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Sharp inverse logarithmic coefficient bounds for starlike functions associated with cosine function

Isha Zahid, Umar Raza and Mohsan Raza

Abstract. Let $\mathcal{S}^*_{cos}$ be the subclass of starlike functions f associated with cosine function defined by $(zf'(z)/f(z)) \prec \cos(z)$. In this paper, we obtain the sharp coefficient bounds and Hankel determinants of second order for the inverse logarithmic function for this class. We also present the best possible bounds of second order Toeplitz determinant for the functions in the same class.

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1. Introduction and preliminaries

Let f be an analytic function having Maclaurin series expansion in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Then f is known to be member of the class $\mathcal{A}$. The family of all analytic functions adhere to the conditions $f(0) = 0$ and $f'(0) = 1$ in $\mathbb{D}$. Denote by $\mathcal{S}$, a subclass of $\mathcal{A}$ which contains all univalent (one-one) functions in $\mathbb{D}$.

The family of analytic functions $p(z)$ in $\mathbb{D}$ that meet the criteria, $Re(p(z)) > 0$ is known as Caratheodory class $\mathcal{P}$ and it's power series expansion is given as

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

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A domain $D \subseteq \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in \mathbb{C}$ if and only if for every $z \in D$, the line segment joining z and z_0 lies entirely in D. An analytic function f is called starlike if $f(\mathbb{D})$ with respect to the origin is a starlike domain and the class of all univalent starlike functions is denoted by $\mathcal{S}^*$. This subclass is defined analytically as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}, \quad z \in \mathbb{D}.$$

Suppose that $\mathcal{B}$ denote a class of analytic functions w in $\mathbb{D}$ which satisfy $w(0) = 0$ and $|w(z)| < 1$. The functions in class $\mathcal{B}$ are known as Schwarz functions. Now, assume two analytic functions g_1 and g_2 in $\mathbb{D}$, the function g_1 is subordinate to g_2 (i.e. $g_1 \prec g_2$), if there exists a function $w(z) \in \mathcal{B}$ such that $g_1(z) = g_2(w(z))$, for $z \in \mathbb{D}$. Furthermore, if g_2 is univalent and $g_1(0) = g_2(0)$, then $g_1 \prec g_2$ is equivalent to the condition $g_1(\mathbb{D}) \subset g_2(\mathbb{D})$. The class $\mathcal{S}^*_{cos}$ defined by

$$\mathcal{S}^*_{cos} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec cos(z) \right\},$$

was thoroughly investigated in [1] by Ali et al. by presenting the sharp coefficient bounds and Hankel determinants of second and third order. They also obtained the same results for logarithmic coefficients but investigating these results for inverse logarithmic function is an open problem.
The inverse f^{-1} of a univalent function f defined on some disk $|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}$ possesses a series expansion given as

$$f^{-1}(z) = z + \sum_{n=2}^{\infty} A_n z^n,$$

For a function $f \in \mathcal{S}$, the logarithmic coefficients γ_n are defined as

$$F_f(z) = log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n.$$

The series expansion of inverse logarithmic coefficients Γ_n is given by

$$F_{f^{-1}}(z) = log\left(\frac{f^{-1}(z)}{z}\right) = 2\sum_{n=1}^{\infty} \Gamma_n z^n. \tag{1.3}$$

By using (1.1) and (1.3), we have

$$\Gamma_1 = -\frac{1}{2}a_2, \quad \Gamma_2 = \frac{3}{4}a_2^2 - \frac{1}{2}a_3, \tag{1.4}$$

$$\Gamma_3 = -\frac{5}{3}a_2^3 + 2a_2a_3 - \frac{1}{2}a_4, \tag{1.5}$$

$$\Gamma_4 = \frac{35}{8}a_2^4 - \frac{15}{2}a_2^2a_3 + \frac{5}{2}a_2a_4 + \frac{5}{4}a_3^2 - \frac{1}{2}a_5. \tag{1.6}$$

