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Global existence, asymptotic behavior, and blow-up for a parabolic $p-$Laplacian type equation with complex interactions at the boundary

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Abstract. In this paper, we study the initial boundary value problem involving the p-Laplacian parabolic equation $u_t - \Delta_p u + \alpha|u|^{p-2}u = 0, \quad (x,t) \in \Omega\times]0,+\infty[$, with logarithmic boundary condition. By using the potential wells method combined with the Nehari Manifold, we establish the existence of a weak global solution. In addition, we also obtain the decay polynomial of the weak solution. Then, by virtue of the differential inequality technique, we prove that the solutions blow up in finite time under suitable initial values.

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1. Introduction and main results

In this paper, we consider the initial-boundary value problem of the following parabolic p-Laplacian equation with a logarithmic non-linear term on the boundary as follows:

$$\begin{cases} u_t - \Delta_p u + \alpha|u|^{p-2}u = 0, & (x,t) \in \Omega\times]0,+\infty[, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \beta|u|^{q-2}u\log(|u|), & (x,t) \in \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\alpha, \beta > 0$ and p, q satisfies

$(H1): \ 1 < p < q < p_\partial,$

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where

$$p_\partial := \begin{cases} \dfrac{(N-1)p}{N-p}, & if\ \ p < N, \\ +\infty, \quad p \geq N. \end{cases}$$

Logarithmic nonlinearities have been widely studied in physics, arising in areas such as supersymmetric field theories, inflationary cosmology, nuclear physics, optics, and quantum mechanics. Parabolic p-Laplacian equations also play a significant role in various fields, such as fluid dynamics and climatology. For more applications involving the parabolic equations in fluids, climate modeling, and climatology , we can refer to [3, 2, 6, 7, 4, 5] and references therein. Furthermore, equation (1.1) models a wide range of phenomena in nature and industry, such as non-Newtonian fluids and diffusive processes that exhibit spatial heterogeneity, as found in porous media and biological systems. Similar models have been discussed in population dynamics, particularly in the work of Murray [12].

The term $\alpha|u|^{p-2}u$ represents reaction-diffusion systems commonly found in chemical reactions and heat propagation models, which are crucial for understanding phenomena like combustion and autocatalytic reactions, as shown in classical works by Vázquez and Lions [15, 11]. The term $\beta|u|^{q-2}u\log(|u|)$, accounting for logarithmic interactions at the boundary, adds a layer of complexity that models surface phenomena such as adsorption and desorption processes, which are relevant to fields like thermodynamics and chemical surface reactions, as discussed by Amann in [1].

Many works have explored similar logarithmic terms in different contexts, such as, Peng and Zhou [13] studied the following heat equation with logarithmic nonlinearity

$$u_t - \Delta u_t = |u|^{p-2}u\log(|u|),$$

under the Dirichlet boundary condition. They obtained the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

Nhan and Truong [14] studied the following Dirichlet nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - \Delta_p u = |u|^{p-2}u\log(|u|).$$

They obtained results regarding the existence or non-existence of global solutions. Also, He et al. [8] proved the decay and the finite time blow-up for weak solutions of the equation.

Xiao and Li [17] considered the initial boundary value problem for nonlinear higher-order heat equations of

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of a weak solution to the static problem by using the potential well technique.

Later Lamaizi et al. [10] studied the following heat equation involving the Laplacian

$$u_t - \Delta u + u = 0,$$

under a nonlinear boundary condition

$$\frac{\partial u}{\partial \nu} = \lambda |u|^{p-1} u.$$

They prove the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

Motivated by the above works, in the present manuscript, using the potential well theory combined with the Nehari manifold, we discuss the global existence and finite time blow-up for the solutions to the problem (1.1) under suitable conditions on the initial data. Meanwhile, we find that the global solution decays polynomially.

The rest of the paper is organized as follows. In Sect. 2, we recall some preliminary results that will be used throughout the paper. In Sect. 3, we introduce a family of potential wells and the invariance of some sets are established. Finaly, in Sect. 4 we presents our main results and its proof.

2. Preliminaries

For $p \in [1, +\infty)$, we denote the L^p-norm of a function $f \in L^p(\Omega)$ by $||f||_p = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}$ and the Lebesgue norm of $L^p(\partial\Omega)$ by $||g||_{p,\partial} = \left(\oint_{\partial\Omega} |g|^p dx\right)^{\frac{1}{p}}$ for $g \in L^p(\partial\Omega)$.
The usual Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{v \in L^p(\Omega) : |\nabla v| \in L^p(\Omega)\},$$

equipped by the norm

$$|| \cdot ||_{W^{1,p}(\Omega)} = \left(|| \cdot ||_p^p + ||\nabla(\cdot)||_p^p\right)^{\frac{1}{p}}.$$

Proposition 2.1. *(See* [16]*) The trace operator $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is continuous if and only if $1 \leq q \leq p_\partial$ if $p \neq N$ and for $1 \leq q < \infty$ if $p = N$. Where*

$$p_\partial = \begin{cases} \dfrac{(N-1)p}{N-p}, & if \;\; 1 < p < N, \\ +\infty, \quad p \geq N. \end{cases}$$

Lemma 2.2. *(See* [9]*) Suppose that a positive, twice-differentiable function ψ satisfies the inequality*

$$\ddot{\psi}(t)\psi(t) - (\mu+1)[\dot{\psi}(t)]^2 > 0,$$

for some $\mu > 0$. Then there exists $T^\star > 0$ such that $\lim_{t \to T^\star} \psi(t) = +\infty$.

Lemma 2.3. *(See* [14]*) For any positive number $s > 0$, we have the following inequality*

$$\log(x) \leq \frac{x^s}{se}, \quad \forall x \geq 1.$$

3. Potential wells

In this section, under the hypothesis $(H1)$, we introduce a family of potential wells for the problem (1.1) and give a series of their properties.

For clarity and simplicity, we will adopt the following notation throughout this paper:

$$u := u(t) := u(x,t),$$

with T representing the maximal existence time of u.

We now introduce two important functionals:

$$J(u) := \frac{1}{p}||u||_{\alpha}^{p} - \frac{\beta}{q}\oint_{\partial\Omega}|u|^q\log(|u|) + \frac{\beta}{q^2}||u||_{q,\partial}^{q} \tag{3.1}$$

$$I(u) := ||u||_{\alpha}^{p} - \beta\oint_{\partial\Omega}|u|^q\log(|u|)dx, \tag{3.2}$$

where $\|u\|_{\alpha} := \left(\|\nabla u\|_p^p + \alpha\|u\|_p^p\right)^{\frac{1}{p}}$, a norm that is equivalent to the standard Sobolev norm $\|u\|_{W^{1,p}(\Omega)}$.

