

Completely inert subgroups of abelian groups

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Dedicated to Patrick W. Keef on the occasion of his 70th birthday

Abstract. We define and study in-depth the so-called *completely inert* and *uniformly completely inert* subgroups of Abelian groups. We curiously show that a subgroup is completely inert exactly when it is characteristically inert. Moreover, we prove that a subgroup is uniformly completely inert precisely when it is uniformly characteristically inert. These two statements somewhat strengthen recent results due to Goldsmith-Salce established for totally inert subgroups in J. Commut. Algebra (2025). Some other closely relevant things are obtained as well.

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1. Some Fundamentals

Throughout the current brief paper, all our groups are *additively* written and *Abelian*. Our notation and terminology are mainly standard and follow those from [14]. In fact, recall the standard concepts that an arbitrary subgroup F of a group G is said to be a *fully invariant* subgroup provided $\phi(F) \subseteq F$ for any endomorphism ϕ of G , and an arbitrary subgroup C of G is said to be a *characteristic* subgroup provided $\psi(C) \subseteq C$ for any automorphism ψ of G . Moreover, in [2], an arbitrary subgroup S of G is said to be a *strongly invariant* subgroup provided $f(S) \subseteq S$ for any homomorphism $f : S \rightarrow G$.

It is obvious that strongly invariant subgroups are fully invariant subgroups, and the latter are always characteristic subgroups, while both reverses fail in general.

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Generalizing these two notions, it is well known that a subgroup N of a group G is called *fully inert* provided $(\phi(N) + N)/N$ is finite for all endomorphisms ϕ of G , and is called *characteristically inert* provided $(\psi(N) + N)/N$ is finite for all automorphisms ψ of G (for the latter see [7]). Besides, in [1], a subgroup N of G is called *strongly inert* provided $(f(N) + N)/N$ is finite for all homomorphisms $f : N \rightarrow G$. If, in both cases, the cardinalities of the finite quotients $(\phi(N) + N)/N$ and $(\psi(N) + N)/N$ are bounded by some fixed positive integer, the subgroup N is termed *uniformly fully inert* and *uniformly characteristically inert*, respectively. The description of these subgroups as being commensurable with fully invariant and, respectively, with characteristic subgroups can be found in [5].

Apparently, the following implications hold:

$$\text{strong inertness} \Rightarrow \text{full inertness} \Rightarrow \text{characteristic inertness.}$$

However, one observes in what follows that the opposite implications are, generally, untrue.

Further, in [13], the new concept of a totally inert subgroup of a group was introduced as follows: A subgroup T of an arbitrary group G is called *totally inert*, provided that the intersection $T \cap \phi(T)$ has finite index in both T and $\phi(T)$ for any non-zero endomorphism ϕ of G .

Clearly, total inertness implies full inertness.

Imitating the "uniformly" property presented above, it is reasonably natural to ask what is the behavior of uniformly totally inert subgroups defined analogously as follows: A subgroup T of an arbitrary group G is termed *uniformly totally inert*, provided that the intersection $T \cap \phi(T)$ has index bounded by a fixed positive integer in both T and $\phi(T)$ for any non-zero endomorphism ϕ of G .

Nevertheless, this does *not* give anything new in the infinite case as the next arguments illustrate – indeed, we claim that these are only the rational torsion-free group \mathbb{Q} and the quasi-cyclic p -group $\mathbb{Z}(p^\infty)$. In fact, looking for infinite subgroups H of a group G different from, respectively, \mathbb{Q} and $\mathbb{Z}(p^\infty)$, which subgroups are uniformly totally inert, looking at [13, Corollary 2.3] in order to admit infinite totally inert subgroups, G must be torsion-free reduced and indecomposable. Furthermore, for such a group G , a subgroup $H \neq \{0\}$ is infinite, but it cannot be uniformly totally inert, because there is a prime p such that $H/pH \neq \{0\}$ (noticing that H is not divisible), so H/pH has cardinality at least p and, therefore, H/p^nH has cardinality at least p^n , whence H cannot be uniformly totally inert, as expected.

