Stud. Univ. Babeş-Bolyai Math. 71(2026), No. 1, 73–82
DOI: 10.24193/subbmath.2026.1.05

On a specific ratio-cosine Hardy-Hilbert-type integral inequality in the entire plane

Christophe Chesneau

Abstract. This article focuses on a specific Hardy-Hilbert-type integral inequality that is defined in the entire plane. The main contribution is the derivation of a ratio-cosine kernel function, which sets it apart from most existing literature on the subject. As a consequence of the main theorem, a related integral inequality of independent interest is also derived. The exposition is self-contained, with full details of all proofs presented, and each step is carefully justified.

Mathematics Subject Classification (2010): 26D15, 33E20.

Keywords: Hardy-Hilbert-type integral inequalities, integral formulas, ratio-cosine kernel function.

1. Introduction

The classical Hardy-Hilbert integral inequality is a key tool in real analysis. It provides a precise upper bound for a double integral involving the product of two functions. More precisely, let $p > 1$, $q = p/(p-1)$ and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then the Hardy-Hilbert integral inequality states that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that both integrals on the right-hand side converge. The constant $\pi/\sin(\pi/p)$ is known to be the best possible and cannot be improved (see [7]). This inequality has found numerous applications in analysis, operator theory, and mathematical physics. It serves as a benchmark for a wide class of Hardy–Hilbert-type integral inequalities.

Received 31 August 2025; Accepted 06 January 2026.

Further details and developments can be found in the monographs of Yang [21, 20], as well as in the classical and modern contributions in [17, 15, 22, 4, 1, 3, 2].

In fact, most of the results in this field have been obtained in the quarter-plane $(0, +\infty)^2 = \{(x, y) \in \mathbb{R}^2;\ x > 0,\ y > 0\}$. In recent years, considerable effort has been devoted to extending Hardy-Hilbert-type integral inequalities beyond this domain to the entire plane $\mathbb{R}^2$, i.e., the set of all points in the plane. See, for instance, the studies in [18, 8, 19, 25, 14, 16, 9, 24, 11, 27, 12, 10, 6, 26, 13, 23]. The established generalizations are quite diverse, involving various classes of kernel functions, and a broad range of parameter settings. As a well-known example, in [19], the result below holds. Let $p > 1$, $q = p/(p-1)$ and $f, g : \mathbb{R} \to (0, +\infty)$ be two functions. Then we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{|1+xy|^{\theta}} f(x)g(y)dxdy$$
$$\leq \kappa \left[\int_{-\infty}^{+\infty} |x|^{(1-\theta/2)p-1} f^p(x)dx\right]^{1/p} \left[\int_{-\infty}^{+\infty} |y|^{(1-\theta/2)q-1} g^q(y)dy\right]^{1/q},$$

where

$$\kappa = B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) + 2B\left(1-\theta, \frac{\theta}{2}\right)$$

and $B(a, b)$ is the standard beta function, i.e., $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, provided that both integrals on the right-hand side converge. The constant κ is known to be the best possible and cannot be improved (see [19]).

Despite the abundance of literature on the subject, inequalities involving ratio-cosine kernel functions have received relatively little attention. This article aims to address this gap. More specifically, our objective is to present a new integral inequality based on the following kernel function:

$$k(x, y) = \frac{\{1 - \tau \cos[\alpha(\beta - x)]\}^{1/q} \{1 - \tau \cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2},$$

where $\alpha, \beta > 0$ and $\tau \in [-1, 1]$ are three independent and adjustable parameters. For the sake of completeness, we provide a detailed proof. A key component of our analysis is an integral formula derived from one originating in [5]. We also prove an integral inequality involving the dependence of only one function.

The remainder of the article is organized as follows: Section 2 presents a preliminary result: the aforementioned new integral formula. Section 3 contains our main results. Section 4 provides a conclusion.

2. A preliminary result

The lemma below corresponds to [5, Formula 3.723 7], with only minor changes to the notation.