Additionally, we observe that:

$$J(u) = \left(\frac{1}{p} - \frac{1}{q}\right)||u||_{\alpha}^{p} + \frac{1}{q}I(u) + \frac{\beta}{q^2}||u||_{q,\partial}^{q}. \tag{3.3}$$

The Nehari manifold is defined by

$$\mathcal{N} := \{u \in W^{1,p} : I(u) = 0,\ ||u||_{\alpha} > 0\},$$

and the mountain pass level d, which represents the depth of the potential well, is given by:

$$d := \inf\{\sup_{\lambda>0} J(\lambda u) :\ u \in\ W^{1,p}(\Omega),\ ||u||_{\alpha}\ > 0\}. \tag{3.4}$$

Moreover, we define the potential well by

$$W := \{u \in W^{1,p}(\Omega) : I(u) > 0,\ J(u) < d\} \cup \{0\},$$

and the set

$$V := \{u \in W^{1,p}(\Omega) : I(u) < 0,\ J(u) < d\}.$$

Lemma 3.1. *For any $u \in W^{1,p}(\Omega)$, $||u||_{q,\partial} > 0$, and $\lambda > 0$ there hold*

1. $\lim_{\lambda\to0^+} J(\lambda u) = 0$ *and* $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$*;*
2. *there exists a unique $\lambda_\star(u) = \lambda_\star$ such that $\frac{d}{d\lambda}J(\lambda_\star u) = 0$;*
3. *the function $\lambda \longmapsto J(\lambda u)$ is increasing on $[0,\lambda_\star]$, decreasing on $[\lambda_\star,+\infty)$ and takes the maximum at $\lambda_\star$;*
4. *the function $\lambda \longmapsto I(\lambda u)$ satisfies $I(\lambda u) > 0$ on $(0,\lambda_\star)$, $I(\lambda u) < 0$ on $(\lambda_\star,+\infty)$ and $I(\lambda_\star u) = 0$.*

Proof. 1. By (3.1) we get

$$J(\lambda u) = \frac{\lambda^p}{p}||u||_{\alpha}^{p} - \frac{\beta}{q}\lambda^q\log(\lambda)||u||_{q,\partial}^{q} - \frac{\beta}{q}\lambda^q\oint_{\partial\Omega}|u|^q\log(|u|)dx + \frac{\lambda^q}{q^2}||u||_{q,\partial}^{q}.$$

Thus, the first statement holds due to $||u||_{q,\partial} > 0$.

2. With a simple calculation, we get the following
$$\frac{d}{d\lambda}J(\lambda u) = \lambda^{p-1}||u||_{\alpha}^{p} - \beta\lambda^{q-1}\log(\lambda)||u||_{q,\partial}^{q} - \beta\lambda^{q-1}\oint_{\partial\Omega}|u|^{q}\log(|u|)dx,$$
making the derivative equal zero, we obtain
$$\lambda^{p-1}||u||_{\alpha}^{p} = \beta\lambda^{q-1}\log(\lambda)||u||_{q,\partial}^{q} + \beta\lambda^{q-1}\oint_{\partial\Omega}|u|^{q}\log(|u|)dx.$$
For $\lambda > 0$, define
$$K(\lambda) := \lambda^{q-p}\log(\lambda)||u||_{q,\partial}^{q} + \lambda^{q-p}\oint_{\partial\Omega}|u|^{q}\log(|u|)dx.$$
It is easy to see that there exists a unique $\lambda_0 > \lambda_1$ such that
$$K(\lambda_0) = 0, \quad K'(\lambda_1) = 0,$$
and the function $\lambda \mapsto K(\lambda)$ is increasing on $[\lambda_1, +\infty)$ and $K(\lambda) > 0$, for all $\lambda > \lambda_0$. Then, for any $u \in W^{1,p}(\Omega), ||u||_{q,\partial} > 0$, and $\beta, \alpha > 0$, there exists a unique $\lambda_\star > \lambda_0$ such that
$$K(\lambda_\star) = \frac{||u||_{\alpha}^{p}}{\beta}, \tag{3.5}$$
that is,
$$\frac{d}{d\lambda}J(\lambda_\star u) = 0.$$
Notice that
$$\lambda\frac{d}{d\lambda}J(\lambda u) = I(\lambda u) = \lambda^{p}(||u||_{\alpha}^{p} - \beta K(\lambda)).$$
Therefore, assertions (3) and (4) are satisfied.

□

Remark 3.2. Based on Lemma 3.1, we observe that $d > 0$, and the depth of the potential well d is further defined by
$$d = \inf_{u\in\mathcal{N}} J(u). \tag{3.6}$$

Next, for $\delta > 0$, we define
$$I_\delta(u) := \delta||u||_{\alpha}^{p} - \beta\oint_{\partial\Omega}|u|^{q}\log(|u|)dx. \tag{3.7}$$
$$\mathcal{N}_\delta := \{u \in W^{1,p}(\Omega) : I_\delta(u) = 0, \ ||u||_{\alpha} > 0\},$$
$$d(\delta) := \inf_{u\in\mathcal{N}_\delta} J(u) \tag{3.8}$$
$$W_\delta := \{u \in W^{1,p}(\Omega) : \ I_\delta(u) > 0, \ J(u) < d(\delta)\} \cup \{0\},$$
and
$$V_\delta := \{u \in W^{1,p}(\Omega) : \ I_\delta(u) < 0, \ J(u) < d(\delta)\}.$$
Note that, from $(H1)$, there exists $s_0 > 0$ such that $s_0 + q < p_\partial$.

We are now ready to demonstrate the following result:

Lemma 3.3. *Let $u \in W^{1,p}(\Omega)$. The following properties hold:*

1. *If $0 < ||u||_\alpha < r(\delta)$, then $I_\delta(u) > 0$.*
2. *If $I_\delta(u) < 0$, then $||u||_\alpha > r(\delta)$.*
3. *If $I_\delta(u) = 0$, then either $||u||_\alpha = 0$ or $||u||_\alpha \geq r(\delta)$.*

Here, $r(\delta)$ is the unique solution to the equation

$$\phi(r) := \beta \frac{C^{q+s_0}}{s_0 e} r^{q-p+s_0} = \delta,$$

where

$$C := \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{||u||_{q+s_0,\partial}}{||u||_{W^{1,p}(\Omega)}}.$$

Proof. 1. For any $u \in W^{1,p}(\Omega) \setminus \{0\}$, we have

$$\oint_{\partial\Omega} |u|^q \log(|u|)dx \leq \frac{1}{s_0 e}||u||^{q+s_0}_{q+s_0,\partial} \leq \frac{C^{q+s_0}}{s_0 e}||u||^{q+s_0}_\alpha.$$

Then, from $||u||_\alpha < r(\delta)$ and the fact that the function $r \mapsto \phi(r)$ is increasing, we get

$$\begin{aligned} \beta \oint_{\partial\Omega} |u|^q \log(|u|)dx &\leq \beta \frac{C^{q+s_0}}{s_0 e}||u||^{q+s_0}_\alpha \\ &= \beta \frac{C^{q+s_0}}{s_0 e}||u||^{q-p+s_0}_\alpha ||u||^p_\alpha \\ &= \phi(||u||_\alpha)||u||^p_\alpha \\ &< \phi(r(\delta))||u||^p_\alpha \\ &= \delta||u||^p_\alpha. \end{aligned}$$

Thus, $I_\delta(u) > 0$.

