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# Hermite–Hadamard type inequalities via $(h, m)$-convexity

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**Abstract.** In this paper, we establish a novel Hermite–Hadamard inequality for $(h, m)$-convex functions using Riemann–Liouville fractional integral operators, right and left. Furthermore, some new Hermite–Hadamard type fractional integral inequalities are proved for differentiable functions whose first derivative is $(h, m)$-convex. We demonstrate that these newly established integral inequalities generalize some existing results.

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## 1. Introduction

The notion of convexity is a fundamental and highly productive idea in contemporary mathematics.

**Definition 1.1.** *A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, $a < b$ is said to be convex if the following inequality holds for all $x, y \in [a, b]$ and $t \in [0, 1]$:*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

From now on, throughout the paper, we always suppose that $a < b$ when we consider intervals $[a, b]$.

Convex functions play an important role in many branches of mathematics, and are of particular interest to mathematicians in inequality theory. The researchers interested in the development of convex functions can get a comprehensive overview in [11].

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Consequently, investigations in this realm have led to the development of numerous novel inequalities, significantly enriching the field. One of the most researched and most important inequalities for convex functions on $[a,b]$ is the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

First established in 1893, the Hermite–Hadamard inequality for convex functions has undergone numerous refinements over the years, with various mathematicians having contributed to its advancement and generalization, see e.g. papers [3, 4, 5, 6, 10, 17] and the references therein for more information and generalizations of inequality (1.1).

Among the many generalizations of the convex functions, we recall the concepts of $m$-convex, $h$-convex and $(h,m)$-convex functions, which coincide with the convex functions for $m=1$, $h(x)=x$ and $m=1$, $h(x)=x$, respectively.

**Definition 1.2.** [19] *Function $f:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ is said to be $m$-convex with $m\in[0,1]$ , if for all $x,y\in[a,b]$ and $t\in[0,1]$:*

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y).$$

**Definition 1.3.** [21] *Let $h:[0,1]\to\mathbb{R}^+$. Then $f:[a,b]\subset\mathbb{R}\to\mathbb{R}^+$ is called an $h$-convex function if the following inequality holds for $x,y\in[a,b]$ and $t\in[0,1]$:*

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y).$$

**Definition 1.4.** [13] *Let $h:[0,1]\to\mathbb{R}^+$. We say that $f:[a,b]\subset\mathbb{R}^+\to\mathbb{R}^+$ is an $(h,m)$-convex function with $m\in[0,1]$ , if for all $x,y\in[a,b]$ and $t\in[0,1]$:*

$$f(tx+m(1-t)y) \leq h(t)f(x)+mh(1-t)f(y). \tag{1.2}$$

Fractional calculus, a generalization of ordinary calculus, has a rich history spanning over about 300 years. Its origins date back to 1695, when Gottfried Wilhelm Leibniz and Guillaume de l'Hôpital exchanged letters discussing the possibility of extending integer-order derivatives to non-integer orders. This conversation sparked the interest of mathematicians, leading to further exploration of fractional calculus concept. Today, fractional calculus has a wide range of applications in various fields, including acoustics, optics, viscoelasticity, chemical, rheology, and control theory, electrical and mechanical engineering, statistical physics, robotics and bioengineering. To address real-world problems, mathematicians have developed numerous fractional integral operators. Generalization of Hermite–Hadamard inequality (1.1) via fractional integrals has been extensively researched, see e.g. [2, 7, 12, 18] and the references therein. We recall one of the most well-known fractional integral operator called Riemann–Liouville integral operator.

**Definition 1.5.** [9, 14] *Let $f \in L^1_{loc}([a,b])$. Then the Riemann–Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ are defined by*

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > a,$$
$$I^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x < b.$$

From now on, throughout the paper, we always assume that $\alpha > 0$.

In [15], Sarikaya et al. proved the following Hermite–Hadamard type inequality for the sum of Riemann–Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$.

**Theorem 1.6.** [16] *Let $f : [a,b] \to \mathbb{R}$ be a positive, convex function on $[a,b]$ and $a \geq 0$. Then we have the following fractional integral inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left(I^{\alpha}_{a^+} f(b) + I^{\alpha}_{b^-} f(a)\right) \leq \frac{f(a)+f(b)}{2}.$$

Recently, Ali et al. gave the following refinements of (1.1) for the Riemann–Liouville fractional integrals.

**Theorem 1.7.** [1] *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be differentiable on $[a,b]$ such that $f' \in L^1_{loc}([a,b])$ is $h$-convex on $[a,b]$. The following inequalities hold:*

$$\begin{aligned}(b-a)^{\alpha} & \frac{\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) + \left(\frac{1}{\alpha} + H - \frac{1}{h\left(\frac{1}{2}\right)}\right) f(a)}{1+H\alpha} - (b-a)^{\alpha+1} \frac{H\alpha f'(a)}{2h\left(\frac{1}{2}\right)(\alpha+1)(1+H\alpha)} \\ & \leq \Gamma(\alpha)\left(I^{\alpha}_{b^-} f(a)\right) \leq (b-a)^{\alpha} \frac{f(a)+H\alpha f(b)}{\alpha(1+H\alpha)} + (b-a)^{\alpha+1} \frac{Hf'(a)}{(\alpha+1)(1+H\alpha)}\end{aligned} \tag{1.3}$$

