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Generalization and refinement of fractional Hermite-Hadamard type inequalities for $m-$convex functions

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Abstract. Fractional Hermite-Hadamard type inequalities are recognized as significant results in the field of convex analysis. In this work, we derive several inequalities of this type for twice differentiable m-convex functions by employing various analytical methods, including the Hölder–İşcan inequality and the improved power mean integral inequality.

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"It seems to me that the notion of a convex function is just as fundamental as that of a positive or an increasing function. If I am not mistaken, this concept deserves a place in elementary treatments of real function theory."

– J.L.W.V. Jensen

1. Introduction

The Hermite-Hadamard inequality is a classical result in convex analysis that provides a double inequality for the integral average of a convex function over an interval. Given its simplicity and elegance, the Hermite-Hadamard inequality has been widely studied and extended in various directions, including discrete, multidimensional, and fractional settings.

In recent years, the development of fractional calculus, a generalization of classical calculus involving derivatives and integrals of arbitrary (non-integer) order, has

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led to renewed interest in extending classical inequalities to the fractional context. This inequality has opened new avenues for applications in mathematical analysis, applied sciences, and engineering, where fractional models often provide more accurate descriptions of memory and hereditary properties.

Parallel to these advancements, the concept of m-convex functions, introduced as a generalization of traditional convex functions, has gained significant attention. As introduced by Toader [20], a function $\xi : [0, \infty) \to \mathbb{R}$ is said to be m-convex for $m \in [0, 1]$ if it satisfies the inequality

$$\xi(v\zeta_1 + m(1-v)\zeta_2) \le v\xi(\zeta_1) + m(1-v)\xi(\zeta_2),$$

for all $\zeta_1, \zeta_2 \in [0, \infty)$ and $v \in [0, 1]$. This class bridges the gap between convex and star-shaped functions, making it highly useful for extending various integral inequalities.

This article focuses on establishing fractional Hermite-Hadamard type inequalities for twice differentiable m-convex functions, employing tools from fractional calculus such as Riemann–Liouville fractional integrals. These new results not only generalize classical Hermite-Hadamard inequalities but also provide refined bounds under the broader assumption of m-convexity. Such generalizations are valuable for both theoretical research and practical applications in areas where fractional models and generalized convexity play a key role, for further investigations see ([1]−[9] and [14]−[18]).

In [12], İşcan obtained the following inequality named as Hölder − İşcan integral inequality which gives better results than the classical Hölder's integral inequality [16] is stated as:

Theorem 1.1. *Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If ξ_1 and ξ_2 are real functions defined on $[\varpi_1, \varpi_2]$ and if $|\xi_1|^p$ and $|\xi_2|^q$ are integrable on $[\varpi_1, \varpi_2]$, then*

$$\int_{\varpi_1}^{\varpi_2} |\xi_1(u)\xi_2(u)|du \le \frac{1}{\varpi_2 - \varpi_1} \left[\left(\int_{\varpi_1}^{\varpi_2} (\varpi_2 - u)|\xi_1(u)|^p du \right)^{\frac{1}{p}} \left(\int_{\varpi_1}^{\varpi_2} (\varpi_2 - u)|\xi_2(u)|^q du \right)^{\frac{1}{q}} + \left(\int_{\varpi_1}^{\varpi_2} (u - \varpi_1)|\xi_1(u)|^p du \right)^{\frac{1}{p}} \left(\int_{\varpi_1}^{\varpi_2} (u - \varpi_1)|\xi_2(u)|^q du \right)^{\frac{1}{q}} \right]. \tag{1.1}$$

Remark 1.2. Note that if we put $p = q = 2$, the above inequality gives us improved Cauchy−Schwarz integral inequality.