Lemma 2.1. [5, Formula 3.723 7] *Let $\alpha, \beta, \eta > 0$. Then we have*

$$\int_{-\infty}^{+\infty} \frac{\cos[\alpha(\beta - x)]}{\eta^2 + x^2} dx = \frac{\pi}{\eta} \cos(\alpha\beta) e^{-\alpha\eta}.$$

Based on this result, we derive the new result below. This mainly involves modifying the integrand of the previous integral to make it non-negative.

Lemma 2.2. *Let $\alpha, \beta > 0$, $\gamma \in \mathbb{R}\backslash\{0\}$ and $\tau \in [-1, 1]$. Then we have*

$$\int_{-\infty}^{+\infty} \frac{1 - \tau\cos[\alpha(\beta - x)]}{\gamma^2 + x^2} dx = \frac{\pi}{|\gamma|} \left[1 - \tau\cos(\alpha\beta) e^{-\alpha|\gamma|}\right].$$

Proof. Combining a linearization of the integral with the standard arctangent primitive and Lemma 2.1 yields

$$\int_{-\infty}^{+\infty} \frac{1 - \tau\cos[\alpha(\beta - x)]}{\gamma^2 + x^2} dx = \int_{-\infty}^{+\infty} \frac{1}{|\gamma|^2 + x^2} dx - \tau \int_{-\infty}^{+\infty} \frac{\cos[\alpha(\beta - x)]}{|\gamma|^2 + x^2} dx$$
$$= \left[\frac{1}{|\gamma|} \arctan\left(\frac{x}{|\gamma|}\right)\right]_{x\to-\infty}^{x\to+\infty} - \tau\frac{\pi}{|\gamma|} \cos(\alpha\beta) e^{-\alpha|\gamma|}$$
$$= \frac{\pi}{|\gamma|} - \tau\frac{\pi}{|\gamma|} \cos(\alpha\beta) e^{-\alpha|\gamma|} = \frac{\pi}{|\gamma|} \left[1 - \tau\cos(\alpha\beta) e^{-\alpha|\gamma|}\right].$$

This ends the proof of the lemma. □

The fact that the integrand is non-negative and is defined on the real line, and that the integral formula is in closed form, makes it suitable for use in Hardy-Hilbert-type integral inequalities in the entire plane. This is the focus of the subsequent section.

3. Main results

Based on Lemma 2.2, the theorem below states a new Hardy-Hilbert-type integral inequality in the entire plane. As outlined in the introduction, it is of ratio-cosine nature.

Theorem 3.1. *Let $p > 1$, $q = p/(p - 1)$, $\alpha, \beta > 0$, $\tau \in [-1, 1]$ and $f, g : \mathbb{R} \to (0, +\infty)$ be two functions. Then we have*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{1 - \tau\cos[\alpha(\beta - x)]\}^{1/q} \{1 - \tau\cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2} f(x)g(y)dxdy$$
$$\leq \pi \left\{\int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau\cos(\alpha\beta) e^{-\alpha|x|}\right] f^p(x)dx\right\}^{1/p}$$
$$\times \left\{\int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 - \tau\cos(\alpha\beta) e^{-\alpha|y|}\right] g^q(y)dy\right\}^{1/q},$$

provided that both integrals on the right-hand side converge.

Proof. Note that the double integral is well-defined because, for any $x, y \in \mathbb{R}$,

$$1 - \tau \cos[\alpha(\beta - x)] \geq 1 - |\tau| \geq 0$$

and

$$1 - \tau \cos[\alpha(\beta - y)] \geq 1 - |\tau| \geq 0.$$

Decomposing the integrand using $1/p + 1/q = 1$ and applying the Hölder integral inequality, we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\{1 - \tau \cos[\alpha(\beta - x)]\}^{1/q}\{1 - \tau \cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2} f(x)g(y)dxdy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left\{\frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2}\right\}^{1/p} f(x) \times \left\{\frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2}\right\}^{1/q} g(y)dxdy \\
&\leq \left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} f^p(x)dxdy\right\}^{1/p} \\
&\quad \times \left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} g^q(y)dxdy\right\}^{1/q}. \qquad (3.1)
\end{aligned}$$