*and*

$$\begin{aligned}(b-a)^{\alpha} & \frac{H\alpha f(a)+f(b)}{\alpha(1+H\alpha)} - (b-a)^{\alpha+1} \frac{Hf'(b)}{(\alpha+1)(1+H\alpha)} \leq \Gamma(\alpha)\left(I^{\alpha}_{a^+} f(b)\right) \\ & \leq (b-a)^{\alpha} \frac{\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) + \left(\frac{1}{\alpha} + H - \frac{1}{h\left(\frac{1}{2}\right)}\right) f(b)}{1+H\alpha} + (b-a)^{\alpha+1} \frac{H\alpha f'(b)}{2h\left(\frac{1}{2}\right)(\alpha+1)(1+H\alpha)},\end{aligned} \tag{1.4}$$

*where $H = \int_0^1 h(t)\, dt$.*

The rest of the paper is organized as follows. In Section 2, we present the extension of Theorem 1.6 for $(h,m)$-convex functions with some consequences for different types of general convexity. After the established results, we give the generalization of Theorem 1.7 via two theorems along with the corresponding consequences. In Section 3, applications are given in order to highlight the main results.

## 2. Main results

First, we formulate an Hermite–Hadamard type result for the sum of Riemann–Liouville fractional integrals $I^{\alpha}_{a^+} f(b)$ and $I^{\alpha}_{b^-} f(a)$ via $(h,m)$-convexity.

**Theorem 2.1.** *Let $f : [0, \infty) \to \mathbb{R}$ be an $(h,m)$-convex function with $m \in (0,1]$. If $f \in L^1[a,b]$, then we have*

$$\begin{aligned} 2\,H_m\, f\left(\frac{m(a+b)}{1+m}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{a^+} f(b) + I^\alpha_{b^-} f(a)\right] \\ &\leq \Gamma(\alpha+1)\left[I^\alpha_{1^-} h(0)(f(a)+f(b)) + I^\alpha_{0^+} h(1) m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right)\right], \end{aligned} \tag{2.1}$$

*where* $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right)\right]^{-1}$.

*Proof.* From the definition of $(h,m)$-convexity, we can write

$$\begin{aligned} f\left(\frac{m(x+y)}{1+m}\right) &= f\left(\frac{m}{1+m}\,x + m\,\frac{1}{1+m}\,y\right) \\ &\leq h\left(\frac{m}{1+m}\right) f(x) + mh\left(\frac{1}{1+m}\right) f(y) \end{aligned}$$

for all $x, y \geq 0$. If we choose $x = ta + (1-t)b$ and $y = tb + (1-t)a$, we get

$$\begin{aligned} f\left(\frac{m(a+b)}{1+m}\right) &= f\left(\frac{m(x+y)}{1+m}\right) \\ &\leq h\left(\frac{m}{1+m}\right) f(ta+(1-t)b) + mh\left(\frac{1}{1+m}\right) f(tb+(1-t)a) \end{aligned} \tag{2.2}$$

for all $t \in [0,1]$. Since

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\,dt = \frac{1}{(b-a)^\alpha}\int_a^b (b-x)^{\alpha-1} f(x)\,dx = \frac{\Gamma(\alpha)}{(b-a)^\alpha} I^\alpha_{a^+} f(b)$$

and

$$\int_0^1 t^{\alpha-1} f(tb+(1-t)a)\,dt = \frac{1}{(b-a)^\alpha}\int_a^b (x-a)^{\alpha-1} f(x)\,dx = \frac{\Gamma(\alpha)}{(b-a)^\alpha} I^\alpha_{b^-} f(a),$$

hence, if we multiply both sides of inequality (2.2) by $t^{\alpha-1}$ and integrate with respect to $t$ on $[0,1]$, we obtain

$$\frac{1}{\alpha} f\left(\frac{m(a+b)}{1+m}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[h\left(\frac{m}{1+m}\right) I^\alpha_{a^+} f(b) + mh\left(\frac{1}{1+m}\right) I^\alpha_{b^-} f(a)\right]. \tag{2.3}$$

Analogously, by choosing $x = tb + (1-t)a$ and $y = ta + (1-t)b$, we get

$$\frac{1}{\alpha} f\left(\frac{m(a+b)}{1+m}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[h\left(\frac{m}{1+m}\right) I^\alpha_{b^-} f(a) + mh\left(\frac{1}{1+m}\right) I^\alpha_{a^+} f(b)\right]. \tag{2.4}$$

Putting together (2.3) and (2.4) yields the first inequality in (2.1).