A different representation of Hölder−İşcan integral inequality is stated as:

Theorem 1.3. *Let ξ_1, ξ_2 are real valued functions defined on $[\varpi_1, \varpi_2]$ and if $|\xi_1|$ and $|\xi_1||\xi_2|^q$ are integrable on $[\varpi_1, \varpi_2]$, then for $q \geq 1$ we have:*

$$\int_{\varpi_1}^{\varpi_2} |\xi_1(u)\xi_2(u)|du \leq$$

$$\leq \frac{1}{\varpi_2 - \varpi_1}\left[\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2 - u)|\xi_1(u)|du\right)^{1-\frac{1}{q}}\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2 - u)|\xi_1(u)||\xi_2(u)|^q du\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\varpi_1}^{\varpi_2}(u - \varpi_1)|\xi_1(u)|du\right)^{1-\frac{1}{q}}\left(\int_{\varpi_1}^{\varpi_2}(u - \varpi_1)|\xi_1(u)||\xi_2(u)|^q du\right)^{\frac{1}{q}}\right]. \quad (1.2)$$

The above inequality is known as Improved power mean integral inequality (see [13]), which is the refinement of power mean integral inequality [18].

Now, we are going to recall some necessary definitions and mathematical results related to fractional calculus which will be use further in this article.

Definition 1.4. [19] *Let $\xi \in L[\varpi_1, \varpi_2]$. The Riemann−Liouville integrals $J^{\alpha}_{\varpi_1+}\xi(\zeta)$ and $J^{\alpha}_{\varpi_2-}\xi(\zeta)$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{\varpi_1+}\xi(\zeta) = \frac{1}{\Gamma(\alpha)}\int_{\varpi_1}^{\zeta}(\zeta - v)^{\alpha-1}\xi(v)dv, \quad \zeta > \varpi_1$$

and

$$J^{\alpha}_{\varpi_2-}\xi(\zeta) = \frac{1}{\Gamma(\alpha)}\int_{\zeta}^{\varpi_2}(v - \zeta)^{\alpha-1}\xi(v)dv, \quad \zeta < \varpi_2,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$ is the Gamma function.

Remark 1.5. Note that if we take $\alpha = 0$, then $J^{0}_{\varpi_1+}\xi(\zeta) = J^{0}_{\varpi_2-}\xi(\zeta) = \xi(\zeta)$ and if we take $\alpha = 1$, then the fractional integral reduces to the classical one.

Recently, we have proved the following two distinct results related to fractional Hermite Hadamard type inequality for the class of twice differentiable $m-$convex functions (see [6]).

Theorem 1.6. *Let $\xi : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some $m \in (0, 1]$ and $q \geq 1$ then the following inequality for* *fractional integrals with* $\alpha > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right| \leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\beta^{\frac{1}{p}}(p+1,\alpha p+1)$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}\right], \qquad (1.3)$$

where β *is the Euler Beta function.*

Theorem 1.7. *Let* $\xi : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I°*. Assume that* $q \geq 1$*, such that* $|\xi''|^q$ *is convex function on* I*. Suppose that* $\varpi_1, \varpi_2 \in I^\circ$ *with* $\varpi_1 < \varpi_2$ *and* $\xi'' \in L[\varpi_1, \varpi_2]$*, then the below stated inequality for fractional integrals with* $\alpha > 0$ *holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{\alpha(\varpi_2-\varpi_1)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|\xi''(\varpi_1)|^q + \frac{\alpha+5}{3\alpha+9}|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{\alpha+5}{3\alpha+9}|\xi''(\varpi_1)|^q + \frac{2\alpha+4}{3\alpha+9}|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right]. \qquad (1.4)$$

The structure of this article unfolds as follows: The subsequent section contains some results related to generalization and refinements of fractional Hermite Hadamard type inequalities applicable to the category of twice differentiable $m-$convex functions. Our approach will leverage diverse techniques, encompassing Hölder-İşcan and improved power mean integral inequalities. These findings are anticipated to exhibit a broader scope compared to those presented in [6], [9], [10] and [11]. The third section will provide a concluding statement, while the final section will offer insights and future prospects for readers interested in further exploration.

2. Various estimations for the right bound of fractional Hermite Hadamard type inequalities for twice differentiable $m-$Convex functions

In order to prove our main results we need to recall following lemma from [10].