2. If $I_\delta(u) < 0$, then $||u||^p_\alpha > 0$ and

$$\delta||u||^p_\alpha < \beta \oint_{\partial\Omega} |u|^q \log(|u|)dx \leq \phi(||u||_\alpha)||u||^p_\alpha,$$

since $\delta = \phi(r(\delta))$ and ϕ is increasing, we get $||u||^p_\alpha > r(\delta)$.

3. If $I_\delta(u) = 0$ and $||u||_\alpha > 0$, then

$$\delta||u||^p_{W^{1,p}(\Omega)} = \beta \oint_{\partial\Omega} |u|^q log(|u|)dx \leq \phi(||u||_\alpha)||u||^p_\alpha,$$

that is, $||u||_\alpha \geq r(\delta)$.
Note that, if $||u||_\alpha = 0$, there holds $I_\delta(u) = 0$.

□

Lemma 3.4. *The continuous function $\delta \longmapsto d(\delta)$ satisfies the following properties:*

1. *For all* $\delta \in (0, \frac{q}{p}]$ *and* $u \in \mathcal{N}_\delta$*, we have* $d(\delta) \geq \theta(\delta) r^p(\delta)$*,*
   where $\theta(\delta) := \frac{1}{p} - \frac{\delta}{q}$*, and* $d(\delta) > 0$ *for every* $\delta \in (0, \frac{q}{p})$*,*
   namely $d \geq d_1 := \left(\frac{1}{p} - \frac{1}{q}\right) r^p(1)$*.*
2. *The function* $\delta \mapsto d(\delta)$ *is increasing on* $(0, 1]$*, decreasing on* $[1, +\infty]$ *and reaches its maximum* $d := d(1)$ *at* $\delta = 1$*.*
3. *There exists* $\delta_0 \geq \frac{q}{p}$ *such that* $d(\delta_0) = 0$ *and* $d(\delta) > 0$ *for each* $\delta \in (0, \delta_0)$*.*

Proof. 1. Let $u \in \mathcal{N}_\delta$, we have

$$\begin{aligned} J(u) &= \frac{1}{p}||u||_\alpha^p - \frac{\beta}{q} \oint_{\partial\Omega} |u|^q log(|u|) + \frac{\beta}{q^2} \|u\|_{q,\partial}^q \\ &\geq \left[\frac{1}{p} - \frac{\delta}{q}\right] ||u||_\alpha^p. \end{aligned}$$

Since $||u||_\alpha > 0$, it follows from Lemma 3.3 that $||u||_\alpha \geq r(\delta)$. The proof of assertion (1) is complet.

2. We prove $d(\delta_1) < d(\delta_2)$ for each $0 < \delta_1 < \delta_2 < 1$ or $1 < \delta_2 < \delta_1 < \infty$. To achieve this, it is enough to show that for any $0 < \delta_1 < \delta_2 < 1$ or $1 < \delta_2 < \delta_1 < \infty$, and for $u \in \mathcal{N}_{\delta_2}$, there exists a function $v \in \mathcal{N}_{\delta_1}$ and a positive constant $\epsilon(\delta_1, \delta_2) > 0$ such that

$$J(u) > J(v) + \epsilon(\delta_1, \delta_2).$$

Indeed, for any $u \in \mathcal{N}_{\delta_2}$, and $\delta > 0$, we can define $\lambda := \lambda(\delta)$ as in equation (3.5) such that:

$$K(\lambda) = \frac{\delta}{\beta}||u||_\alpha^p \quad \text{with} \quad \lambda(\delta_2) = 1,$$

which ensurest $I_\delta(\lambda u) = 0$. By taking $v := \lambda(\delta_1)u$, we observe that $I_{\delta_1}(\lambda(\delta_1)u) = I_{\delta_1}(v) = 0$ and since $||u||_\alpha > 0$, it follows that $v \in \mathcal{N}_{\delta_1}$.
Additionally, we have:

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u) &= \frac{1}{\lambda} I(\lambda u) = \frac{1}{\lambda}\left[(1-\delta)||\lambda u||_\alpha^p + I_\delta(\lambda u)\right] \\ &= (1-\delta)\lambda^{p-1}(\delta)||u||_\alpha^p. \end{aligned} \tag{3.9}$$

Note that, the function $\delta \mapsto \lambda(\delta)$ is increasing on $(0, \infty)$, as shown in the proof of assertion (3).

(a) If $0 < \delta_1 < \delta_2 < 1$, we have $\lambda(\delta_1) < \lambda(\delta_2)$. Furthermore, $u \in \mathcal{N}_{\delta_2}$ and Lemma 3.3 (3) says $||u||_\alpha \geq r(\delta_2)$. Integrating (3.9) over $[\delta_1, \delta_2]$, we get

$$\begin{aligned} J(u) - J(v) &= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} \frac{d}{d\lambda} J(\lambda u) d\lambda \\ &= J(\lambda(\delta_2)u) - J(\lambda(\delta_1 u)) \\ &= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1-\delta)\lambda^{p-1}(\delta)||u||_\alpha^p d\lambda \\ &\geq \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1-\delta_2)\lambda^{p-1}(\delta_1) r^p(\delta_1) d\lambda \\ &= (\lambda(\delta_2) - \lambda(\delta_1)(1-\delta_2)\lambda^{p-1}(\delta_1) r^p(\delta_2) \\ &= \epsilon(\delta_1, \delta_2) > 0. \end{aligned}$$

(b) If $1 < \delta_2 < \delta_1 < +\infty$, using the semme argument in (a) to arrive at

$$J(u) - J(v) \geq (\lambda(\delta_1) - \lambda(\delta_2))(\delta_2 - 1)\lambda^{p-1}(\delta_2) r^p(\delta_2) = \epsilon(\delta_1, \delta_2) > 0.$$

Thus, assertion (2) follows.