To see the second inequality of (2.1), first note that (1.2) implies

$$f(ta+(1-t)b) \leq h(t)f(a) + mh(1-t) f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a) \leq h(t)f(b) + mh(1-t) f\left(\frac{a}{m}\right).$$

After multiplying the sides of the previous inequalities by $t^{\alpha-1}$ and integrating with respect to $t$ on $[0,1]$, we get

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\,dt \le f(a)\int_0^1 t^{\alpha-1}h(t)\,dt + mf\left(\frac{b}{m}\right)\int_0^1 (1-t)^{\alpha-1}h(t)\,dt$$

and

$$\int_0^1 t^{\alpha-1} f(tb+(1-t)a)\,dt \le f(b)\int_0^1 t^{\alpha-1}h(t)\,dt + mf\left(\frac{a}{m}\right)\int_0^1 (1-t)^{\alpha-1}h(t)\,dt,$$

which inequalities together give us the second inequality of (2.1). □

If in Theorem 2.1, we suppose $m$-convexity or $h$-convexity, we get the following consequence for the sum of Riemann–Liouville fractional integrals.

**Corollary 2.2.** *Let us consider an $m$-convex or $h$-convex function $f$.*

- *If $f \in L^1[a,b]$ is $m$-convex with $m \in (0,1]$, then we have*

$$\begin{aligned}\frac{1+m}{m} f\left(\frac{m(a+b)}{1+m}\right) &\le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{a+}f(b)+I^{\alpha}_{b-}f(a)\right]\\ &\le \frac{\alpha}{\alpha+1}[f(a)+f(b)] + \frac{m}{\alpha+1}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right].\end{aligned}$$

- *If $f \in L^1[a,b]$ is $h$-convex, then*

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{a+}f(b)+I^{\alpha}_{b-}f(a)\right]\\ &\le \Gamma(\alpha+1)\left[f(a)+f(b)\right]\left[I^{\alpha}_{0+}h(1)+I^{\alpha}_{1-}h(0)\right].\end{aligned}$$

By analyzing the proof of Theorem 2.1, one can easily see that upper bounds for the individual Riemann–Liouville fractional integrals can be given.

**Remark 2.3.** For $(h,m)$-convex function $f \in L^1[a,b]$ with $m \in (0,1]$, we have

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} I^{\alpha}_{a+}f(b) \le \Gamma(\alpha)\left[I^{\alpha}_{1-}h(0)f(a) + I^{\alpha}_{0+}h(1)mf\left(\frac{b}{m}\right)\right]$$

and

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} I^{\alpha}_{b-}f(a) \le \Gamma(\alpha)\left[I^{\alpha}_{1-}h(0)f(b) + I^{\alpha}_{0+}h(1)mf\left(\frac{a}{m}\right)\right].$$

If in Theorem 2.1, we suppose $\alpha = 1$ and different types of general convexity, keeping in mind Remark 2.3 or using [13, Theorem 8], we get the following corollary.

**Corollary 2.4.** *Let us consider an $(h,m)$-convex, $m$-convex, $h$-convex or convex function $f$.*

- *If $f \in L^1[a,b]$ is $(h,m)$-convex function with $m \in (0,1]$. Then we have*

$$\begin{aligned}H_m f\left(\frac{m(a+b)}{1+m}\right) &\le \frac{1}{b-a}\int_a^b f(x)\,dx\\ &\le H \min\left\{f(a)+mf\left(\frac{b}{m}\right), f(b)+mf\left(\frac{a}{m}\right)\right\},\end{aligned} \tag{2.5}$$

*where* $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right)\right]^{-1}$ *and* $H = \int_0^1 h(t)\,dt$.

- *If* $f \in L^1[a,b]$ *is* $m$*-convex with* $m \in (0,1]$*, then we have*

$$\frac{1+m}{2m} f\left(\frac{m(a+b)}{1+m}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\leq \frac{1}{2}\min\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\}.$$

- *For* $h$*-convex functions, Theorem 2.1 becomes* [15, Theorem 6].
- *For convex functions, Theorem 2.1 implies inequality* (1.1).

We present the extension of Theorem 1.7 for Riemann–Liouville fractional integral $I_{b-}^{\alpha} f(a)$ via $(h,m)$-convexity.

**Theorem 2.5.** *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* $[a,b]$ *such that* $f' \in L^1_{loc}([a,b])$ *is* $(h,m)$*-convex on* $[a,b]$*, then the following inequality holds:*

$$(b-a)^{\alpha}\frac{\left(\frac{1+m}{m}\right)H_m\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2ma}{1+m}\right)\right]+\left(\frac{1}{\alpha}+H\right)f(a)}{1+H\alpha} - (b-a)^{\alpha+1}\frac{mH_mH\alpha f'\left(\frac{2a}{1+m}\right)}{(\alpha+1)(1+H\alpha)}$$
$$\leq \Gamma(\alpha)(I_{b-}^{\alpha} f(a)) \leq (b-a)^{\alpha}\frac{f(a)+H\alpha f(b)}{\alpha(1+H\alpha)} + (b-a)^{\alpha+1}\frac{mHf'\left(\frac{a}{m}\right)}{(\alpha+1)(1+H\alpha)}, \tag{2.6}$$

*where* $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right)\right]^{-1}$ *and* $H = \int_0^1 h(t)\,dt$.