Lemma 2.1. *Let* $\xi : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I°*, the interior of* I*. Assume that* $\varpi_1, \varpi_2 \in I^\circ$ *with* $\varpi_1 < \varpi_2$ *and* $\xi'' \in L[\varpi_1, \varpi_2]$*, then the below stated* *identity for fractional integral with $\alpha > 0$ holds:*

$$\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]$$
$$= \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 v(1-v^\alpha)[\xi''(v\varpi_1+(1-v)\varpi_2)+\xi''((1-v)\varpi_1+v\varpi_2)]dv, \quad (2.1)$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$ *is the Gamma function.*

Now, we are going to state and prove our new results related to fractional Hermite Hadamard type inequalities for twice differentiable $m-$convex functions.

Theorem 2.2. *Let $\xi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1,\varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some $m \in (0,1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ and $\frac{1}{p}+\frac{1}{q}=1$ holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left\{(\beta(p+1,\alpha p+1)-\beta(p+2,\alpha p+1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p+2,\alpha p+1)\right\}$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q + 2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right]. \quad (2.2)$$

Proof. By using Lemma 2.1 and the property of absolute value, we have,

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 |v(1-v^\alpha)|[|\xi''(v\varpi_1+(1-v)\varpi_2)|+|\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \quad (2.3)$$

Applying (1.1) to $\int_0^1 |v(1-v^{\alpha})||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$ and $\int_0^1 |v(1-v^{\alpha})||\xi''((1-v)\varpi_1+v\varpi_2)|dv$ implies

$$\begin{aligned}&\int_0^1 |v(1-v^{\alpha})||\xi''(v\varpi_1+(1-v)\varpi_2)|dv\\ &\leq \left[\left(\int_0^1 (1-v)|v(1-v^{\alpha})|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-v)|\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\int_0^1 v|v(1-v^{\alpha})|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 v|\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}\right]\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 |v(1-v^{\alpha})||\xi''((1-v)\varpi_1+v\varpi_2)|dv\\ &\leq \left[\left(\int_0^1 (1-v)|v(1-v^{\alpha})|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-v)|\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\int_0^1 v|v(1-v^{\alpha})|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 v|\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}\right]\end{aligned}$$

As we have $|\xi''|^q$ is an $m-$convex function, so we can take

$$|\xi''\left(v\varpi_1+(1-v)\varpi_2\right)|^q \leq v\left|\xi''\left(\varpi_1\right)\right|^q + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q$$

and

$$|\xi''\left((1-v)\varpi_1+v\varpi_2\right)|^q \leq (1-v)|\xi''(\varpi_1)|^q + mv\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q.$$

Utilizing the above four results (2.3) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1^+}\xi(\varpi_2)+J^{\alpha}_{\varpi_2^-}\xi(\varpi_1)]\right|$$

$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left[\left(\int_0^1 v^p(1-v^{\alpha})^p dv-\int_0^1 v^{p+1}(1-v^{\alpha})^p dv\right)^{\frac{1}{p}}\right.$$

$$\times\left\{\left(|\xi''(\varpi_1)|^q\int_0^1 v(1-v)dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1(1-v)^2dv\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(|\xi''(\varpi_1)|^q\int_0^1(1-v)^2dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v(1-v)dv\right)^{\frac{1}{q}}\right\}$$

$$+\left(\int_0^1 v^{p+1}(1-v^{\alpha})^p dv\right)^{\frac{1}{p}}$$

$$\times\left\{\left(|\xi''(\varpi_1)|^q\int_0^1 v^2dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v(1-v)dv\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(|\xi''(\varpi_1)|^q\int_0^1 v(1-v)dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v^2dv\right)^{\frac{1}{q}}\right\}\right].$$

After using the following facts, the result of Theorem 2.2 is accomplished.

$$\int_0^1 v^p(1-v^{\alpha})^p dt \leq \int_0^1 v^p(1-v)^{\alpha p}dv=\beta(p+1,\alpha p+1),$$

$$\int_0^1 v^{p+1}(1-v^{\alpha})^p dt \leq \int_0^1 v^{p+1}(1-v)^{\alpha p}dv=\beta(p+2,\alpha p+1),$$

$$\int_0^1 v(1-v)dv=\frac{1}{6}$$

and

$$\int_0^1 v^2dv=\int_0^1(1-v)^2dv=\frac{1}{3}.$$

□

Remark 2.3. If we choose $\alpha = m = 1$ in Theorem 2.2, then we get Theorem 2.1 of [9].