In order to strengthen this, we come to the following basic tool (see the initial version in [8] as well).

Definition 1.1. *A subgroup C of an arbitrary group G is called completely inert provided the intersection $C \cap \psi(C)$ has finite index in both C and $\psi(C)$ for any automorphism ψ of G .*

Evidently, total inertness forces complete inertness, but the reciprocal implication is manifestly non-true as we will illustrate in the sequel.

As above, a reasonably logical question is to ask what happens with *uniformly completely inert subgroups* defined by analogy thus: The subgroup C has an intersection $C \cap \psi(C)$ bounded by a fixed positive integer in C and $\psi(C)$ for any automorphism ψ of G .

And so, the objective of this article is to give a systematic exploration of the so-introduced concept of complete inertness by finding its crucial properties and comparing them with these of the defined above total inertness.

2. Main results

Before establishing the principal achievements, our preliminaries here are the following ones distributed subsequently into two subsections.

2.1. Completely inert subgroups

The first technicality can easily be established by analogy with [13, Proposition 2.1].

Lemma 2.1. *A subgroup commensurable with a completely inert subgroup is again completely inert.*

Proof. It is straightforwardly analogous to Proposition 2.1 from [13]. \square

We now arrive at the following quite surprising assertion.

Proposition 2.2. *A subgroup C of a group G is completely inert subgroup if, and only if, it is characteristically inert.*

Proof. Necessity is evident, so we omit the arguments. As for sufficiency, by assumption, the quotient $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ is finite for each $\psi \in \text{Aut}(G)$. We claim that the factor-group $C/(C \cap \psi(C))$ is likewise finite as being an isomorphic image of $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ under the action of ψ .

In fact, mapping

$$\bar{\psi} : \psi^{-1}(c) + (C \cap \psi^{-1}(C)) \mapsto c + (\psi(C) \cap C),$$

one concludes that it is an isomorphism between $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ and $C/(C \cap \psi(C))$. Indeed, ψ maps $C \cap \psi^{-1}(C)$ into $\psi(C) \cap C$, so that $\bar{\psi}$ is obviously a well-defined homomorphism. Furthermore, if $c = \psi(c_1)$ for some $c, c_1 \in C$, then $c_1 = \psi^{-1}(c) \in C$, and hence $\bar{\psi}$ is an injection. It is also routinely seen that $\bar{\psi}$ is a surjection. Whence, $\bar{\psi}$ is an isomorphism, as asserted. \square

We now proceed by proving a series of technicalities as our next technical statement is the following known fact.

Lemma 2.3. ([6, Lemma 2.2]) *Let H be a fully inert subgroup of the group $G = \bigoplus_{i \in I} G_i$, where the index set I is infinite, and let each π_i denote the canonical projection from G onto G_i . Then, H is commensurable with $\bigoplus_{i \in I} \pi_i(H)$, the images $\pi_i(H)$ are fully inert in G_i , and almost all $\pi_i(H)$ are fully invariant in G_i . Furthermore, there is a finite subset $S \subset I$, such that $\bigoplus_{i \in I \setminus S} \pi_i(H)$ is fully invariant in $\bigoplus_{i \in I \setminus S} G_i$.*

We shall use, hereafter, the symbol " \sim " to designate the following symmetric relation between two fixed subgroups of a given group: We call two subgroups K and L of a group A *commensurable* and indicate that as $K \sim L$ whenever the quotients $(K + L)/K$ and $(K + L)/L$ are simultaneously finite.

We are, thereby, ready to attack the following.

Proposition 2.4. *Let H be a fully inert subgroup of the group $G = \bigoplus_{i \in I} G_i$, where $G_i \cong G_j$ for every $i, j \in I$ and the index set I is infinite. Then, H is commensurable with some fully invariant subgroup of G .*

Proof. Invoking Lemma 2.3, with no loss of generality we can assume that $H = \bigoplus_{i \in I} \pi_i(H)$. For $j \in I \setminus S$, denoting $F_j = \pi_j(H)$, we derive that $\bigoplus_{j \in I \setminus S} F_j$ is fully invariant in $\bigoplus_{j \in I \setminus S} G_j$, where it is clear that all F_j are isomorphic ($j \in I \setminus S$).