*Proof.* Writing the inequality (2.5) for $f'$ yields

$$H_m f'\left(\frac{m(a+b)}{1+m}\right) \leq \frac{1}{b-a}\int_a^b f'(x)\,dx$$
$$\leq H\min\left\{f'(a) + mf'\left(\frac{b}{m}\right), f'(b) + mf'\left(\frac{a}{m}\right)\right\}. \tag{2.7}$$

We choose to write

$$\min\left\{f'(a) + mf'\left(\frac{b}{m}\right), f'(b) + mf'\left(\frac{a}{m}\right)\right\} = f'(b) + mf'\left(\frac{a}{m}\right).$$

Then inequality (2.7) can be written as

$$H_m f'\left(\frac{m(a+b)}{1+m}\right) \leq \frac{1}{b-a}\int_a^b f'(x)\,dx \leq H\left[f'(b) + mf'\left(\frac{a}{m}\right)\right].$$

By replacing $b$ by $x$, multiplying all the three terms by $(x-a)^{\alpha} > 0$ and then integrating with respect to $x$ over $[a,b]$, we have

$$S_1 = H_m\int_a^b (x-a)^{\alpha} f'\left(\frac{m(a+x)}{1+m}\right)dx$$
$$\leq S_2 = \int_a^b (x-a)^{\alpha-1}\left(\int_a^x f'(t)\,dt\right)dx$$
$$\leq S_3 = H\int_a^b (x-a)^{\alpha}\left[f'(x) + mf'\left(\frac{a}{m}\right)\right]dx.$$

After some simplification, we get

$$\begin{aligned} S_2 &= \int_a^b (x-a)^{\alpha-1}\left(f(x)-f(a)\right)\,dx \\ &= \int_a^b (x-a)^{\alpha-1} f(x)\,dx - \frac{1}{\alpha}(b-a)^{\alpha} f(a), \end{aligned} \tag{2.8}$$

and integration by parts yields

$$\begin{aligned} S_3 &= H\left[\int_a^b (x-a)^{\alpha} f'(x)\,dx + m f'\left(\frac{a}{m}\right)\int_a^b (x-a)^{\alpha}\,dx\right] \\ &= H\left[(b-a)^{\alpha} f(b) - \alpha\int_a^b (x-a)^{\alpha-1} f(x)\,dx + m\frac{(b-a)^{\alpha+1}}{\alpha+1} f'\left(\frac{a}{m}\right)\right]. \end{aligned} \tag{2.9}$$

By combining (2.8) and (2.9), we get

$$\begin{aligned} \int_a^b (x-a)^{\alpha-1} f(x)\,dx \leq{} & \frac{1}{\alpha}(b-a)^{\alpha} f(a) + H\left[(b-a)^{\alpha} f(b)\right. \\ & \left. -\alpha\int_a^b (x-a)^{\alpha-1} f(x)dx + m\frac{(b-a)^{\alpha+1}}{\alpha+1} f'\left(\frac{a}{m}\right)\right], \end{aligned}$$

from where we obtain

$$\begin{aligned} & (1+H\alpha)\int_a^b (x-a)^{\alpha-1} f(x)\,dx \\ & \leq (b-a)^{\alpha}\left(\frac{1}{\alpha} f(a) + H f(b)\right) + (b-a)^{\alpha+1}\frac{mHf'\left(\frac{a}{m}\right)}{\alpha+1}, \end{aligned} \tag{2.10}$$

hence we have the right inequality in (2.6).

Now, integrating $S_1$ by parts and substituting $t = \frac{m(a+x)}{1+m}$ yield

$$\begin{aligned} S_1 &= H_m\left[\frac{1+m}{m}(b-a)^{\alpha} f\left(\frac{m(a+b)}{1+m}\right) - \frac{1+m}{m}\alpha\int_a^b (x-a)^{\alpha-1} f\left(\frac{m(a+x)}{1+m}\right)dx\right] \\ &= (b-a)^{\alpha}\left(\tfrac{1+m}{m}\right) H_m f\left(\tfrac{m(a+b)}{1+m}\right) - \alpha\left(\tfrac{1+m}{m}\right)^{\alpha+1} H_m \int_{\frac{2ma}{1+m}}^{\frac{m(a+b)}{1+m}} \left(t-\tfrac{2ma}{1+m}\right)^{\alpha-1} f(t)\,dt. \end{aligned}$$

Using inequality (2.10) for $\frac{2ma}{1+m}$ and $\frac{m(a+b)}{1+m}$ instead of $a$ and $b$, we have

$$S_1 \geq (b-a)^\alpha \left(\frac{1+m}{m}\right) H_m f\left(\frac{m(a+b)}{1+m}\right) - \alpha\left(\frac{1+m}{m}\right)^{\alpha+1} H_m$$
$$\times \left[(b-a)^\alpha \left(\frac{m}{1+m}\right)^\alpha \frac{Hf\left(\frac{m(a+b)}{1+m}\right)+\frac{1}{\alpha}f\left(\frac{2ma}{1+m}\right)}{1+H\alpha} + (b-a)^{\alpha+1}\left(\frac{m}{1+m}\right)^{\alpha+1} \frac{mHf'\left(\frac{2a}{1+m}\right)}{(\alpha+1)(1+H\alpha)}\right]$$
$$= (b-a)^\alpha \left(\frac{1+m}{m}\right) H_m f\left(\frac{m(a+b)}{1+m}\right) - (b-a)^\alpha\left(\frac{1+m}{m}\right) H_m \frac{H\alpha f\left(\frac{m(a+b)}{1+m}\right)+f\left(\frac{2ma}{1+m}\right)}{1+H\alpha}$$
$$-(b-a)^{\alpha+1}\frac{mH_m H\alpha f'\left(\frac{2a}{1+m}\right)}{(\alpha+1)(1+H\alpha)}.$$