3. For $u \in W^{1,p}(\Omega)$, $||u||_\alpha > 0$, and $\delta > 0$, from (3.5), we can define $\lambda := \lambda(\delta)$ such that

$$K(\lambda) = \frac{\delta}{\beta} ||u||_\alpha^p,$$

so that $I_\delta(\lambda u) = 0$. The next step is to prove that the continuous function $\delta \mapsto \lambda(\delta)$ is increasing on $(0, \infty)$. Taking the derivative of $K(\lambda)$ with respect to δ, we obtain $\lambda'(\delta)K'(\lambda(\delta)) = \frac{1}{\beta}||u||_\alpha^p > 0$. If λ is decreasing, we have for $\delta_1 < \delta_2$, $\lambda(\delta_2) < \lambda(\delta_1)$ and $\log(\lambda(\delta_2)) < \log(\lambda(\delta_1))$, then

$$\frac{K(\lambda(\delta_2))}{\lambda^{q-p}(\delta_2)} < \frac{K(\lambda(\delta_1))}{\lambda^{q-p}(\delta_1)},$$

that is

$$\frac{\lambda^{q-p}(\delta_1)}{\lambda^{q-p}(\delta_2)} < \frac{\delta_1}{\delta_2} < 1,$$

this implies that $\delta_2 < \delta_1$, which contradict the fact that $\delta_1 < \delta_2$. According to Lemma 3.1, we have $\delta \mapsto J(\lambda u)$ is increasing on $(0, \lambda_\star]$ and decreasing on $[\lambda_\star, +\infty)$, thus there exists $\delta_0 > 0$ such that $d(\delta_0) = 0$ and from the assertion (1) of this lemma, we deduce that $\delta_0 \geq \frac{q}{p}$.

□

Remark 3.5. In vertue of Lemma 3.4, we infer that $d > d_0 := \lim_{\delta \mapsto 0} d(\delta) \geq 0$ and for any $\mu \in (d_0, d)$, the equation $d(\delta) = \mu$ has two solutions δ_1, δ_2 such that $\delta_1 < 1 < \delta_2 < \delta_0$.

Lemma 3.6. *Assume that $d_0 < J(u) < d$ for some $u \in W^{1,p}(\Omega)$. Let δ_1, δ_2 be two roots of the equation $d(\delta) = J(u)$ with $\delta_1 < 1 < \delta_2$. Then the sign of $I_\delta(u)$ is unchangeable on $\delta \in (\delta_1, \delta_2)$.*

Proof. Arguing by contradiction. Assume that there exists $\delta_0 \in (\delta_1, \delta_2)$ such that $I_{\delta_0}(u) = 0$. From $J(u) > 0$, we have $||u||_\alpha > 0$, then $u \in \mathcal{N}_{\delta_0}$ and $J(u) \geq d(\delta_0)$. This leads to a contradiction with $J(u) = d(\delta_1) = d(\delta_2) < d(\delta_0)$. ☐

Note that by the Definition 4.1, the weak solution u satisfies the following energy equality:

$$\int_0^t ||u_\tau||_2^2 d\tau + J(u) = J(u_0), \quad \forall t \in [0, T). \tag{3.10}$$

Next, by using the potential wells above, we show the invariance of sets W_δ and V_δ in the subcritical case $0 < J(u_0) < d$ and critical case $J(u_0) = d$.

Proposition 3.7. *Let* $u_0 \in W^{1,p}(\Omega)$, *such that* $J(u_0) \in (0, d)$. *Then there exits* $\delta_1 \in (1, \delta_0)$ *such that* $J(u_0) = d(\delta_1)$. *Furthermore, for any weak solution of problem* (1) *with the initial data* u_0 *there hold:*
for each $\delta \in [1, \delta_1)$ *and* $t \in [0, T)$,

1. *if* $I(u_0) > 0$, *then* $u \in W_\delta$,
2. *if* $I(u_0) < 0$, *then* $u \in V_\delta$.

Proof. Since $J(u_0) \in (0, d)$, then from Lemma 3.4, we get the existence of $\delta_1 \in (1, \delta_0)$ such that $J(u_0) = d(\delta_1)$.

1. Let $\delta \in [1, \delta_1)$ and suppose that $I(u_0) > 0$. We have

   $$I_\delta(u_0) = (\delta - 1)||u_0||_\alpha^p + I(u_0),$$

   then $I_\delta(u_0) > 0$. Further, Lemma 3.4 asserts that $J(u_0) = d(\delta_1) < d(\delta)$ since $\delta \mapsto d(\delta)$ is decreasing on $[1, +\infty)$. Thus $u_0 \in W_\delta$, for each $\delta \in [1, \delta_1)$.

   Next, we prove $u \in W_\delta$ for all $\delta \in (1, \delta_1)$ and $t \in (0, T)$. By contradiction, if it is false, from the continuity of I_δ and $d(\delta)$ with respect to t and δ, there exists $t_0 \in (0, T)$ and $\delta_2 \in [1, \delta_1)$ such that $u(t_0) \in \partial W_{\delta_2}$, which means that either $I_{\delta_2}(u(t_0)) = 0, ||u(t_0)||_{W^{1,p}(\Omega)} > 0$ or $J(u(t_0)) = d(\delta_2)$(since 0 is an interior point of W). If $J(u(t_0)) = d(\delta_2)$, it follows from (3.10) that

   $$d(\delta_2) = J(u(t_0)) \leq J(u_0) = d(\delta_1),$$

   which is impossible, thus,

   $$u(t_0) \in \mathcal{N}_{\delta_2} \quad \text{and} \quad J(u(t_0)) \geq d(\delta_2).$$

   Again, from (3.10), we get a contradiction.
2. Let $\delta \in [1, \delta_1)$ and suppose that $I(u_0) < 0$. By contradiction, we prove that $u_0 \in V_\delta$ for all $\delta \in (1, \delta_1)$. From the continuity of I_δ and $d(\delta)$, there exists $\delta_2 \in [1, \delta_1)$ such that $u_0 \in V_\delta$ for all $\delta \in [1, \delta_2)$ and $u_0 \in \partial V_{\delta_2}$, that means that

   $$I_\delta(u_0) < 0, J(u_0) < d(\delta), \ \forall \delta \in [1, \delta_2) \quad \textit{and} \quad I_{\delta_2}(u_0) = 0, \ \textit{or} \ J(u_0) = d(\delta_2).$$

   Since $J(u_0) = d(\delta_1)$, we get $J(u_0) \neq d(\delta_2)$. Thus,

   $$I_{\delta_2}(u_0) = 0 \quad \textit{and} \quad I_\delta(u_0) < 0, \forall \delta \in [1, \delta_2).$$

   In vertue of Lemma 3.3, we get $||u_0||_\alpha > r(\delta)$ for all $\delta \in [1, \delta_2)$. Letting $\delta \to \delta_2$, we obtain $||u_0||_\alpha \geq r(\delta_2) > 0$, thus $u_0 \in \mathcal{N}_{\delta_2}$ and $J(u_0) = d(\delta_1) \geq d(\delta_2)$, which is impossible.