Corollary 2.4. *Under the assumptions of Theorem* 2.2*,*

1. *If we put $p = q = 2$, then we get the refined result obtained by using Cauchy–Schwarz integral inequality as:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left\{(\beta(3,2\alpha+1)-\beta(4,2\alpha+1))^{\frac{1}{2}}+\beta^{\frac{1}{2}}(4,2\alpha+1)\right\}$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^2+2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^2}{6}\right)^{\frac{1}{2}}+\left(\frac{2|\xi''(\varpi_1)|^2+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^2}{6}\right)^{\frac{1}{2}}\right].$$

which is the refinement of first result of Corollary 2 *of* [6].

2. *If we choose $m = 1$, then we get following refined result related to fractional Hermite Hadamard type inequality for twice differentiable convex function:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left\{(\beta(p+1,\alpha p+1)-\beta(p+2,\alpha p+1))^{\frac{1}{p}}+\beta^{\frac{1}{p}}(p+2,\alpha p+1)\right\}$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q+2\,|\xi''\,(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\xi''(\varpi_1)|^q+|\xi''\,(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}\right]$$

which is the refinement of third result of Theorem 3 *of* [10].

3. *If we choose $\alpha = 1$, then we get following refined result related to Hermite Hadamard type inequality for twice differentiable $m-$convex function:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{4}\left\{(\beta(p+1,p+1)-\beta(p+2,p+1))^{\frac{1}{p}}+\beta^{\frac{1}{p}}(p+2,p+1)\right\}$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q+2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\xi''(\varpi_1)|^q+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right].$$

which is the refinement of third result of Corollary 2 *of* [6].

Theorem 2.5. *Here we claim that inequality* (2.2) *of Theorem* 2.2 *is better than the inequality* (1.6).

Proof. Since the function $h : [0,\infty) \to \mathbb{R}$, $h(\zeta) = \zeta^n$, $n \in (0,1]$ is concave, we can write:

$$\frac{{\varpi_1}^n + {\varpi_2}^n}{2} = \frac{h(\varpi_1) + h(\varpi_2)}{2} \leq h\left(\frac{\varpi_1 + \varpi_2}{2}\right) = \left(\frac{\varpi_1 + \varpi_2}{2}\right)^n \tag{2.4}$$

$\forall\ \varpi_1, \varpi_2 \geq 0$. For the above inequality if we choose

$$\varpi_1 = \frac{|\xi''(\varpi_1)|^q + 2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}, \quad \varpi_2 = \frac{2|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}$$

and $n = \frac{1}{q}$. Applying the inequality (2.4) to the inequality (2.2), we have

$$\begin{aligned}
&\left|\frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^\alpha}[J^\alpha_{\varpi_1^+}\xi(\varpi_2) + J^\alpha_{\varpi_2^-}\xi(\varpi_1)]\right| \\
&\leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha+1)}\left\{(\beta(p+1,\alpha p+1) - \beta(p+2,\alpha p+1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p+2,\alpha p+1)\right\} \\
&\times \left[\left(\frac{|\xi''(\varpi_1)|^q + 2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(\varpi_2 - \varpi_1)^2}{(\alpha+1)}\left\{(\beta(p+1,\alpha p+1) - \beta(p+2,\alpha p+1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p+2,\alpha p+1)\right\} \\
&\times \left[\frac{1}{2}\left(\frac{|\xi''(\varpi_1)|^q + 2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q + 2|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}\right)\right]^{\frac{1}{q}} \\
&= \frac{(\varpi_2 - \varpi_1)^2}{2^{\frac{1}{q}}(\alpha+1)}\left\{(\beta(p+1,\alpha p+1) - \beta(p+2,\alpha p+1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p+2,\alpha p+1)\right\} \\
&\times \left[\frac{|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Again applying the inequality (2.4) to the above inequality by taking