Considering $S_1 \leq S_2$ along with the above inequality implies the left inequality in (2.6). □

If in Theorem 2.5, we suppose $m$-convexity or $h$-convexity, we get the following consequence for Riemann–Liouville fractional integral $I^\alpha_{b-}f(a)$.

**Corollary 2.6.** *Let us consider a function $f$ which has an $m$-convex, $h$-convex or convex derivative.*

- *If $f' \in L^1_{loc}([a,b])$ is $m$-convex with $m \in (0,1]$, then we have*

$$(b-a)^\alpha \frac{\alpha\left(\frac{1+m}{m}\right)^2\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2ma}{1+m}\right)\right]+(\alpha+2)f(a)}{\alpha(\alpha+2)} - (b-a)^{\alpha+1}\frac{\alpha(1+m)f'\left(\frac{2a}{1+m}\right)}{2(\alpha+1)(\alpha+2)}$$
$$\leq \Gamma(\alpha)(I^\alpha_{b-}f(a)) \leq (b-a)^\alpha \frac{2f(a)+\alpha f(b)}{\alpha(\alpha+2)} + (b-a)^{\alpha+1}\frac{mf'\left(\frac{a}{m}\right)}{(\alpha+1)(\alpha+2)}.$$

- *If $f' \in L^1_{loc}([a,b])$ is $h$-convex, then inequality* (2.6) *coincides with* (1.3)*.*
- *If $f' \in L^1_{loc}([a,b])$ is convex, then inequality* (2.6) *coincides with* [1, Corollary 2.1]*.*

Moreover, if in Theorem 2.5, we suppose $\alpha = 1$ and different types of general convexity, we get the following corollary.

**Corollary 2.7.** *Let us consider a function $f$ which has an $(h,m)$-convex, $m$-convex, $h$-convex or convex derivative.*

- *If $f' \in L^1_{loc}([a,b])$ is $(h,m)$-convex function with $m \in (0,1]$, then we have*

$$\frac{\left(\frac{1+m}{m}\right)H_m\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2ma}{1+m}\right)\right]+(1+H)f(a)}{1+H} - \frac{mH_m H(b-a)f'\left(\frac{2a}{1+m}\right)}{2(1+H)} \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\leq \frac{f(a)+Hf(b)}{1+H} + \frac{mH(b-a)f'\left(\frac{a}{m}\right)}{2(1+H)},$$

*where $H_m = \left[h\left(\frac{m}{1+m}\right)+mh\left(\frac{1}{1+m}\right)\right]^{-1}$ and $H = \int_0^1 h(t)\,dt$.*

- *If $f' \in L^1_{loc}([a,b])$ is $m$-convex with $m \in (0,1]$, then we have*

$$\frac{\left(\frac{1+m}{m}\right)^2\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2ma}{1+m}\right)\right]+3f(a)}{3}-\frac{(1+m)(b-a)f'\left(\frac{2a}{1+m}\right)}{12}\le\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\le\frac{2f(a)+f(b)}{3}+\frac{m(b-a)f'\left(\frac{a}{m}\right)}{6}.$$

- *If $f' \in L^1_{loc}([a,b])$ is $h$-convex, then*

$$\frac{2f\left(\frac{a+b}{2}\right)+(-1+H)f(a)}{1+H}-\frac{H(b-a)f'(a)}{2(1+H)}\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+Hf(b)}{1+H}+\frac{H(b-a)f'(a)}{2(1+H)}.$$

- *If $f' \in L^1_{loc}([a,b])$ is convex, then inequality* (2.6) *coincides with* [8, Corollary 3].

**Theorem 2.8.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$ such that $f' \in L^1_{loc}([a,b])$ is $(h,m)$-convex on $[a,b]$, then the following inequality holds:*

$$(b-a)^\alpha\frac{H\alpha f(a)+f(b)}{\alpha(1+H\alpha)}-(b-a)^{\alpha+1}\frac{mHf'\left(\frac{b}{m}\right)}{(\alpha+1)(1+H\alpha)}\le\Gamma(\alpha)(I^\alpha_{a^+}f(b))$$
$$\le(b-a)^\alpha\frac{\left(\frac{1+m}{m}\right)H_m\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2mb}{1+m}\right)\right]+\left(\frac{1}{\alpha}+H\right)f(b)}{1+H\alpha}+(b-a)^{\alpha+1}\frac{mH_mH\alpha f'\left(\frac{2b}{1+m}\right)}{(\alpha+1)(1+H\alpha)},\tag{2.11}$$