Let us examine these integrals. It follows from the Fubini-Tonelli integral theorem and Lemma 2.2 with $\gamma = x$ that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} f^p(x)dxdy \\
&= \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} dy\right\} f^p(x)dx \\
&= \int_{-\infty}^{+\infty} \left\{\frac{\pi}{|x|}\left[1 - \tau \cos(\alpha\beta)e^{-\alpha|x|}\right]\right\} f^p(x)dx \\
&= \pi \int_{-\infty}^{+\infty} \frac{1}{|x|}\left[1 - \tau \cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx. \qquad (3.2)
\end{aligned}$$

In a similar way, we find that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} g^q(y)dxdy \\
&= \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} dx\right\} g^q(y)dy \\
&= \int_{-\infty}^{+\infty} \left\{\frac{\pi}{|y|}\left[1 - \tau \cos(\alpha\beta)e^{-\alpha|y|}\right]\right\} g^q(y)dy \\
&= \pi \int_{-\infty}^{+\infty} \frac{1}{|y|}\left[1 - \tau \cos(\alpha\beta)e^{-\alpha|y|}\right] g^q(y)dy. \qquad (3.3)
\end{aligned}$$

It follows from Equations (3.1), (3.2) and (3.3), and $1/p+1/q=1$ that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2}f(x)g(y)dxdy$$
$$\leq \pi\left\{\int_{-\infty}^{+\infty}\frac{1}{|x|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right]f^p(x)dx\right\}^{1/p}$$
$$\times\left\{\int_{-\infty}^{+\infty}\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]g^q(y)dy\right\}^{1/q}.$$

This completes the proof of the theorem. □

To the best of our knowledge, this is a new addition to the literature on the Hardy-Hilbert integral inequality in the entire plane.

The constant factor, π, was obtained in the development process with the minimum possible number of steps. However, it has not been proven rigorously that this is the best possible approach. The question of its optimality remains unanswered.

Assuming that both integrals on the right-hand side converge implies that the functions f and g must behave in a specific way near 0. More precisely, near 0, they must satisfy the equivalences $f(x)\sim|x|^{\iota}$ and $g(y)\sim|y|^{\omega}$ with $\iota,\omega>0$, with reference to the convergence of the Riemann integral in the interval $(-1,1)$.

Some specific examples of Theorem 3.1 are provided below.

- If we take $\tau=0$, we get

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{x^2+y^2}f(x)g(y)dxdy\leq\pi\left[\int_{-\infty}^{+\infty}\frac{1}{|x|}f^p(x)dx\right]^{1/p}\left[\int_{-\infty}^{+\infty}\frac{1}{|y|}g^q(y)dy\right]^{1/q}.$$

- If we take $\tau=1$, using the following trigonometric formula: $\sin^2(a)=[1-\cos(2a)]/2$, $a\in\mathbb{R}$, and $1/p+1/q=1$, we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\left\{\sin^2[\alpha(\beta-x)/2]\right\}^{1/q}\left\{\sin^2[\alpha(\beta-y)/2]\right\}^{1/p}}{x^2+y^2}f(x)g(y)dxdy$$
$$\leq\frac{\pi}{2}\left\{\int_{-\infty}^{+\infty}\frac{1}{|x|}\left[1-\cos(\alpha\beta)e^{-\alpha|x|}\right]f^p(x)dx\right\}^{1/p}$$
$$\times\left\{\int_{-\infty}^{+\infty}\frac{1}{|y|}\left[1-\cos(\alpha\beta)e^{-\alpha|y|}\right]g^q(y)dy\right\}^{1/q}.$$