Now, we prove $u \in V_\delta$ for all $\delta \in [1, \delta_1)$ and $t \in (0, T)$. By contradiction, assume that there exists $t_0 \in (0, T)$ and $\delta_2 \in [1, \delta_1)$ such that $u \in V_\delta$ for all $(t, \delta) \in [0, t_0) \times [1, \delta_2)$ and $u(t_0) \in \partial V_{\delta_2}$. Then,

$$I_{\delta_2}(u(t_0)) = 0 \quad or \quad J(u(t_0)) = d(\delta_2).$$

From $u \in V_\delta$ for all $(t, \delta) \in [0, t_0) \times [1, \delta_2)$ and Lemma 3.3 we deduce that $||u||_\alpha > r(\delta)$ for all $(t, \delta) \in [0, t_0) \times [1, \delta_2)$. When t is near t_0, we obtain $\|u(t_0)\|_\alpha \geq r(\delta) > 0$. This leads to $I_{\delta_2}(u(t_0)) \neq 0$. Thus $J(u(t_0)) = d(\delta_2) < d(\delta_1)$, which is impossible . Following the same argument used in the proof of $u_0 \in V_{\delta_2}$, we deduce that $u \in V_\delta$ for each $\delta \in [1, \delta_1)$ and $t \in [0, T)$.

□

In view of Proposition 3.7 and Lemma 3.4, we deduce the following results:.

Corollary 3.8. *Let $u_0 \in W^{1,p}(\Omega)$, $\mu \in [d_0, d)$, and δ_1, δ_2 two roots of the equation $d(\delta) = \mu$ with $\delta_1 < 1 < \delta_2$. suppose that u is a weak solution of (1.1) with $d_0 < J(u_0) \leq \mu$, then for each $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, there hold*

1. *if $I(u_0) > 0$, then $u \in W_\delta$,*
2. *if $I(u_0) < 0$, then $u \in V_\delta$.*

Proposition 3.9. *Let u be a weak solution to the problem (1.1). Suppose that $J(u_0) \leq 0$; then*

$$||u||_\alpha \geq \left(\frac{s_0 eq}{p\beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}}.$$

Moreover, if $||u_0||_\alpha > 0$, then for any $\delta \in (0, \frac{q}{p})$, the weak solution belongs to V_δ.

Proof. Let u be any solution of problem (1.1) with the initial data that satisfies $J(u_0) \leq 0$. From the conservation of energy, we deduce that

$$J(u) \leq 0, \quad \forall t \in [0, T),$$

which implies that

$$\frac{1}{p}||u||_\alpha^p \leq \frac{\beta}{s_0 eq} C^{q+s_0} ||u||_\alpha^{q+s_0}, \quad \forall t \in [0, T). \tag{3.11}$$

(*i*) If $J(u_0) < 0$, we get $||u_0||_\alpha > 0$ and $J(u) < 0$ implies that $||u||_\alpha > 0$. Thus from (3.11) we get

$$||u||_\alpha \geq \left(\frac{s_0 eq}{p\beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}}.$$

(*ii*) If $J(u_0) = 0$, then by (3.11) we must have either $||u_0||_\alpha \geq \left(\frac{s_0 eq}{p\beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}}$ or $||u_0||_\alpha = 0$.

(*a*) If $||u_0||_\alpha = 0$, we infer $||u||_\alpha = 0$ for all $t \in [0, T)$. If it is false, there exists $t_0 \in (0, t)$ such that

$$0 < ||u(t_0)||_\alpha < \left(\frac{s_0 eq}{p\beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}},$$

which leads to a contradiction with (3.11).

(b) If $||u_0||_\alpha \geq \left(\dfrac{s_0 eq}{p\beta C^{q+s_0}}\right)^{\frac{1}{q-p+s_0}}$, by a similar argument in (a) we can prove that $||u||_\alpha \geq \left(\dfrac{s_0 eq}{p\beta C^{q+s_0}}\right)^{\frac{1}{q-p+s_0}}$. for all $t \in [0, t)$.

On the other hand, for any $\delta \in (0, \frac{q}{p})$ and $t \in [0, T)$ we have

$$\theta(\delta)||u||_\alpha^p + \frac{1}{q} I_\delta(u) \leq J(u) \leq J(u_0). \tag{3.12}$$

1. If $J(u_0) < 0$, from (3.12), we get

$$I_\delta(u) < 0, \quad \textit{and} \quad J(u) < d(\delta), \quad \forall \delta \in (0, \frac{q}{p}), t \in [0, T).$$

   Thus, $u \in V_\delta$.

2. If $J(u_0) = 0$, from (ii) we have $||u||_\alpha \geq \left(\dfrac{s_0 eq}{p\beta C^{q+s_0}}\right)^{\frac{1}{q-p+s_0}}$ for all $t \in [0, T)$. Together with (3.12) we also have

$$I_\delta(u) < 0, \quad \textit{and} \quad J(u) < d(\delta), \quad \forall \delta \in (0, \frac{q}{p}), t \in [0, T).$$

Thus, $u \in V_\delta$. □

Proposition 3.10. *Suppose that* $u_0 \in W^{1,p}(\Omega)$, $J(u_0) = d$ *and* u *is a weak solution of the problem (1.1). Then for all* $t \in [0, T)$ *there holds*

1. *if* $I(u_0) > 0$, *we have* $I(u) > 0$,
2. *if* $I(u_0) < 0$, *we have* $I(u) < 0$.

Proof.

1. If it is false, there exists $t_0 \in (0, T)$ such that $I(u(t)) > 0$ for all $t \in (0, t_0)$ and $I(u(t_0)) = 0$. which, together with the definition of d means that $J(u(t_0)) \geq d$. Furthermore, by (3.10) we get

$$J(u(t_0)) = J(u_0) - \int_0^{t_0} ||u_s||_2^2 ds \leq d,$$

   then $J(u(t_0)) = d$ and $\int_0^{t_0} \int_\Omega |u_s|^2 dx ds = 0$, i.e., $u_t = 0$ for all $t \in [0, t_0]$. Which is a contradiction with $I(u_0) > 0$.
2. If the result is false, then there exists $t_0 \in (0, T)$ such that $I(u(t)) < 0$ for all $t \in (0, t_0)$ and $I(u(t_0)) = 0$.
   Similar to the proof above, we also have a contradiction. Thus, Proposition 3.10 follows.

□

4. Main results

In this section, we will present three theorems concerning the global existence, asymptotic behavior, and blow-up of weak solutions to problem (1.1).

We begin by defining the weak solution, maximal existence time, and blow-up of solutions for problem (1.1).

