximum at $\lambda = \lambda^*$. We have that:*

1) if $I_\delta(u) < 0$ then $\lambda^* \in (0,\ 1)$;
2) if $I_\delta(u) = 0$ then $\lambda^* = 1$;
3) if $I_\delta(u) > 0$ then $\lambda^* > 1$.

Proof. We have that $I_\delta(\lambda u) = \lambda \Psi'_\delta(\lambda)$, where

$$\Psi'_\delta(\lambda) = \delta \int_\Omega \lambda^{p(x)-1} |\nabla u|^{p(x)}\, dx - \theta \int_\Omega \lambda^{p(x)-1} |u|^{p(x)} \ln |\lambda u|\, dx$$

and

$$I_\delta(u) = \delta \int_\Omega |\nabla u|^{p(x)}\, dx - \theta \int_\Omega |u|^{p(x)} \ln |u|\, dx.$$

Therefore, we obtain

$$\Psi'_\delta(\lambda) \le \lambda^{p_S - 1} I_\delta(u)$$

for $\lambda \in (0,\ 1)$, and

$$\Psi'_\delta(\lambda) \ge \lambda^{p_S - 1} I_\delta(u)$$

for $\lambda > 1$. The lemma is proven. □

Definition 5.4. *The Nehari manifold N_δ associated with the energy functional J_δ is defined by*

$$N_\delta = \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\} \ :\ I_\delta(u) = \langle J'_\delta(u),\ u \rangle = 0 \right\}.$$

Straightforwardly, we obtain that the set N_δ is not empty, and the operator J_δ is coercive on the set N_δ. We denote

$$\begin{aligned} d_\delta &= \inf_{u \in N_\delta} J_\delta(u) \\ &= \inf \left\{ \sup_{\lambda \ge 0} J_\delta(u) \ :\ u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\} \right\}. \end{aligned}$$

We define

$$W_\delta = \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) \ :\ J_\delta(u) < d_\delta,\ I_\delta(u) > 0 \right\} \cup \{0\},$$
$$V_\delta = \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) \ :\ J_\delta(u) < d_\delta,\ I_\delta(u) < 0 \right\}.$$

By the parametric logarithmic Sobolev inequality, we estimate

$$\begin{aligned} I_\delta(u) &= \delta \int_\Omega |\nabla u|^{p(x)}\, dx - \theta \int_\Omega |u|^{p(x)} \ln|u|\, dx \geq \\ &\geq (\delta - \theta\mu) \int_\Omega |\nabla u|^{p(x)}\, dx \\ &\quad - \left(\theta c(\mu) + \theta\left(c_1 \frac{p_S}{p_m} + 1\right) \ln\left(\left(\int_\Omega |u|^{p(x)}\, dx\right)\right)\right) \int_\Omega |u|^{p(x)}\, dx. \end{aligned}$$

We obtain that

1) if $\rho_{p(\cdot)}(u) \leq \exp\left(-c(\mu)\left(c_1 \frac{p_S}{p_m} + 1\right)^{-1}\right) = \ell$ and $\mu < \frac{\delta}{\theta}$, then $I_\delta(u) > 0$;

2) if $I_\delta(u) < 0$ and $\mu < \frac{\delta}{\theta}$, then $\rho_{p(\cdot)}(u) > \exp\left(-c(\mu)\left(c_1 \frac{p_S}{p_m} + 1\right)^{-1}\right)$;

3) if $I_\delta(u) = 0$ and $\mu < \frac{\delta}{\theta}$, then $\rho_{p(\cdot)}(u) \geq \exp\left(-c(\mu)\left(c_1 \frac{p_S}{p_m} + 1\right)^{-1}\right)$.

Theorem 5.5. *Let $\phi_0 \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$, $\phi_0 \in W_\delta$, $0 < \delta < \ell$ and $\phi_1 \in L^2(\Omega)$ be given. We assume $E_0 < d$, then the hyperbolic problem (1) has a global weak solution $u \in L^\infty\left([0, \infty), W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}\right)$, $\partial_t u \in L^\infty\left([0, \infty), L^2(\Omega)\right)$ and the identity*

$$\begin{aligned} &\int_\Omega \frac{\partial^2 u(x, t)}{\partial t^2} \varphi(x)\, dx + \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx \\ &\quad + \int_\Omega \frac{\partial u}{\partial t} \varphi dx = \theta \int_\Omega |u|^{p(x)-2} u \ln|u|\, \varphi dx \end{aligned}$$

holds for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$. Moreover, the weak solution u grows as an exponential function as time approaches infinity. Since $L_t(u)$ is equivalent to the energy norm, this implies exponential growth of the solution in the energy norm.