$$\varpi_1 = \beta(p+1,\alpha p+1) - \beta(p+2,\alpha p+1), \quad \varpi_2 = \beta(p+2,\alpha p+1)$$

and $n = \frac{1}{p}$, we have

$$\begin{aligned}
&\left|\frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^\alpha}[J^\alpha_{\varpi_1^+}\xi(\varpi_2) + J^\alpha_{\varpi_2^-}\xi(\varpi_1)]\right| \\
&\leq \frac{(\varpi_2 - \varpi_1)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left(\frac{|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.5}$$

Similarly. we can write

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq\frac{(\varpi_2-\varpi_1)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left(\frac{m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q+|\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.6}$$

By adding (2.5) and (2.6) we get our required result i.e., inequality (1.6). □

Theorem 2.6. *Let $\xi : I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1,\varpi_2\in I^\circ$ with $\varpi_1<\varpi_2$ and $\xi''\in L[\varpi_1,\varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some $m\in(0,1]$ and $q\geq 1$ then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq\frac{\alpha(\varpi_2-\varpi_1)^2}{12(\alpha+1)(\alpha+3)(\alpha+4)}\left[\left(\frac{(\alpha+4)(\alpha+5)}{(\alpha+2)}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left\{\left(\frac{(\alpha+7)}{2}|\xi''(\varpi_1)|^q+m\frac{(\alpha^2+9\alpha+26)}{2(\alpha+2)}\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\alpha^2+9\alpha+26)}{2(\alpha+2)}|\xi''(\varpi_1)|^q+m\frac{(\alpha+7)}{2}\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}$$
$$+(2(\alpha+4))^{1-\frac{1}{q}}\left\{\left(\frac{3(\alpha+3)}{2}|\xi''(\varpi_1)|^q+m\frac{(\alpha+7)}{2}\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{(\alpha+7)}{2}|\xi''(\varpi_1)|^q+m\frac{3(\alpha+3)}{2}\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}\right]. \tag{2.7}$$

Proof. By using Lemma 2.1 and the property of absolute value, we have,

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq\frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1|v(1-v^\alpha)|[|\xi''(v\varpi_1+(1-v)\varpi_2)|+|\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \tag{2.8}$$

Applying (1.2) to $\int_0^1 |v(1-v^{\alpha})||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$ and $\int_0^1 |v(1-v^{\alpha})||\xi''((1-v)\varpi_1+v\varpi_2)|dv$ implies

$$\begin{aligned}
&\int_0^1 |v(1-v^{\alpha})||\xi''(v\varpi_1+(1-v)\varpi_2)|dv \\
&\leq \left[\left(\int_0^1 v(1-v)(1-v^{\alpha})dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v(1-v)(1-v^{\alpha})|\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1 v(v(1-v^{\alpha}))dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v(v(1-v^{\alpha}))|\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |v(1-v^{\alpha})||\xi''((1-v)\varpi_1+v\varpi_2)|dv \\
&\leq \left[\left(\int_0^1 v(1-v)(1-v^{\alpha})dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v(1-v)(1-v^{\alpha})|\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1 v(v(1-v^{\alpha}))dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v(v(1-v^{\alpha}))|\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}\right]
\end{aligned}$$

As we have $|\xi''|^q$ is an $m-$convex function, so we can take

$$|\xi''(v\varpi_1+(1-v)\varpi_2)|^q \leq v|\xi''(\varpi_1)|^q + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q$$

and

$$|\xi''((1-v)\varpi_1+v\varpi_2)|^q \leq (1-v)|\xi''(\varpi_1)|^q + mv\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q.$$