In [10], the concept of Hankel determinant was introduced by Pommerenke as a technique for investigating analytic functions and their subclasses in geometric function theory. It is defined as

$$H_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix}$$

where $m, n \geq 1$. We see that we can obtain different Hankel determinants by changing the values of m and n. We note that

$$H_{2,1}(f) = a_1 a_3 - a_2^2, \qquad H_{2,2}(f) = a_2 a_4 - a_3^2. \tag{1.7}$$

$H_{2,1}(f)$ demonstrates the Fekete-Szegö functional. Riaz et al. [12] investigated the bounds of Hankel determinants for starlike and convex functions associated with sigmoid functions. In [9], M. Obradović and N. Tunesk calculated the upper bounds of second and third orders Hankel determinants for the class $\mathcal{S}$ of univalent functions. Allu et al. [3], studied the upper bound of second order Hankel determinant for logarithmic coefficients of univalent functions and In 2020, Arif et al. [4], investigated the fifth Hankel determinant for the classd of bounded turning function. Similarly, many other authors studied Hankel determinant for different subclasses of analytic functions (see [1, 11]).
The symmetric Toeplitz determinant $T_m(n)(f)$ was introduced by Ali et al. [2] is defined as

$$T_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix}$$

$$T_2(1)(f) = a_1^2 - a_2^2, \qquad T_2(2)(f) = a_2^2 - a_3^2. \tag{1.8}$$

In [2], Ali et al. examined the bounds of $|T_2(n)(f)|, |T_3(1)(f)|$ and $|T_3(2)(f)|$ within the classes $\mathcal{S}^*$ and $\mathcal{S}^c$. Cudna et al. [6] calculated sharp upper and lower estimates for $|T_2(1)(f)|$ and $|T_3(1)(f)|$ concerning the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$, where $0 \leq \alpha < 1$. Toeplitz determinant for initial values of m and n are estimated for starlike and close-to-convex functions in [2].
In this paper, we estimate the bounds of inverse logarithmic coefficients $|\Gamma_2|, |\Gamma_3|$ and $|\Gamma_4|$ for the class $\mathcal{S}^*_{cos}$. We also present the upper bounds for Hankel determinants $|H_{2,1}(f^{-1}/2)|$ and $|H_{2,2}(f^{-1}/2)|$ as well as symmetric Toeplitz determinant $|T_2(1)(f^{-1}/2)|$ and $|T_2(2)(f^{-1}/2)|$ for the same class.
The following lemmas will be used to obtain our main results. The first lemma is the well known Carathéodory lemma.

Lemma 1.1. [7] *Let $p \in \mathcal{P}$ be given in the form* (1.2). *Then*

$$|c_n| \leq 2, \ n \geq 1.$$

The inequality holds for all $n \geq 1$ if and only if $p(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| = 1$.

Lemma 1.2. [5] *Let* $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ *be the closed unit disk, and* $p \in \mathcal{P}$ *be given by* (1.2). *Then,*

$$\begin{aligned} c_1 =& 2\zeta_1, \\ c_2 =& 2\zeta_1^2 + 2\big(1 - |\zeta_1|^2\big)\,\zeta_2, \\ c_3 =& 2\zeta_1^3 + 4\big(1 - |\zeta_1|^2\big)\,\zeta_1\zeta_2 - 2\big(1 - |\zeta_1|^2\big)\,\overline{\zeta_1}\zeta_2^2 + 2\big(1 - |\zeta_1|^2\big)\big(1 - |\zeta_2|^2\big)\,\zeta_3, \\ c_4 =& 2\zeta_1^4 + 2\big(1 - |\zeta_1|^2\big)\big(3\zeta_1^2 + \overline{\zeta_1}^2\zeta_2^2 - 3|\zeta_1|^2\zeta_2 + \zeta_2\big)\zeta_2 \\ &+ 2\big(1 - |\zeta_1|^2\big)\big(1 - |\zeta_2|^2\big)\big(2\zeta_1 - 2\overline{\zeta_1}\zeta_2 - \overline{\zeta_2}\zeta_3\big)\zeta_3 \\ &+ 2\big(1 - |\zeta_1|^2\big)\big(1 - |\zeta_2|^2\big)\big(1 - |\zeta_3|^2\big)\zeta_4, \end{aligned}$$

for some $\zeta_i \in \overline{\mathbb{D}}$ *and* $i \in \{1, 2, 3, 4\}$.