- If we take $\tau = -1$, using the following trigonometric formula: $\cos^2(a) = [1 + \cos(2a)]/2$, $a \in \mathbb{R}$, and $1/p + 1/q = 1$, we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\left\{\cos^2[\alpha(\beta-x)/2]\right\}^{1/q}\left\{\cos^2[\alpha(\beta-y)/2]\right\}^{1/p}}{x^2+y^2} f(x)g(y)dxdy$$
$$\leq \frac{\pi}{2}\left\{\int_{-\infty}^{+\infty} \frac{1}{|x|}\left[1+\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx\right\}^{1/p}$$
$$\times \left\{\int_{-\infty}^{+\infty} \frac{1}{|y|}\left[1+\cos(\alpha\beta)e^{-\alpha|y|}\right] g^q(y)dy\right\}^{1/q}.$$

The proposition below is a non-trivial consequence of Theorem 3.1. It has the distinctive feature of involving only a single function.

Proposition 3.2. *Let $p > 1$, $q = p/(p-1)$, $\alpha, \beta > 0$, $\tau \in [-1, 1]$ and $f : \mathbb{R} \to (0, +\infty)$ be a function. Then we have*

$$\int_{-\infty}^{+\infty}\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-(p-1)}$$
$$\times\left\{\int_{-\infty}^{+\infty} \frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx\right\}^{p} dy$$
$$\leq \pi^p\left\{\int_{-\infty}^{+\infty} \frac{1}{|x|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx\right\},$$

provided that the integral on the right-hand side converges.

Proof. To ease the developments, let us set

$$\mathfrak{A} = \int_{-\infty}^{+\infty}\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-(p-1)}$$
$$\times\left\{\int_{-\infty}^{+\infty} \frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx\right\}^{p} dy.$$

Then we can write

$$\mathfrak{A} = \int_{-\infty}^{+\infty}\left\{\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-1}\right.$$
$$\left.\times\int_{-\infty}^{+\infty} \frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx\right\}^{p-1}$$
$$\times\left\{\int_{-\infty}^{+\infty} \frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx\right\} dy$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)g_{\diamond}(y)dxdy, \tag{3.4}$$

with

$$g_{\diamond}(y)=\left\{\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-1}\right.$$
$$\left.\times\int_{-\infty}^{+\infty}\frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2}f(x)dx\right\}^{p-1}.$$

Applying Theorem 3.1 to f and $g_{\diamond}$, we get

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2}f(x)g_{\diamond}(y)dxdy$$
$$\leq\pi\left\{\int_{-\infty}^{+\infty}\frac{1}{|x|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right]f^p(x)dx\right\}^{1/p}$$
$$\times\left\{\int_{-\infty}^{+\infty}\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]g_{\diamond}^q(y)dy\right\}^{1/q}. \tag{3.5}$$

Let us now examine this last integral. Using $1/p+1/q=1$ and the expression of $\mathfrak{A}$, we obtain

$$\int_{-\infty}^{+\infty}\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]g_{\diamond}^q(y)dy$$
$$=\int_{-\infty}^{+\infty}\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\left\{\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-1}\right.$$
$$\left.\times\int_{-\infty}^{+\infty}\frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2}f(x)dx\right\}^{q(p-1)}dy$$
$$=\int_{-\infty}^{+\infty}\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\left\{\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-1}\right.$$
$$\left.\times\int_{-\infty}^{+\infty}\frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2}f(x)dx\right\}^{p}dy$$
$$=\int_{-\infty}^{+\infty}\left\{\frac{1}{|y|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right]\right\}^{-(p-1)}$$
$$\times\left\{\int_{-\infty}^{+\infty}\frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2}f(x)dx\right\}^{p}dy$$
$$=\mathfrak{A}. \tag{3.6}$$

It follows from Equations (3.4), (3.5) and (3.6) that

$$\mathfrak{A}\leq\pi\left\{\int_{-\infty}^{+\infty}\frac{1}{|x|}\left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right]f^p(x)dx\right\}^{1/p}\mathfrak{A}^{1/q}.$$

Using $1/p+1/q=1$, we obtain

$$\mathfrak{A}^{1/p} \leq \pi \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx \right\}^{1/p},$$

so that

$$\mathfrak{A} \leq \pi^p \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx.$$

This can be written as

$$\int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1-\tau\cos(\alpha\beta)e^{-\alpha|y|}\right] \right\}^{-(p-1)} \times \left\{ \int_{-\infty}^{+\infty} \frac{\{1-\tau\cos[\alpha(\beta-x)]\}^{1/q}\{1-\tau\cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx \right\}^p dy \leq \pi^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1-\tau\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx \right\}.$$

The proof of the proposition ends. □

This proposition can be typically used in the study of specific ratio-cosine integral operators. Some particular examples are given below.