Proof. The existence can be proven similarly to the previous theorem.
We have proven that

$$-E_t'(u) = \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 \geq 0.$$

We assume that $0 < E_0(u) < E_1 < \min\left(\frac{\ell^{p_m}}{p_S}, \frac{\ell^{p_S}}{p_S}\right)$, and denote

$$L_t(u) = E_1 - E_t(u) + \varepsilon \int_\Omega u \frac{\partial u}{\partial t} dx + \frac{\varepsilon}{2} \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2$$

for all $t \geq 0$. We differentiate with respect to time and obtain

$$L_t'(u) = -E_t'(u) + \\ +\varepsilon \int_\Omega u \left(div \left(|\nabla u|^{p(x)-2} \nabla u \right) + \theta |u|^{p(x)-2} u \ln |u| \right) dx \\ +\varepsilon \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx + \varepsilon \int_\Omega u \frac{\partial u}{\partial t} dx = (1+\varepsilon) \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx - \\ -\varepsilon \int_\Omega |\nabla u|^{p(x)} dx + \varepsilon \int_\Omega \theta |u|^{p(x)-2} u \ln |u| \, dx.$$

For a positive number γ, we have

$$L_t'(u) = \varepsilon\gamma E_1 - \varepsilon\gamma E_t(u) - \varepsilon\gamma E_1 \\ + \left(1 + \varepsilon + \frac{1}{2}\varepsilon\gamma \right) \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx + \varepsilon \int_\Omega \frac{\gamma - p(x)}{p(x)} |\nabla u|^{p(x)} dx \\ -\varepsilon \int_\Omega \frac{\gamma - p(x)}{p(x)} \theta |u|^{p(x)-2} u \ln |u| \, dx + \theta\varepsilon\gamma \int_\Omega \frac{1}{p(x)^2} |u|^{p(x)} dx.$$

Since $\phi_0 \in W_\delta$, we obtain $u \in W$ and $I(u) < 0$, therefore, we have $\|u\|_{L^{p(\cdot)}} > \ell$. So, we obtain

$$L_t'(u) \geq \varepsilon\gamma E_1 - \varepsilon\gamma E_t(u) + \\ + \left(1 + \varepsilon + \frac{1}{2}\varepsilon\gamma \right) \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx + \varepsilon \int_\Omega \frac{\gamma - p(x)}{p(x)} |\nabla u|^{p(x)} dx \\ -\varepsilon\theta \frac{\gamma - p_m}{p_m} \left(\mu \int_\Omega |\nabla u|^{p(x)} dx + c(\mu) \int_\Omega |u|^{p(x)} dx + \right. \\ \left. + \left(\frac{p_S}{p_m} + 1 \right) \ln \left(\int_\Omega |u|^{p(x)} dx \right) \int_\Omega |u|^{p(x)} dx \right) \geq \\ \geq \varepsilon\gamma E_1 - \varepsilon\gamma E_t(u) + \left(1 + \varepsilon + \frac{1}{2}\varepsilon\gamma \right) \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx \\ +\varepsilon \left(\frac{\gamma - p_S}{p_S} - \theta \frac{\gamma - p_m}{p_m} \mu \right) \int_\Omega |\nabla u|^{p(x)} dx \\ -\varepsilon\theta \frac{\gamma - p_m}{p_m} \left(c(\mu) - \left(\frac{p_S}{p_m} + 1 \right) \ln \left(\int_\Omega |u|^{p(x)} dx \right) \right) \int_\Omega |u|^{p(x)} dx.$$

For $\|u\|_{L^{p(\cdot)}} > \ell$, we choose $\gamma = 4p_S$, then we obtain

$$L_t'(u) \geq \beta \left(E_1 - E_t(u) + \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{p(x)} dx \right),$$

where we denote

$$\beta = \min \left(E_1 - E_t(u), \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx, \int_\Omega |\nabla u|^{p(x)} dx, \int_\Omega |u|^{p(x)} dx \right) > 0.$$

for $\varepsilon > 0$.

By the Young and generalized Holder inequalities, we have

$$L_t(u) \le E_1 - E_t(u) + \varepsilon c_1 \left(\int_\Omega |u|^{p(x)}\, dx + E_1 - E_t(u) \right)$$
$$+\frac{1}{2}\varepsilon \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx \le$$
$$\le c \left(E_1 - E_t(u) + \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |u|^{p(x)}\, dx \right),$$

where $c_1 = 1 + \frac{1}{E_1 - E_0(u)}$. Thus, we obtain $L_t'(u) \ge const\ \ L_t(u)$ so $L_t(u) \ge L_0(u) \exp(ct)$. □

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