Definition 4.1. *(Weak solution) A function $u(x,t)$ is called a weak solution of* (1.1) *on $\Omega \times [0,T[$, if $u \in L^{\infty}(0,T;W^{1,p}(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ and satisfies* (1.1) *in the distribution sense, that is, $\forall v \in W^{1,p}(\Omega)$, and $t \in [0,T)$,*

$$\int_{\Omega} u_t v dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + \alpha \int_{\Omega} |u|^{p-2} u v dx - \beta \oint_{\partial\Omega} |u|^{q-2} u \log(|u|) v = 0, \quad (4.1)$$

$$u(x,0) = u_0(x) \in W^{1,p}(\Omega). \quad (4.2)$$

Definition 4.2. *(Maximal existence time). Let u(x,t) be a weak solution of* (1.1)*. We define the maximal existence time T of $u(x,t)$ as follows:*

1. *if $u(x,t)$ exists for all $t \geq 0$, then $T = +\infty$,*
2. *if there exists $0 < t_0 < +\infty$ such that $u(x,t)$ exists for $t \in [0,t_0[$, but doesn't exist at t_0, then $T = t_0$.*

Definition 4.3. *(Finite time blow-up). Let $u(x,t)$ be a weak solution of* (1.1)*. We say that u blows-up in finite time, if the maximal existence time T is finite and*

$$\lim_{t \longrightarrow T^-} \int_0^t ||u(.,s)||_2^2 ds = +\infty.$$

4.1. Global existence and polynomial decay

Theorem 4.4. *Under the Hypothesis $(H1)$, if $u_0 \in W^{1,p}(\Omega)$ such that $J(u_0) \leq d$ and $I(u_0) \geq 0$, then the problem (1.1) has a global weak solution $u \in L^{\infty}(0,+\infty;W^{1,p}(\Omega))$ with $u_t \in L^2(0,+\infty;L^2(\Omega))$.*

Theorem 4.5. *Under the hypothesis of Theorem 4.4, if $I(u_0) > 0$ then the weak solution of problem (1.1) satisfies the following estimates :*

1. *if $J(u_0) < d_1$, we have $u \in W$ for all $t \geq 0$ and there exists a constant $K > 0$, such that*

$$||u(.,t)||_2 \leq \Big(||u_0||_2^{2-p} + (p-2)Kt\Big)^{\frac{1}{2-p}}, \ \forall\, t \geq 0. \quad (1.1)$$

2. *if $d_1 < J(u_0) \leq d$, we have $u \in \bar{W}$ for all $t \geq 0$, and for each $\eta \in (0,d)$, there exists $t_\eta > 0$ such that*

$$||u(.,t)||_2 \leq \Big(||u_0||_2^{2-p} + (p-2)K't\Big)^{\frac{1}{2-p}}, \ \forall\, t \geq t_\eta. \quad (1.2)$$

Here $K, K' > 0$ are two constants that will be determined later.

Proof of Theorem 4.4. We split the proof into two cases.

Case 1 : Global existence for ($J(u_0) < d$).

First, we can exclude some special cases as follows:

1. $0 < J(u_0) < d$, $I(u_0) = 0$. This is contradictory with the definition of d.
2. $J(u_0) = 0$, $I(u_0) = 0$. If $||u_0||_\alpha > 0$, it follows that $u_0 \in \mathcal{N}$ and $J(u_0) \geq d$, which is contradictive. Otherwise, $u_0 = 0$, which is a trivial case.
3. $J(u_0) = 0$, $I(u_0) > 0$. This contradict (3.3).
4. $J(u_0) < 0$, $I(u_0) \geq 0$. Similar to case (3).

It remains for us to consider the case $0 < J(u_0) < d$ and $I(u_0) > 0$.

We choose a sequence of eigenfunctions $(\phi_k(x))_{k\geq 1}$ as an orthogonal basis of the Sobolev space $W^{1,p}(\Omega)$ and construct an approximate solution $u^{(m)}(x,t)$ of problem (1.1),

$$u^{(m)}(x,t) = \sum_{k=1}^{m} g_{km}(t)\phi_k(x), \; m \geq 1,$$

which satisfy

$$\begin{aligned}&\textstyle\int_\Omega u_t^{(m)}\phi_j dx + \int_\Omega |\nabla u^{(m)}|^{p-2}\nabla u^{(m)}\nabla\phi_j dx + \alpha\int_\Omega |u^{(m)}|^{p-2}u^{(m)}\phi_j dx \\ &-\beta \textstyle\oint_{\partial\Omega} |u^{(m)}|^{q-2}u^{(m)}\log(|u^{(m)}|)\phi_j dx = 0, \quad j = 1, ..., m,\end{aligned} \tag{4.3}$$

and as $m \to +\infty$,

$$u^{(m)}(x,0) = \sum_{k=1}^{m} a_{km}\phi_k(x) \to u_0(x) \; in \; W^{1,p}(\Omega). \tag{4.4}$$

Multiplying (4.3) by $g'_{jm}(t)$, summing for j and integrating with respect to t from 0 to t, we get

$$\int_0^t \int_\Omega ||u_s^{(m)}||_2^2 ds + J(u^{(m)}) = J(u^{(m)}(0)), \quad \forall t \geq 0. \tag{4.5}$$

By (4.4) we have $J(u^{(m)}(0)) \to J(u_0)$, which together with (4.5) implies that

$$\int_0^t \int_\Omega ||u_s^{(m)}||_2^2 ds + J(u^{(m)}) = J(u^{(m)}(0)) < d, \; \forall t \geq 0, \tag{4.6}$$

for sufficiently large m.

Following an argument similar to that in the proof of Proposition 3.7 together with (4.6) we can prove that $u_m(x,t) \in W$ for any $t \geq 0$ and sufficiently large m. Thus, from (3.1) and (4.6) we deduce that

$$\textstyle\int_0^t ||u_s^{(m)}||_2^2 ds dx < d, \;\; ||u^{(m)}||_\alpha^p < \frac{qpd}{q-p}.$$

On the other hand, from Lemma 2.3, we get

$$\oint_{\partial\Omega} \left| |u^{(m)}|^{q-2}u^{(m)}\log(|u^{(m)}|)\right|^{\frac{q+s_0}{q+s_0-1}} dx < \frac{C^{q+s_0}}{s_0 e}\left(\frac{qpd}{q-p}\right)^{\frac{q+s_0}{p}}.$$

Denote by $\rightharpoonup_\star$ the weakly star convergence. Then the estimates above imply that there exists a subsequence, still denoted by (u_m), such that as $m \to +\infty$,

$u^{(m)} \rightharpoonup_\star u$ in $L^\infty(0,\infty; W^{1,p}(\Omega))$ and a.e. in $\Omega \times [0,+\infty)$,

$u_t^{(m)} \rightharpoonup u_t,$ in $L^2(0,\infty; L^2(\Omega))$,

$|u^{(m)}|^{q-2}u^{(m)}\log(|u^{(m)}|) \rightharpoonup_\star |u|^{q-2}u\log(|u|)$ in $L^\infty(0,\infty; L^{\frac{q+s_0}{q+s_0-1}}(\partial\Omega))$ and a.e. in $\Omega \times [0,+\infty)$.

Hence in (4.3), for j fixed, letting $m \to +\infty$, there holds

$$\int_\Omega u_t\phi_j dx+\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi_j dx+\alpha\int_\Omega |u|^{p-2}u\phi_j dx = \beta\oint_{\partial\Omega}|u|^{q-2}u\log(|u|)\phi_j dx,\ \forall j\geq 1.$$

Furthermore,

$$\int_\Omega u_t v dx+\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx+\alpha\int_\Omega |u|^{p-2}uv dx = \beta\oint_{\partial\Omega}|u|^{q-2}u\log(|u|)v dx,\ \forall v\in H^1_{X,0}(\Omega).$$

Meanwhile, from (4.4) we obtain $u(x,0) = u_0(x)$ in $W^{1,p}(\Omega)$. It follows from Proposition 3.7 that $u \in W$ for all $t \geq 0$. Therefore, u is a global weak solution to the problem (1.1).