*where $H_m=\left[h\left(\frac{m}{1+m}\right)+mh\left(\frac{1}{1+m}\right)\right]^{-1}$ and $H=\int_0^1 h(t)\,dt$.*

*Proof.* If in (2.7), we choose to write

$$\min\left\{f'(a)+mf'\left(\frac{b}{m}\right),f'(b)+mf'\left(\frac{a}{m}\right)\right\}=f'(a)+mf'\left(\frac{b}{m}\right),$$

then inequality (2.7) can be written as

$$H_mf'\left(\frac{m(a+b)}{1+m}\right)\le\frac{1}{b-a}\int_a^b f'(x)\,dx\le H\left[f'(a)+mf'\left(\frac{b}{m}\right)\right].$$

By replacing $a$ by $x$, multiplying all the three terms by $(b-x)^\alpha>0$ and then integrating with respect to $x$ over $[a,b]$, we have

$$T_1=H_m\int_a^b(b-x)^\alpha f'\left(\frac{m(x+b)}{1+m}\right)dx$$
$$\le T_2=\int_a^b(b-x)^{\alpha-1}\left(\int_x^b f'(t)\,dt\right)dx$$
$$\le T_3=H\int_a^b(b-x)^\alpha\left[f'(x)+mf'\left(\frac{b}{m}\right)\right]dx.$$

Some simplification yields

$$\begin{aligned}T_2&=\int_a^b(b-x)^{\alpha-1}\left(f(b)-f(x)\right)dx\\&=\frac{1}{\alpha}(b-a)^\alpha f(b)-\int_a^b(b-x)^{\alpha-1}f(x)\,dx,\end{aligned}\tag{2.12}$$

while, by integration by parts, we have

$$\begin{aligned} T_3 &= H\left[\int_a^b (b-x)^\alpha f'(x)\,dx + mf'\left(\frac{b}{m}\right)\int_a^b (b-x)^\alpha\,dx\right] \\ &= H\left[-(b-a)^\alpha f(a) + \alpha\int_a^b (b-x)^{\alpha-1} f(x)\,dx + mf'\left(\frac{b}{m}\right)\frac{(b-a)^{\alpha+1}}{\alpha+1}\right]. \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13) yields

$$\begin{aligned} &\frac{1}{\alpha}(b-a)^\alpha f(b) - H\Big[-(b-a)^\alpha f(a) + \alpha\int_a^b (b-x)^{\alpha-1} f(x)\,dx + mf'\left(\tfrac{b}{m}\right)\tfrac{(b-a)^{\alpha+1}}{\alpha+1}\Big] \\ &\le \int_a^b (b-x)^{\alpha-1} f(x)\,dx, \end{aligned}$$

from where we obtain

$$(b-a)^\alpha \frac{H\alpha f(a) + f(b)}{\alpha} - (b-a)^{\alpha+1}\frac{mHf'\left(\frac{b}{m}\right)}{(\alpha+1)} \le (1+H\alpha)\int_a^b (b-x)^{\alpha-1} f(x)\,dx, \tag{2.14}$$

hence we got the left inequality in (2.11).

After integration by parts and substituting $t = \frac{m(x+b)}{1+m}$ yield, we calculate $T_1$ as

$$\begin{aligned} T_1 &= H_m\left[\left(\tfrac{1+m}{m}\right)(b-x)^\alpha f\left(\tfrac{m(x+b)}{1+m}\right)\Big|_a^b + \alpha\left(\tfrac{1+m}{m}\right)\int_a^b (b-x)^{\alpha-1} f\left(\tfrac{m(x+b)}{1+m}\right)dx\right] \\ &= -(b-a)^\alpha\left(\tfrac{1+m}{m}\right)H_m f\left(\tfrac{m(a+b)}{1+m}\right) + \alpha\left(\tfrac{1+m}{m}\right)^{\alpha+1} H_m \int_{\frac{m(a+b)}{1+m}}^{\frac{2mb}{1+m}} \left(\tfrac{2mb}{1+m} - t\right)^{\alpha-1} f(t)\,dt. \end{aligned}$$

Using inequality (2.14) for $\frac{m(a+b)}{1+m}$ and $\frac{2mb}{1+m}$ instead of $a$ and $b$, we have

$$\begin{aligned} T_1 \ge &-(b-a)^\alpha\left(\frac{1+m}{m}\right)H_m f\left(\frac{m(a+b)}{1+m}\right) + \alpha\left(\frac{1+m}{m}\right)^{\alpha+1} H_m \\ &\times\left[(b-a)^\alpha\left(\frac{m}{1+m}\right)^\alpha \frac{H\alpha f\left(\frac{m(a+b)}{1+m}\right) + f\left(\frac{2mb}{1+m}\right)}{\alpha(1+H\alpha)}\right. \\ &\left.-(b-a)^{\alpha+1}\left(\frac{m}{1+m}\right)^{\alpha+1}\frac{mHf'\left(\frac{2b}{1+m}\right)}{(\alpha+1)(1+H\alpha)}\right] \\ = &-(b-a)^\alpha\left(\frac{1+m}{m}\right)H_m f\left(\frac{m(a+b)}{1+m}\right) \\ &+(b-a)^\alpha\left(\frac{1+m}{m}\right)H_m\frac{H\alpha f\left(\frac{m(a+b)}{1+m}\right) + f\left(\frac{2mb}{1+m}\right)}{1+H\alpha} \end{aligned}$$

$$- (b-a)^{\alpha+1}\alpha H_m \frac{mHf'\left(\frac{2b}{1+m}\right)}{(\alpha+1)(1+H\alpha)}.$$

Considering $T_1 \le T_2$ along with the above inequality implies the right inequality in (2.11). □

If in Theorem 2.8, we suppose $m$-convexity or $h$-convexity, we get the following consequence for Riemann–Liouville fractional integral $I^{\alpha}_{a^+}f(b)$.