After that, in each G_i , where $i \in S$, there exists a fully invariant subgroup $F_i \cong F_j$ with $j \in I \setminus S$. Thus, $F = \bigoplus_{i \in I} F_i$ is a fully invariant subgroup of G . It next suffices to show that, for each $i \in S$, the subgroup F_i is commensurable with $\pi_i(H)$. To this aim, let i be some index in S and $\varphi \in E(G)$ such that $\varphi(G_i) = G_{j_0}$, where j_0 is a fixed index from $I \setminus S$ with $\varphi(G_k) = \{0\}$ whenever $k \neq i$. So, one checks that

$$H + \varphi(H) = \left(\bigoplus_{s \in S} \pi_s(H) \right) \oplus (F_{j_0} + F'_{j_0}) \oplus \left(\bigoplus_{l \in I \setminus (S \cup \{j_0\})} F_l \right),$$

where $F'_{j_0} = \varphi \pi_i(H)$, whence $F_{j_0} + F'_{j_0} \sim F_{j_0}$.

Let us now $\psi \in E(G)$ such that $\psi(G_{j_0}) = G_i$ and $\varphi(G_k) = \{0\}$ whenever $k \neq j_0$. Therefore, one infers that

$$H + \psi(H) = \left(\bigoplus_{s \in S \setminus \{i\}} \pi_s(H) \right) \oplus (F'_i + \pi_i(H)) \oplus \left(\bigoplus_{l \in I \setminus S} F_l \right),$$

where $F'_i = \psi(F_{j_0}) = F_i$, whence $F_i + \pi_i(H) \sim \pi_i(H)$.

But since $F'_{j_0} \cong \pi_i(H)$, $F_{j_0} \cong F_i$ and these isomorphisms both induce endomorphisms of the group G , we then arrive at the relations $F_i + \pi_i(H) \sim F_i$ and $F_i + \pi_i(H) \sim \pi_i(H)$, i.e., $\pi_i(H) \sim F_i$, as asked for. \square

The next technical assertion is well-known (compare with Lemma 2.3 as listed above).

Lemma 2.5. ([3, Lemma 7]) *Let $G = \bigoplus_{i \in I} G_i$, and let $\pi_i : G \rightarrow G_i$ be the corresponding projections. If H is a fully inert subgroup of G , then H is of finite index in the subgroup $\bigoplus_{i \in I} \pi_i(H)$.*

It is worthy of noticing that, if $G := A^n$ for some natural number n and a group A , then *not* every fully inert subgroup of G is commensurable with a fully invariant subgroup (see, for example, [4]).

We finish the series of technical claims with the last one.

Lemma 2.6. *Let $G = A_1 \oplus \cdots \oplus A_n$, $A_i \cong A$ for $i = 1, \dots, n$, where A is a non-zero group, and suppose in A each fully inert subgroup is commensurable with a fully invariant subgroup. Then, each fully inert subgroup H of G is commensurable with a fully invariant subgroup of G . In particular, the subgroup H is uniformly fully inert.*

Proof. Utilizing Lemma 2.5, it can be assumed that

$$H = \pi_1(H) \oplus \cdots \oplus \pi_n(H).$$

It is now easy to see that each $\pi_i(H)$ is fully inert in A_i for $i = 1, \dots, n$. So, each $\pi_i(H)$ is commensurable with a fully invariant subgroup F_i of A_i . Likewise, it is evident that each F_i is commensurable with $\phi(F_j)$ for each isomorphism $\phi : A_j \rightarrow A_i$. Thus, it can be supposed that $F_i \cong F_j$ for all $i, j = 1, \dots, n$. Consequently, in this case, $F = F_1 \oplus \cdots \oplus F_n$ is a fully invariant subgroup of G which is commensurable with H , as required. \square

The following construction is worthwhile, although it was documented in [7, Example 4.1] that there is a group with a characteristically inert subgroup which is *not* fully inert. Thus, Proposition 2.2 yields that there is a group with completely inert subgroup which is *not* fully inert and so, manifestly, *not* totally inert.