Case 2 : Global existence for ($J(u_0) = d$).

Set $u_0^{(m)} = \mu_m u_0$, where $\mu_m = 1 - \frac{1}{m}$, $m \geq 2$.
We consider the initial boundary value problem

$$\begin{cases} u_t - \Delta_p u + \alpha|u|^{p-2}u = 0, \quad \in \Omega\times]0,+\infty[,\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial\nu} = \beta|u|^{q-2}u\log(|u|), \quad (x,t)\in\partial\Omega\times[0,+\infty[,\\ u(x,0) = u_0^{(m)}(x), \quad x\in\Omega \end{cases} \tag{4.7}$$

From Lemma 3.1 and the fact $I(u_0) \geq 0$, we have

$$\lambda_\star(u_0) = \lambda_\star \geq 1.$$

Together with $\mu_m < 1$, it follows that

$$I(u_0^{(m)}) > 0,$$
$$J(u_0^{(m)}) < J(u_0) = d.$$

Thus, it follows from the results of **Case 1** that, for any m, the problem (4.7) admits a global weak solution $u^{(m)} \in L^\infty(0,\infty; W^{1,p}(\Omega))$ with $u_t^{(m)} \in L^2(0,\infty; L^2(\Omega))$ and $u^{(m)} \in W$, $\forall t \geq 0$, and satisfies

$$\int_\Omega u_t^{(m)}v dx + \int_\Omega |\nabla u^{(m)}|^{p-2}\nabla u^{(m)}\nabla v dx + \alpha\int_\Omega |u^{(m)}|^{p-2}u^{(m)}v dx$$
$$= \beta\oint_{\partial\Omega}|u^{(m)}|^{q-2}u^{(m)}\log(|u^{(m)}|)v dx,$$

for any $v \in W^{1,p}(\Omega),\ t > 0$, and

$$\int_0^t \int_\Omega ||u_s^{(m)}||_2^2 ds + J(u^{(m)}) = J(u^{(m)}(0)) < J(u_0) = d, \ \forall t \geq 0.$$

The proof left similar to that in the **Case 1**. The proof of Theorem 4.4 is complet. □

Proof of Theorem 4.5. (1) If $J(u_0) < d_1$ and $I(u_0) > 0$ we see that $J(u_0) > 0$, then from Proposition 3.7, we have $u \in W_\delta$ for each $\delta \in [1, \delta_1)$, which implies that $I(u) > 0, \forall t \geq 0$.

Note that by (3.10), one can has $J(u) \leq J(u_0)$, together with (3.3) we deduce that

$$||u||_\alpha < \left[\left(\frac{1}{p} - \frac{1}{q}\right)^{-1} J(u_0)\right]^{\frac{1}{p}}.$$

On the other hand, we have

$$\begin{aligned} I(u) &= ||u||_\alpha^p - \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx \\ &\geq ||u||_\alpha^p - \beta \frac{C^{q+s_0}}{s_0 e} ||u||_\alpha^{q+s_0} \\ &= \left(1 - \beta \frac{C^{q+s_0}}{s_0 e} ||u||_\alpha^{q-p+s_0}\right) ||u||_\alpha^p \\ &\geq \left(1 - \beta \frac{C^{q+s_0}}{s_0 e} \left[\left(\frac{1}{p} - \frac{1}{q}\right)^{-1} J(u_0)\right]^{\dfrac{q-p+s_0}{p}}\right) ||u||_\alpha^p \\ &= \sigma ||u||_\alpha^p, \end{aligned}$$

where

$$\sigma = \left(1 - \beta \frac{C^{q+s_0}}{s_0 e} \left[\left(\frac{1}{p} - \frac{1}{q}\right)^{-1} J(u_0)\right]^{\dfrac{q-p+s_0}{p}}\right) > 0.$$

Furthermore, from Definition 4.1, we get

$$I(u) = -\frac{1}{2} \frac{d}{dt} ||u||_2^2, \quad \forall t \geq 0.$$

Thus, we obtain

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} ||u||_2^2 &\geq \sigma ||u||_\alpha^p \\ &\geq \sigma C_1^p ||u||_2^p, \end{aligned}$$

where $C_1 > 0$, is the optimal constant of the embdding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

Integrating the last inequality from 0 to t, we get the decay polynomially (1.1).

(2) In this case, we see from Proposition 3.10 that

$$\int_\Omega u_t u dx = -I(u) < 0.$$

By Hôlder's inequality, we have

$$0 < -\int_\Omega u_t u dx \leq ||u_t||_2 ||u||_2,$$

then $||u_t||_2 > 0$ and $t \mapsto \int_0^t ||u_s||_2^2 ds$ is increasing on $[0, \infty)$. According that to (3.10), we deduce that $0 < \int_0^t ||u_s||_2^2 ds < d$ and for any $\eta \in (0, d)$, there exists $T_\eta > 0$ such that

$$\int_0^{T_\eta} ||u_s|| ds = \eta,$$

i.e.,

$$d - \eta = J(u(T_\eta)).$$

Taking T_η as the initial time, similar to (1), we obtain the decay estimate (1.2). This complet the proof of Theorem 4.5.

□

4.2. Blow-up in finite time

Theorem 4.6. *Under the hypothesis* $(H1)$, *if* $u_0 \in W^{1,p}(\Omega)$ *such that* $J(u_0) \leq d$ *and* $I(u_0) < 0$. *Then, for* $q > 2$*(resp.* $q \leq 2$*) the weak solution* u *of problem (1.1) blows up in finite time(resp. at* ∞*) in the sense of Definition 4.3.*

Proof of Theorem 4.6. We divide the proof into two cases.

Case 1 : Blow-up for $J(u_0) < d$**.**

Let u be a weak solution of problem (1.1) with $J(u_0) < d$ and $I(u_0) < 0$, then $||u_0||_\alpha > 0$. Arguing by contradiction, suppose that u is global(i.e., $T = \infty$).

For $t \geq 0$, define

$$\psi(t) = \int_0^t ||u(s)||_2^2 ds + \|u_0\|_2^2.$$

Then

$$\dot{\psi}(t) = ||u(t)||_2^2,$$
$$\ddot{\psi}(t) = \int_\Omega 2uu_t dx = -2I(u),$$

where $\dot{\phi}(t) := \dfrac{d}{dt}\phi(t)$.