- If we take $\tau=0$, we get

$$\int_{-\infty}^{+\infty} |y|^{p-1} \left[\int_{-\infty}^{+\infty} \frac{1}{x^2+y^2} f(x)dx \right]^p dy \leq \pi^p \left[\int_{-\infty}^{+\infty} \frac{1}{|x|} f^p(x)dx \right].$$

- If we take $\tau=1$, using the following trigonometric formula: $\sin^2(a)=[1-\cos(2a)]/2$, $a\in\mathbb{R}$, and $1/p+1/q=1$, we obtain

$$\int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1-\cos(\alpha\beta)e^{-\alpha|y|}\right] \right\}^{-(p-1)} \times \left\{ \int_{-\infty}^{+\infty} \frac{\left\{\sin^2[\alpha(\beta-x)/2]\right\}^{1/q}\left\{\sin^2[\alpha(\beta-y)/2]\right\}^{1/p}}{x^2+y^2} f(x)dx \right\}^p dy \leq \left(\frac{\pi}{2}\right)^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1-\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx \right\}.$$

- If we take $\tau=-1$, using the following trigonometric formula: $\cos^2(a)=[1+\cos(2a)]/2$, $a\in\mathbb{R}$, and $1/p+1/q=1$, we obtain

$$\int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1+\cos(\alpha\beta)e^{-\alpha|y|}\right] \right\}^{-(p-1)} \times \left\{ \int_{-\infty}^{+\infty} \frac{\left\{\cos^2[\alpha(\beta-x)/2]\right\}^{1/q}\left\{\cos^2[\alpha(\beta-y)/2]\right\}^{1/p}}{x^2+y^2} f(x)dx \right\}^p dy \leq \left(\frac{\pi}{2}\right)^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1+\cos(\alpha\beta)e^{-\alpha|x|}\right] f^p(x)dx \right\}.$$

4. Conclusion

In conclusion, we have derived a new Hardy-Hilbert-type integral inequality involving a ratio-cosine kernel function in the entire plane, alongside a related inequality that depends on only one function. The presence of three independent parameters makes our results notably flexible and distinguishes them from existing inequalities in the literature. One limitation of our study is that we do not prove that π is the best possible constant factor, which remains an open problem. Proposed areas for future research include exploring multidimensional analogues, extending the method to other classes of oscillatory kernel functions, and investigating potential applications in harmonic analysis and mathematical physics.

References

[1] Chesneau, C., *General inequalities of the Hilbert integral type using the method of switching to polar coordinates,* Hilbert J. Math. Anal., **3**(2024), 7-26.

[2] Chesneau, C., *Theoretical results on new Hardy-Hilbert-type inequalities,* Ann. Com. Math., **8**(2025), 253-274.

[3] Chesneau, C., *A collection of new and flexible modifications of the Hardy-Hilbert integral inequality,* Asian J. Math. Appl., **10**(2025), 1-58.

[4] Debnath, L., Yang, B. C., *Recent developments of Hilbert-type discrete and integral inequalities with applications,* Int. J. Math. Math. Sci., **2012**(2012), 871845.

[5] Gradshteyn, I. S., Ryzhik, I. M., *Table of Integrals, Series, and Products*, 7th ed., Academic Press, 2007.

[6] Gu, Z. H., Yang, B. C., *A Hilbert-type integral inequality in the whole plane with a non-homogeneous kernel and a few parameters,* J. Inequal. Appl., **2015**(2015), 314.