**Corollary 2.9.** *Let us consider a function $f$ which has an $m$-convex, $h$-convex or convex derivative.*

- *If $f' \in L^1_{loc}([a,b])$ is $m$-convex with $m \in (0,1]$, then we have*

$$(b-a)^{\alpha}\tfrac{\alpha f(a)+2f(b)}{\alpha(\alpha+2)} - (b-a)^{\alpha+1}\tfrac{mf'\left(\frac{b}{m}\right)}{(\alpha+1)(\alpha+2)} \le \Gamma(\alpha)(I^{\alpha}_{a^+}f(b))$$
$$\le (b-a)^{\alpha}\tfrac{\alpha\left(\frac{1+m}{m}\right)^2\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2mb}{1+m}\right)\right]+(\alpha+2)f(b)}{\alpha(\alpha+2)} + (b-a)^{\alpha+1}\tfrac{\alpha(1+m)f'\left(\frac{2b}{1+m}\right)}{2(\alpha+1)(\alpha+2)}.$$

- *If $f' \in L^1_{loc}([a,b])$ is $h$-convex, then inequality* (2.11) *coincides with* (1.4).
- *If $f' \in L^1_{loc}([a,b])$ is convex, then inequality* (2.11) *coincides with* [1, Corollary 2.3].

Moreover, if in Theorem 2.8, we suppose $\alpha = 1$ and different types of general convexity, we get the following corollary.

**Corollary 2.10.** *Let us consider a function $f$ which has an $(h,m)$-convex, $m$-convex, $h$-convex or convex derivative.*

- *If $f' \in L^1_{loc}([a,b])$ is $(h,m)$-convex function with $m \in (0,1]$, then we have*

$$\tfrac{Hf(a)+f(b)}{1+H} + \tfrac{mH(b-a)f'\left(\frac{b}{m}\right)}{2(1+H)} \le \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\le \tfrac{\left(\frac{1+m}{m}\right)H_m\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2mb}{1+m}\right)\right]+(1+H)f(b)}{1+H} + \tfrac{mH_mH(b-a)f'\left(\frac{2b}{1+m}\right)}{2(1+H)},$$

*where $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right)\right]^{-1}$ and $H = \int_0^1 h(t)\,dt$.*

- *If $f' \in L^1_{loc}([a,b])$ is $m$-convex, then we have*

$$\tfrac{f(a)+2f(b)}{3} - \tfrac{m(b-a)f'\left(\frac{b}{m}\right)}{6} \le \tfrac{1}{b-a}\int_a^b f(x)\,dx$$
$$\le \tfrac{\left(\frac{1+m}{m}\right)^2\left[f\left(\frac{m(a+b)}{1+m}\right)-f\left(\frac{2mb}{1+m}\right)\right]+3f(b)}{3} + \tfrac{(1+m)(b-a)f'\left(\frac{2b}{1+m}\right)}{12}.$$

- *If $f' \in L^1_{loc}([a,b])$ is $h$-convex, then*

$$\tfrac{Hf(a)+f(b)}{1+H} - \tfrac{H(b-a)f'(b)}{2(1+H)} \le \tfrac{1}{b-a}\int_a^b f(x)\,dx \le \tfrac{2f\left(\frac{a+b}{2}\right)+(-1+H)f(b)}{1+H} + \tfrac{H(b-a)f'(b)}{2(1+H)}.$$

- *If* $f' \in L^1_{loc}([a,b])$ *is convex, then inequality* (2.11) *coincides with* [1, Corollary 2.4].

## 3. Conclusions

We present two examples which demonstrate the usability of our main result. The first example includes an $m$-convex function for which we calculate the bounds for the sum of the Riemann–Liouville fractional integrals. The second one includes a function with $m$-convex derivative.

**Example 3.1.** Let us consider the function $f : [0,\infty) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{12}(x^4 - 5x^3 + 9x^2 - 5x),$$

which is known to be $\frac{16}{17}$-convex (see [20]), but not convex or concave.