Utilizing the above four results (2.8) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$

$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left[\left(\int_0^1 v(1-v^\alpha)dv-\int_0^1 v^2(1-v^\alpha)dv\right)^{1-\frac{1}{q}}\right.$$

$$\times\left\{\left(|\xi''(\varpi_1)|^q\int_0^1 v^2(1-v)(1-v^\alpha)dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v(1-v)^2(1-v^\alpha)dv\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(|\xi''(\varpi_1)|^q\int_0^1 v(1-v)^2(1-v^\alpha)dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v^2(1-v)(1-v^\alpha)dv\right)^{\frac{1}{q}}\right\}$$

$$+\left(\int_0^1 v^2(1-v^\alpha)dv\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(|\xi''(\varpi_1)|^q\int_0^1 v^3(1-v^\alpha)dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v^2(1-v)(1-v^\alpha)dv\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(|\xi''(\varpi_1)|^q\int_0^1 v^2(1-v)(1-v^\alpha)dv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v^3(1-v^\alpha)dv\right)^{\frac{1}{q}}\right\}\right].$$

After arranging and using the following facts the result of Theorem 2.6 is accomplished.

$$\int_0^1 v(1-v^\alpha)dv=\frac{\alpha}{2(\alpha+2)},$$

$$\int_0^1 v^2(1-v^\alpha)dv=\frac{\alpha}{3(\alpha+3)},$$

$$\int_0^1 v^2(1-v)(1-v^\alpha)dv=\frac{\alpha(\alpha+7)}{12(\alpha+3)(\alpha+4)},$$

$$\int_0^1 v(1-v)^2(1-v^\alpha)dv=\frac{\alpha(\alpha^2+9\alpha+26)}{12(\alpha+3)(\alpha+4)}$$

and

$$\int_0^1 v^3(1-v^\alpha)dv = \frac{\alpha}{4(\alpha+4)}.$$

□

Remark 2.7. If we choose $\alpha = m = 1$ in Theorem 2.6, then we get Theorem 2.3 of [9].

Corollary 2.8. *Under the assumptions of Theorem* 2.6*,*

1. *If we choose $m = 1$, then we get following refined result related to fractional Hermite Hadamard type inequality for twice differentiable convex function:*

$$\begin{aligned}
&\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right| \\
&\leq \frac{\alpha(\varpi_2-\varpi_1)^2}{12(\alpha+1)(\alpha+3)(\alpha+4)}\left[\left(\frac{(\alpha+4)(\alpha+5)}{(\alpha+2)}\right)^{1-\frac{1}{q}}\right. \\
&\times\left\{\left(\frac{(\alpha+7)}{2}|\xi''(\varpi_1)|^q + \frac{(\alpha^2+9\alpha+26)}{2(\alpha+2)}\left|\xi''\left(\varpi_2\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&+\left.\left(\frac{(\alpha^2+9\alpha+26)}{2(\alpha+2)}|\xi''(\varpi_1)|^q + \frac{(\alpha+7)}{2}\left|\xi''\left(\varpi_2\right)\right|^q\right)^{\frac{1}{q}}\right\} \\
&+(2(\alpha+4))^{1-\frac{1}{q}}\left\{\left(\frac{3(\alpha+3)}{2}|\xi''(\varpi_1)|^q + \frac{(\alpha+7)}{2}\left|\xi''\left(\varpi_2\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&+\left.\left.\left(\frac{(\alpha+7)}{2}|\xi''(\varpi_1)|^q + \frac{3(\alpha+3)}{2}\left|\xi''\left(\varpi_2\right)\right|^q\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}$$

2. *If we choose $\alpha = 1$, then we get following Hermite Hadamard type inequality for twice differentiable $m-$convex function:*

$$\begin{aligned}
&\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{24} \\
&\times\left[\left(\frac{2|\xi''(\varpi_1)|^q + 3m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{5}\right)^{\frac{1}{q}} + \left(\frac{3|\xi''(\varpi_1)|^q + 2m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{5}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is the refinement of Corollary 3 *of* [6].