[7] Hardy, G. H., Littlewood, J. E., Pólya, G., *Inequalities*, Cambridge University Press, Cambridge, MA, 1934.

[8] He, B., Yang, B. C., *On a Hilbert-type integral inequality with the homogeneous kernel of 0-degree and the hypergeometric function,* Math. Pract. Theory, **40**(2010), 105-211.

[9] He, B., Yang, B. C., *On an inequality concerning a non-homogeneous kernel and the hypergeometric function,* Tamsul Oxford J. Inf. Math. Sci., **27**(2011), 75-88.

[10] Huang, X. Y., Cao, J. F., He, B., Yang, B. C.,*Hilbert-type and Hardy-type integral inequalities with operator expressions and the best constants in the whole plane,* J. Inequal. Appl., **2015**(2015), 129.

[11] Huang, Q. L., Wu, S. H., Yang, B. C., *Parameterized Hilbert-type integral inequalities in the whole plane,* Sci. World J., **2014**(2014), 169061.

[12] Rassias, M. T., Yang, B. C., *A Hilbert-type integral inequality in the whole plane related to the hypergeometric function and the beta function,* J. Math. Anal. Appl., **428**(2015), 1286-1308.

[13] Rassias, M. T., Yang, B. C., Raigorodskii, A., *On Hardy-type integral inequality in the whole plane related to the extended Hurwitz-zeta function,* J. Inequal. Appl., **2020**(2020), 94.

[14] Wang, A. Z., Yang, B. C., *A new Hilbert-type integral inequality in whole plane with the non-homogeneous kernel,* J. Inequal. Appl., **2011**(2011), 123.

[15] Xin, D. M., *A Hilbert-type integral inequality with the homogeneous kernel of zero degree,* Math. Theory Appl., **30**(2010), 70-74.

[16] Xin, D. M., Yang, B. C., *A Hilbert-type integral inequality in whole plane with the homogeneous kernel of degree -2,* J. Inequal. Appl., **2011**(2011), 401428.

[17] Xu, J. S., *Hardy-Hilbert's inequalities with two parameters,* Adv. Math., **36**(2007), 63-76.

[18] Yang, B. C., A new Hilbert-type integral inequality, *Soochow J. Math.*, **33**(2007), 849-859.

[19] Yang, B. C., *A new Hilbert-type integral inequality with some parameters,* J. Jilin Univ. (Sci. Ed.), **46**(2008), 1085-1090.

[20] Yang, B. C., *Hilbert-Type Integral Inequalities*, Bentham Science Publishers Ltd., Sharjah, United Arab Emirates, 2009.

[21] Yang, B. C., *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, 2009.

[22] Yang, B. C., *A Hilbert-type integral inequality with the homogeneous kernel of degree 0,* J. Shandong Univ. (Nat.), **45**(2010), 103-106.

[23] Yang, B. C., Andrica, D., Bagdasar, O., Rassias, M. T., *On a Hilbert-type integral inequality in the whole plane with the equivalent forms,* Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **117**(2023), 5.

[24] Xie, Z. T., Zeng, Z., Sun, Y. F., *A new Hilbert-type inequality with the homogeneous kernel of degree -2,* Adv. Appl. Math. Sci., **12**(2013), 391-401.

[25] Zeng, Z., Xie, Z. T., *On a new Hilbert-type integral inequality with the homogeneous kernel of degree 0 and the integral in whole plane,* J. Inequal. Appl., **2010**(2010), 256796.

[26] Zhong, Y. R., Huang, M. F., Yang, B. C., *A Hilbert-type integral inequality in the whole plane related to the kernel of exponent function,* J. Inequal. Appl., **2018**(2018), 234.

[27] Zhen, Z., Gandhi, K. R. R., Xie, Z. T., *A new Hilbert-type inequality with the homogeneous kernel of degree -2 and with the integral,* Bull. Math. Sci. Appl., **3**(2014), 11-20.

Christophe Chesneau
Department of Mathematics, LMNO, University of Caen-Normandie,
14032 Caen, France.
e-mail: `christophe.chesneau@gmail.com`