Applying Corollary 2.2 on the interval $[0,3]$ yields

$$\begin{aligned}
&\frac{1+m}{m} f\left(\frac{m(a+b)}{1+m}\right) = \frac{33}{16} f\left(\frac{48}{33}\right) = 0.1474 \\
&\leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{a+} f(b) + I^\alpha_{b-} f(a)\right] = \frac{\alpha}{3^\alpha}\left[\int_0^3 (3-t)^{\alpha-1} f(t)\,dt + \int_0^3 t^{\alpha-1} f(t)\,dt\right] \\
&\leq \frac{\alpha}{\alpha+1}[f(a)+f(b)] + \frac{m}{\alpha+1}\left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right] = \frac{\alpha+1.318}{\alpha+1}
\end{aligned}$$

and by applying Corollary 2.4 on $[0,3]$, we have

$$\begin{aligned}
&\frac{1+m}{2m} f\left(\frac{m(a+b)}{1+m}\right) = 0.0737 \\
&\leq \frac{1}{b-a}\int_a^b f(x)\,dx = 0.1625 \\
&\leq \frac{1}{2}\min\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\} = \frac{1}{2}\min\{1.3181, 1\} = 0.5.
\end{aligned}$$

**Example 3.2.** Let us consider the function $g : [0,\infty) \to \mathbb{R}$ given by

$$g(x) = \frac{1}{60}x^5 - \frac{5}{48}x^4 + \frac{1}{4}x^3 - \frac{5}{24}x^2,$$

which has a $\frac{16}{17}$-convex derivative $g'(x) = f(x)$, where $f$ is from Example 3.1.

Applying Corollary 2.6 for $g$ on $[0,3]$ yields

$$\begin{aligned}
&(b-a)^\alpha \frac{\alpha\left(\frac{1+m}{m}\right)^2\left[g\left(\frac{m(a+b)}{1+m}\right) - g\left(\frac{2ma}{1+m}\right)\right] + (\alpha+2)g(a)}{\alpha(\alpha+2)} \\
&\quad - (b-a)^{\alpha+1}\frac{\alpha(1+m)g'\left(\frac{2a}{1+m}\right)}{2(\alpha+1)(\alpha+2)} = \frac{-0.1241}{\alpha+2}\,3^\alpha \leq \Gamma(\alpha)(I^\alpha_{b-} g(a)) = \int_0^3 t^{\alpha-1} g(t)\,dt \\
&\leq (b-a)^\alpha \frac{2g(a)+\alpha g(b)}{\alpha(\alpha+2)} + (b-a)^{\alpha+1}\frac{mg'\left(\frac{a}{m}\right)}{(\alpha+1)(\alpha+2)} = \frac{0.4875}{\alpha+2}\,3^\alpha,
\end{aligned}$$

and by applying Corollary 2.7 for $g$ on $[0,3]$, we have

$$\frac{\left(\frac{1+m}{m}\right)^2\left[g\left(\frac{m(a+b)}{1+m}\right)-g\left(\frac{2ma}{1+m}\right)\right]+3g(a)}{3}-\frac{(1+m)(b-a)g'\left(\frac{2a}{1+m}\right)}{12}=-0.0414$$

$$\leq\frac{1}{b-a}\int_a^b g(x)\,dx=0.05\leq\frac{2g(a)+g(b)}{3}+\frac{m(b-a)g'\left(\frac{a}{m}\right)}{6}=0.1625.$$

Similarly, by applying Corollary 2.9 for $g$ on $[0,3]$, we obtain

$$(b-a)^\alpha\frac{\alpha g(a)+2g(b)}{\alpha(\alpha+2)}-(b-a)^{\alpha+1}\frac{mg'\left(\frac{b}{m}\right)}{(\alpha+1)(\alpha+2)}=\frac{0.975}{\alpha(\alpha+2)}3^\alpha-\frac{1.3181}{(\alpha+1)(\alpha+2)}3^{\alpha+1}$$

$$\leq\Gamma(\alpha)(I_{a^+}^\alpha g(b))=\int_0^3(3-t)^{\alpha-1}g(t)\,dt$$

$$\leq(b-a)^\alpha\frac{\alpha\left(\frac{1+m}{m}\right)^2\left[g\left(\frac{m(a+b)}{1+m}\right)-g\left(\frac{2mb}{1+m}\right)\right]+(\alpha+2)g(b)}{\alpha(\alpha+2)}+(b-a)^{\alpha+1}\frac{\alpha(1+m)g'\left(\frac{2b}{1+m}\right)}{2(\alpha+1)(\alpha+2)}$$

$$=\frac{-1.3544\,\alpha+0.975}{\alpha(\alpha+2)}3^\alpha+\frac{1.1448\,\alpha}{(\alpha+1)(\alpha+2)}3^{\alpha+1},$$

and Corollary 2.10 yields

$$\frac{g(a)+2g(b)}{3}-\frac{m(b-a)g'\left(\frac{b}{m}\right)}{6}=-0.334$$

$$\leq\frac{1}{b-a}\int_a^b g(x)\,dx=0.05$$

$$\leq\frac{\left(\frac{1+m}{m}\right)^2\left[g\left(\frac{m(a+b)}{1+m}\right)-g\left(\frac{2mb}{1+m}\right)\right]+3g(b)}{3}+\frac{(1+m)(b-a)g'\left(\frac{2b}{1+m}\right)}{12}=0.446.$$

We note that the extension of our main results in case of other fractional integrals, such as Katugampola or Raina remains open.

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