The following Lemma presents the initial part of the result from [8, Remark p. 162]

Lemma 1.3. [8] *Let* $p \in \mathcal{P}$ *be given in the form* (1.2). *Then*

$$\left|c_2 - vc_1^2\right| \leq 2 - v\,|c_1|^2\,,\ 0 < v \leq \frac{1}{2}.$$

2. Main results

Let $f \in \mathcal{S}^*_{\cos}$, so by using the relations (1.1) and (1.2), we get

$$a_2 = 0, \quad a_3 = -\frac{1}{16}c_1^2, \quad a_4 = \frac{1}{24}c_1^3 - \frac{1}{12}c_1c_2, \tag{2.1}$$

$$a_5 = -\frac{1}{48}c_1^4 - \frac{1}{32}c_2^2 + \frac{3}{32}c_1^2c_2 - \frac{1}{16}c_1c_3. \tag{2.2}$$

In our first theorem, we calculate the sharp bounds of inverse logarithmic coefficients for functions in the class $\mathcal{S}^*_{cos}$.

Theorem 2.1. *Consider* $f \in \mathcal{S}^*_{cos}$ *is of the form* (1.1). *Then*

$$|\Gamma_1| = 0, \quad |\Gamma_2| \leq \frac{1}{8}, \quad |\Gamma_3| \leq \frac{\sqrt{3}}{27}, \quad |\Gamma_4| \leq \frac{1}{16}.$$

These inequalities are sharp.

Proof. Using values of (2.1) and (2.2) in (2.3)-(2.5), we deduce that

$$\Gamma_1 = 0, \qquad \Gamma_2 = \frac{1}{32}c_1^2, \tag{2.3}$$

$$\Gamma_3 = \frac{1}{24}c_1c_2 - \frac{1}{48}c_1^3, \tag{2.4}$$

$$\Gamma_4 = \frac{47}{3072}c_1^4 + \frac{1}{64}c_2^2 - \frac{3}{64}c_1^2c_2 + \frac{1}{32}c_1c_3. \tag{2.5}$$

$\mathbf{A_1}$: From the second equality of (2.3), it follows that

$$|\Gamma_2| \leq \frac{1}{32}|c_1|^2.$$

Denote $c := |c_1|$, the above inequality becomes

$$|\Gamma_2| \leq \frac{1}{32}c^2 := F(c).$$

We can see that $F(c)$ attains maximum value at $c = 2$, then

$$|\Gamma_2| \leq \frac{1}{8}.$$

This inequality is sharp for the function f_1 given by

$$f_1(z) := z\exp\left(\int_0^z \frac{\cos(x) - 1}{x}dx\right) = z - \frac{1}{4}z^3 + \frac{1}{24}z^5 + \dots. \tag{2.6}$$

Here, $a_2 = 0$ and $a_3 = -\frac{1}{4}$. Thus from (1.4), we have $|\Gamma_2| = \frac{1}{8}$.

$\mathbf{A_2}$: Now, (2.4) can be expressed as

$$\Gamma_3 = \frac{1}{24}c_1\left(c_2 - \frac{c_1^2}{2}\right),$$

by applying Lemma 1.3, we obtain

$$|\Gamma_3| \leq \frac{1}{12}|c_1| - \frac{1}{48}|c_1|^3.$$

Set $c := |c_1|$, for $c \in [0, 2]$, we have

$$|\Gamma_3| \leq \frac{1}{12}c - \frac{1}{48}c^3 =: L(c).$$

We can see that $L'(c)$ has zero $c_0 = \dfrac{2\sqrt{3}}{3}$ that lies in $(0, 2)$, satisfying $L''(c_0) < 0$, thus

$$|\Gamma_3| \leq L(c_0) = \frac{\sqrt{3}}{27}.$$

To demonstrate sharpness, consider $t_1 = \frac{2}{3}\sqrt{3}$ and $p_1(z) = \frac{1+2t_1z+z^2}{1-z^2}$ and

$$w_1(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{(\sqrt{3} + 3z)z}{z\sqrt{3} + 3}.$$

Clearly, we can see that $w_1(0) = 0$ and $|w_1(z)| < 1$. Thus, the function

$$f_2(z) = z\exp\left(\int_0^z \frac{\cos(w_1(x)) - 1}{x}dx\right) = z - \frac{1}{12}z^3 - \frac{2}{27}\sqrt{3}\,z^4 + \dots \tag{2.7}$$

belongs to the class $\mathcal{S}^*_{cos}$ with $a_2 = 0, a_3 = -\frac{1}{12}$ and $a_4 = -\frac{2}{27}\sqrt{3}$. Therefore from (1.5), we have $|\Gamma_3| = \frac{\sqrt{3}}{27}$.