Theorem 2.9. *Under the same assumptions of Theorem* 2.6*, we have:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}\left[J^{\alpha}_{\varpi_1+}\xi(\varpi_2)+J^{\alpha}_{\varpi_2-}\xi(\varpi_1)\right]\right|$$
$$\leq \frac{\alpha(\varpi_2-\varpi_1)^2}{4(\alpha+1)(\alpha+2)}$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q+|\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}\right]. \quad (2.9)$$

Proof. The proof of the above theorem will be followed in the same way as in Theorem 2.5 by applying inequality (2.4) to the result of Theorem 2.6. □

Remark 2.10. Following well-known results would be captured as special cases of our obtained result by varying different values of m and α in Theorem 2.9:

1. If we choose $\alpha = 1$, then we get Corollary 3 of [6].
2. If we choose $\alpha = m = 1$, then we get Remark 13 of [11].

Corollary 2.11. *If we choose* $m = 1$ *in Theorem* 2.9*, then we get following refined result related to fractional Hermite Hadamard type inequality for twice differentiable convex function:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}\left[J^{\alpha}_{\varpi_1+}\xi(\varpi_2)+J^{\alpha}_{\varpi_2-}\xi(\varpi_1)\right]\right|$$
$$\leq \frac{\alpha(\varpi_2-\varpi_1)^2}{2(\alpha+1)(\alpha+2)}\left(\frac{|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}},$$

which is the same result as we have obtained by applying (2.4) *to the result of Theorem* 1.7*.*

3. Conclusion

The fractional Hermite-Hadamard inequality stands out as one of the most renowned inequality within the realm of inequalities, boasting numerous generalizations across different classes of convex functions in existing literature. In this article, we have presented a comprehensive study on the generalization and refinement of fractional Hermite-Hadamard type inequalities for the class of twice differentiable m-convex functions. The main results, outlined in Section 2, extend classical inequalities by incorporating the framework of fractional calculus and the broader notion of m-convexity. These results not only generalize the existing Hermite-Hadamard type inequalities but also provide sharper and more refined bounds compared to those available in the current literature, i.e., [6], [9], [10] and [11]. Finally, last section discusses several promising directions for future research.

Overall, this study contributes significantly to the growing body of work in fractional integral inequalities and convex analysis, opening up new possibilities for both theoretical exploration and practical implementation.

Now, we are going to summarize the results of Section 2 in tabular form.

3.1. Results Summary of Section 2

S. No	m	α	**Results**	**Found in**
1	1	–	FHHTI for Convex Functions	[10]This Article
2	–	1	HHTI for $m-$Convex Functions	[6], This Article
3	1	1	HHTI for Convex Functions	[9, 11]

TABLE 1. Result Summary of Section 2

In the above table FHHTI, HHTI and – stands for Fractional Hermite-Hadamard type inequality, Hermite Hadamard type inequality and for any value, respectively.

4. Remarks and future ideas

In this section, we present several observations and propose potential avenues for future research based on the results established in this article.

1. All inequalities discussed throughout this study can be reformulated in the reverse direction for concave functions, utilizing the well-known equivalence that a function ξ is concave if and only if $-\xi$ is convex.
2. An interesting direction for future work is the extension of fractional Hermite-Hadamard type inequalities by incorporating weights, thereby deriving Fejér-type inequalities.
3. Similar analytical frameworks may be applied to explore other generalized classes of convex functions beyond the m-convex class considered here.
4. The results presented in this work could also be adapted to the discrete setting, providing analogous inequalities in discrete calculus.
5. Another promising extension involves generalizing these results to multidimensional settings, which may broaden their applicability.
6. The theory can further be developed within the framework of time scale calculus or quantum calculus, allowing for unified treatment of discrete and continuous cases.
7. Lastly, the ideas and methods employed in this article may be extended to the context of fuzzy analysis, opening new perspectives in uncertainty modeling and fuzzy optimization.

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