It follows from (3.1), (3.10), Lemma 3.3 (2) and the embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ that

$$\begin{aligned}
\ddot{\psi}(t) &= -2qJ(u) + 2\left(\frac{q}{p} - 1\right)||u||_\alpha^p + \frac{2\beta}{q}||u||_{q,\partial}^q \\
&\geq 2q\int_0^t ||u_s||_2^2 ds - 2qJ(u_0) + 2\left(\frac{q}{p} - 1\right)C_1^2||u||_2^2 r^{p-2}(1) \\
&= 2q\int_0^t ||u_s||_2^2 ds - 2qJ(u_0) + 2\left(\frac{q}{p} - 1\right)C_1^2 r^{p-2}(1)\dot{\psi}(t).
\end{aligned}$$

Note that

$$\left(\int_0^t\int_\Omega uu_t dxds\right)^2 = \frac{1}{4}\left(\int_0^t \frac{d}{ds}||u(s)||_2^2 ds\right)^2 = \frac{1}{4}\left([\dot{\psi}(t)]^2 - 2||u_0||_2^2\dot{\psi}(t) + ||u_0||_2^4\right),$$

then

$$\begin{aligned}[\dot{\psi}(t)]^2 &= 4\left(\int_0^t\int_\Omega uu_t dxds\right)^2 + 2||u_0||_2^2\dot{\psi}(t) - ||u_0||_2^4\\ &\leq 4\left(\int_0^t\int_\Omega uu_t dxds\right)^2 + 2||u_0||_2^2\dot{\psi}(t).\end{aligned}$$

Thus

$$\begin{aligned}\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 \geq 2q&\left[\left(\int_0^t ||u||_2^2 dxds\right)\left(\int_0^t ||u_t||_2^2 dxds\right) - \left(\int_0^t\int_\Omega uu_t dxds\right)^2\right]\\ &- 2qJ(u_0)\psi(t) + 2\left(\frac{q}{p} - 1\right)C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q||u_0||_2^2\dot{\psi}(t).\end{aligned}$$

Using Hölder's inequality, we obtain

$$\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 \geq -2qJ(u_0)\psi(t) + 2\left(\frac{q}{p} - 1\right)C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q||u_0||_2^2\dot{\psi}(t). \tag{4.8}$$

1. If $J(u_0) \leq 0$, we see from Proposition 3.9 that $I(u) < 0$. Then, by (4.8) we get

$$\begin{aligned}\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 &\geq 2\left(\frac{q}{p} - 1\right)C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q||u_0||_2^2\dot{\psi}(t)\\ &= \left(\frac{q}{p} - 1\right)C_1^2 r^{p-2}(1)\left[\psi(t) - \frac{qp}{(q-p)C_1^p r^{p-2}(1)}||u_0||_2^2\right]\dot{\psi}(t).\end{aligned}$$

Moreover, $\ddot{\psi}(t) > 0$, since $I(u) < 0$. Hence,

$$\psi(t) \geq t\dot{\psi}(t) + \psi(0), \qquad t \geq 0,$$

i.e.,

$$\psi(t) \geq t||u_0||_2^2, \qquad t \geq 0.$$

For $t > \frac{qp}{(q-p)C_1^2 r^{p-2}(1)}$, we have $\psi(t) - \frac{qp}{(q-p)C_1^2 r^{p-2}(1)}||u_0||_2^2 > 0$, that is,

$$\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 > 0, \quad \forall t > \frac{qp}{(q-p)C_1^2 r^{p-2}(1)}.$$

2. If $0 < J(u_0) < d$, by (4.8) we write

$$\begin{aligned}\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 &\geq -2qJ(u_0)\psi(t) + 2\left(\frac{q}{p} - 1\right)C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q||u_0||_2^2\dot{\psi}(t)\\ &= \frac{q-p}{2p}C_1^2 r^{p-2}(1)\left[\dot{\psi}(t) - \frac{4pq}{(q-p)C_1^2 r^{p-2}(1)}J(u_0)\right]\psi(t)\\ &\quad + \frac{q-p}{2p}C_1^2 r^{p-2}(1)\left[\psi(t) - \frac{2pq}{(q-p)C_1^2 r(1)^{p-2}}||u_0||_2^2\right]\dot{\psi}(t).\end{aligned}$$

It follows from Proposition 3.7 that $I_{\delta_2}(u) < 0$, where $\delta_2 \in [1, \delta_1)$ and Lemma 3.3 implies that $||u||_\alpha^p > r^p(\delta_2)$. Now, By

$$I(u) = (1 - \delta_2)||u||_\alpha^p + I_{\delta_2}(u)$$

we get that

$$\ddot{\psi}(t) > 2(\delta_2 - 1)r^p(\delta_2), \quad \forall t \geq 0. \tag{4.9}$$

Integrating (4.9) over $[0, t]$, we get

$$\dot{\psi}(t) > 2(\delta_2 - 1)r^p(\delta_2)t.$$

and

$$\psi(t) > (\delta_2 - 1)r^p(\delta_2)t^2, \forall t \geq 0.$$

Therefore, for sufficiently large t, we can obtain

$$\begin{aligned} \dot{\psi}(t) - \frac{4pq}{(q-p)C_1^2 r^{p-2}(1)} J(u_0) > 0 \\ \psi(t) - \frac{2pq}{(q-p)C_1^2 r^{p-2}(1)} ||u_0||_2^2 > 0. \end{aligned}$$

Thus,

$$\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 > 0.$$

Finally, applying Lemma 2.2 for $q > 2$, we deduce that the function ψ goes to ∞ in finite time, This contradicts the assumption $T = \infty$. In other words, the weak solution blows up in finite time in the sense of Definition 4.3.

For $q \leq 2$ and $\|u_0\|_2 > 0$, we have

$$\frac{\ddot{\psi}(t)}{\dot{\psi}(t)} > \frac{q}{2} \frac{\dot{\psi}(t)}{\psi(t)}, \quad t \geq 0.$$

Solving the differential inequality above we obtain for $t \geq 0$

$$\psi(t) \geq e^{ct + \log(\psi(0))},$$

where $c := \frac{\dot{\psi}(0)}{\psi(0)^{\frac{q}{2}}}$. This shown that u blow up at ∞.

Case 2 : Blow-up for $J(u_0) = d$.
Let u be a weak solution to problem (1.1) with $J(u_0) = d$ and $I(u_0) < 0$. By Proposition 3.10 we have $I(u) < 0$ for each $t \in [0, T)$, this combined with $-I(u) = \int_\Omega u_t u dx$ and using Hölder's inequality, we get $||u_t||_2 > 0$ and $t \mapsto \int_0^t ||u_t||_2^2 ds$ is increasing for $t \geq 0$.
In the energy equality (3.10), choosing $t_0 > 0$ so that

$$0 < J(u(t_0)) = d - \int_0^{t_0} ||u_t||_2^2 ds < d,$$

and taking t_0 as the initial time, we can prove that the weak solution u blows up in finite time, by using a similar argument in the **case 1**. The proof of Theorem 4.6 is complete. □

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