$\mathbf{A_3}$: In the end, we will find the bound of $|\Gamma_4|$ by solving (2.5), we get

$$\Gamma_4 = \frac{47}{3072}c_1^4 + \frac{1}{64}c_2^2 - \frac{3}{64}c_2c_1^2 + \frac{1}{32}c_1c_3,$$

and by utilizing Lemma 1.2, we have

$$\Gamma_4 = \frac{11}{192}\zeta_1^4 + \frac{1}{16}\left(1 - |\zeta_1|^2\right)^2\zeta_2^2 - \frac{1}{8}\left(1 - |\zeta_1|^2\right)\zeta_2^2|\zeta_1|^2 + \frac{1}{8}\left(1 - |\zeta_1|^2\right)\left(1 - |\zeta_2|^2\right)\zeta_1\zeta_3.$$

Applying the triangle inequality and since $\zeta_i \in \mathbb{D}$, which implies that $|\zeta_3| \leq 1$, we obtain the following inequality

$$\begin{aligned}|\Gamma_4| \leq & \frac{11}{192}|\zeta_1|^4 + \frac{1}{16}\left(1-|\zeta_1|^2\right)^2|\zeta_2|^2 + \frac{1}{8}\left(1-|\zeta_1|^2\right)|\zeta_2|^2|\zeta_1|^2 \\ & + \frac{1}{8}\left(1-|\zeta_1|^2\right)\left(1-|\zeta_2|^2\right)|\zeta_1|.\end{aligned}$$

Indicating $x := |\zeta_1|$ and $y := |\zeta_2|$, where $x, y \in (0,1)$, we obtain

$$|\Gamma_4| \leq \frac{11}{192}x^4 + \frac{1}{16}(1-x^2)^2y^2 + \frac{1}{8}(1-x^2)x^2y^2 + \frac{1}{8}(1-x^2)(1-y^2)x := Q(x,y). \tag{2.8}$$

Denoting $\Theta = [0,1] \times [0,1]$ is the closed unit square, now our aim is to find the maximum value of $Q(x,y)$ in Θ.

I. Firstly, we will evaluate the highest value of $Q(x,y)$ on the four edges of Θ.
(i) On edge $x = 0$, $Q(x,y)$ deduce to

$$q_1(y) := Q(0,y) = \frac{y^2}{16} \leq \frac{1}{16}, \quad \text{for} \quad y \in (0,1).$$

(ii) At edge $x = 1$, we get

$$Q(1,y) = \frac{11}{192}.$$

(iii) At the edge $y = 0$, $Q(x,y)$ becomes

$$q_2(x) := Q(x,0) = \frac{11}{192}x^4 + \frac{x}{8}(1-x^2), \quad x \in (0,1).$$

It is easy to check that the zero of q_2' is $x_0 = 0.8148 \in (0,1)$, satisfying $q_2''(x_0) < 0$. Thus

$$Q(x,0) \leq Q(x_0,0) = 0.0595.$$

(iv) On edge $y = 1$, we obtain

$$q_3(x) := Q(x,1) = \frac{47}{192}x^4 + \frac{1}{16} - \frac{1}{4}x^2, \quad x \in (0,1).$$

We can see that $q_3' = 0$ yields $x = \frac{2\sqrt{282}}{47}$ but $q_3''(\frac{2\sqrt{282}}{47}) = 1 > 0$. Therefore, q_3 has no maximum value in$(0,1)$.
II. Now, we will maximize $Q(x,y)$ on the vertices of $\Theta := [0,1] \times [0,1]$.

$$Q(0,0) = 0, \quad Q(0,1) = \frac{1}{16}, \quad Q(1,0) = Q(1,1) = \frac{11}{192}.$$

We conclude that

$$max\{Q(x,y) : (x,y) \in \Theta\} = Q(0,y) = Q(0,1) = \frac{1}{16}.$$

Hence, (2.8) becomes

$$|\Gamma_4| \leq \frac{1}{16}.$$

This inequality is sharp for the function f_3 given by

$$f_3(z) = z\exp\left(\int_0^z \frac{\cos(x^2)-1}{x}dx\right) = z - \frac{1}{8}z^5 + \dots.$$

Here, $a_2 = a_3 = a_4 = 0$ and $a_5 = -\frac{1}{8}$. Thus, from (1.6), we have $|\Gamma_4| = \frac{1}{16}$. □

Next, we calculate the sharp upper bound on $|H_{2,1}(f^{-1}/2)|$.

Theorem 2.2. *Consider $f \in \mathcal{S}^*_{cos}$ is given by the form* (1.1). *Then*

$$|H_{2,1}(f^{-1}/2)| \leq \frac{1}{64}.$$

This inequality is sharp.

Proof. From (2.3), (2.4) and (1.7), we get

$$H_{2,1}(f^{-1}/2) = -\frac{1}{1024}c_1^4$$

It implies

$$|H_{2,1}(f^{-1}/2)| \leq \frac{1}{1024}|c_1|^4.$$

Since $|c_1| \leq 2$, thus

$$|H_{2,1}(f^{-1}/2)| \leq \frac{1}{64}.$$

This inequality is sharp for the function f_1 given by (2.6). □

In next theorem, we find the sharp bound for $|H_{2,2}(f^{-1}/2)|$.

Theorem 2.3. *Consider $f \in \mathcal{S}^*_{cos}$ is of the form* (1.1). *Then*

$$|H_{2,2}(f^{-1}/2)| \leq \frac{11}{1536}.$$

The inequality is sharp.

Proof. By utilizing (1.7), (2.3), (2.4) and (2.5), we obtain

$$H_{2,2}(f^{-1}/2) = \frac{13}{294912}c_1^6 - \frac{23}{18432}c_1^2c_2^2 + \frac{5}{18432}c_1^4c_2 + \frac{1}{1024}c_1^3c_3,$$

by applying Lemma (1.2), the above equality becomes

$$\begin{aligned} H_{2,2}(f^{-1}/2) =& \frac{11}{1536}\zeta_1^6 - \frac{23}{1152}\left(1-|\zeta_1|^2\right)^2\zeta_1^2\zeta_2^2 - \frac{1}{64}(1-|\zeta_1|^2)\zeta_1^2\zeta_2^2|\zeta_1|^2 \\ &+ \frac{1}{64}\left(1-|\zeta_1|^2\right)\left(1-|\zeta_2|^2\right)\zeta_1^3\zeta_3. \end{aligned}$$

Using triangular inequality and $|\zeta_3| \leq 1$ because $\zeta_i \in \mathbb{D}$, we have

$$\begin{aligned} |H_{2,2}(f^{-1}/2)| \leq& \frac{11}{1536}|\zeta_1|^6 + \frac{23}{1152}\left(1-|\zeta_1|^2\right)^2|\zeta_1|^2|\zeta_2|^2 + \frac{1}{64}\left(1-|\zeta_1|^2\right)|\zeta_1|^4|\zeta_2|^2 \\ &+ \frac{1}{64}\left(1-|\zeta_1|^2\right)\left(1-|\zeta_2|^2\right)|\zeta_1|^3. \end{aligned}$$

Indicating $x := |\zeta_1|$ and $y := |\zeta_2|$, where $x, y \in [0,1]$, we obtain

$$|H_{2,2}(f^{-1}/2)| \leq \frac{11}{1536}x^6 + \frac{23}{1152}(1-x^2)^2x^2y^2 + \frac{1}{64}(1-x^2)x^4y^2 \qquad (2.9)$$
$$+ \frac{1}{64}(1-x^2)(1-y^2)x^3 = L(x,y).$$

Denoting $\Theta := [0,1] \times [0,1]$ be the closed unit square, we now aim to maximize $L(x,y)$ on the edges and vertices of Θ.

I. Firstly, we will evaluate the highest value of $L(x,y)$ on the four edges of Θ.
(i) On the edge $x = 0$, we get

$$L(0,y) = 0.$$

(ii) At edge $x = 1$, we get

$$L(1,y) = \frac{11}{1536}.$$

(iii) At the edge $y = 0$, $L(x,y)$ becomes

$$r_1(x) := L(x,0) = \frac{11}{1536}x^6 - \frac{1}{64}x^5 + \frac{1}{64}x^3, \quad x \in (0,1).$$

We note that $r_1' > 0$, it follows that r_1 is an increasing function on $(0,1)$, so

$$L(x,0) \leq \frac{11}{1536}.$$

(iv) On edge $y = 1$, we obtain

$$r_2(x) := L(x,1) = \frac{53}{4608}x^6 - \frac{7}{288}x^4 + \frac{23}{1152}x^2, \quad x \in (0,1).$$

Since, $r_2' > 0$, it follows that r_2 is an increasing function on $(0,1)$, then

$$L(x,1) \leq \frac{11}{1536}.$$

II. Lastly, we will maximize $L(x,y)$ on the vertices of $\Theta := [0,1] \times [0,1]$.

$$L(0,0) = L(0,1) = 0, \quad L(1,0) = L(1,1) = \frac{11}{1536}.$$

We conclude that

$$max\{L(x,y) : (x,y) \in \Theta\} = L(x,0) = L(x,1) = L(1,y) = L(1,0) = L(1,1) = \frac{11}{1536}.$$

Hence, becomes

$$|H_{2,2}(f^{-1}/2)| \leq \frac{11}{1536}.$$

The equality exists for the function f_1 given by (2.6). Here, $a_2 = a_4 = 0, a_3 = -\frac{1}{4}$ and $a_5 = \frac{1}{24}$. Therefore, after some simplifications we get $|H_{2,2}(f^{-1}/2)| = \frac{11}{1536}$. □

The sharp bound of $|T_2(1)(f^{-1}/2)| := |T_2(1)|$ is presented in this theorem.

Theorem 2.4. *Consider $f \in \mathcal{S}^*_{cos}$ is given by the form* (1.1)*. Then*

$$|T_2(1)| \leq \frac{1}{64}.$$

The inequality is sharp.

Proof. By using (2.3) in (1.8), we deduce that

$$T_2(1) = -\frac{1}{1024}c_1^4,$$

applying modulus, the above expression reduces to

$$|T_2(1)| \leq \frac{1}{1024}|c_1|^4 \leq \frac{1}{64},$$

since $c_1 \in [0, 2]$.
This inequality is sharp for the function f_1 given by (2.6). □

In our last theorem, we calculate the sharp bound on $|T_2(2)(f^{-1}/2)| := |T_2(2)|$.

Theorem 2.5. *Consider $f \in \mathcal{S}^*_{cos}$ is of the form* (1.1)*. Then*

$$|T_2(2)| \leq \frac{1}{64}.$$

The inequality is sharp.

Proof. By utilizing (2.3) in (1.8), we obtain

$$T_2(2) = \frac{c_1^4}{1024} - \left(\frac{c_1}{24}\left(c_2 - \frac{c_1^2}{2}\right)\right)^2,$$

by applying Lemma 1.3, the above equality becomes

$$|T_2(2)| \leq \frac{|c_1|^4}{1024} + \left(\frac{|c_1|}{12} - \frac{|c_1|^3}{48}\right)^2.$$

Denoting $c := |c_1|$, we have $c \in [0, 2]$

$$|T_2(2)| \leq \frac{c^4}{1024} + \left(\frac{c}{12} - \frac{c^3}{48}\right)^2 =: S(c), \quad c \in [0, 2].$$

We can observe that $S'(c) > 0$. Thus $S(c)$ is an increasing function and has maximum value at $c = 2$, it implies

$$|T_2(2)| \leq S(2) = \frac{1}{64}.$$

The function f_1 defined by (2.6) gives equality for this bound. □

3. Conclusion

In this article by applying the Cho et. al. [5] results, we presented the coefficient bounds of initial inverse logarithmic coefficients for the starlike functions in the class $\mathcal{S}^*_{cos}$. Additionally, we determined the bounds of second order Hankel and Toeplitz determinants for the same class. The bounds presented here are sharp and the extremel function for each bound is also given.

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Isha Zahid
Department of Mathematics, University of Jhang, Pakistan
e-mail: `ishazahid2019@gmail.com`

Umar Raza
Department of Mathematics, University of Jhang, Pakistan
e-mail: `umarraza55@gmail.com`

Mohsan Raza
Department of Mathematics, Government College University Faisalabad, Pakistan
e-mail: `mohsan976@yahoo.com`