Example 2.7. There exists a group with a completely inert subgroup which is *not* totally inert.

Proof. Let G be a torsion-free group of infinite rank such that $E(G) \cong \mathbb{Z}$, the group of integers. Therefore, each its subgroup is characteristic and hence is completely inert. But $H \leq G$ will be totally inert exactly when the factor-group H/nH is finite for all non-zero integers n , that is, if it is a narrow subgroup in terms of [13]. But, because G must have infinite rank, in it will exist subgroups that are not totally inert, as promised. (Notice that this could also be deduced by consulting with [13, Theorem 2.8].) \square

Remark 2.8. Concerning the preceding example, note that if G is a torsion-free group each subgroup of which is completely inert and $pG = G$ for at least one prime p , then the rank of G is necessarily finite. Indeed, if the rank of G is infinite, then G will have a free subgroup H of infinite rank with the property $H \approx pH$.

Likewise, there is an abundance of completely inert subgroups which are not totally inert: in fact, such are all infinite characteristic subgroups of decomposable groups A (e.g., the subgroups nA and $(p^m A)[n]$ for some naturals m, n) as well as, if a group G is torsion-free, then such subgroups are $G(t)$ and $G(\chi)$, where t is the type and χ is the characteristic of G .

In the other vein, since each totally inert subgroup of a decomposable group is finite (see [13, Proposition 2.2]), it must be that each infinite characteristic subgroup of a decomposable group will be completely inert but *not* totally inert subgroup.

We now proceed by showing more statements in the following aspect.

Lemma 2.9. *Suppose that in a group G every subgroup is completely inert. The following two points hold:*

- (1) *if $G = A \oplus B$, then $f(B)$ is finite for every homomorphism $f : B \rightarrow A$;*
- (2) *if G is torsion-free, then G is indecomposable.*

Proof. (1) It is evident since $\varphi = 1_G + f \in \text{Aut } G$ and $\varphi(B) = f(B) \oplus B \sim B$.

(2) Let $H = \langle a + b \rangle$, where $0 \neq a \in A$, $0 \neq b \in B$. If $f = 1_A \oplus (-1_B)$, then $f(H) = \langle a - b \rangle$ and $H \cap f(H) = \{0\}$, and so $H \not\sim f(H)$, as required. \square

Thus, from this lemma, it immediately follows that in any divisible group D each subgroup is completely inert if, and only if, either $D \cong \mathbb{Q}$, the group of all rational numbers, or D is torsion and the rank of each its p -component is ≤ 1 .

Lemma 2.10. *The following three items hold:*

- (1) *If G is a reduced p -group in which each subgroup is completely inert, then G is finite.*
- (2) *In a non-reduced p -group G all subgroups are completely inert if, and only if, G is a direct sum of a quasi-cyclic group and a finite group.*
- (3) *In a torsion group all subgroups are completely inert if, and only if, almost all its p -components are co-cyclic and the non co-cyclic p -components are either finite or a direct sum of a quasi-cyclic group and a finite group.*

Proof. (1) From Lemma 2.9, it follows that the Ulm-Kaplansky invariant $f_n(G)$ is finite for all $n < \omega$. So, if we assume the contrary that G is infinite, then both G/pG and $G[p]$ are also infinite, and thus there exists a homomorphism $G/pG \rightarrow G[p]$ with infinite image.

Now, if $\pi : G \rightarrow G/pG$ is the usual canonical surjection and $\varphi = f \circ \pi$, then one checks that $\varphi^2 = 0$ and thus $\psi = 1 + \varphi \in \text{Aut } G$. But, as $H \cap \psi(H) = pH$, we infer $H \cap \psi(H) \not\sim H$, proving that G is finite, as promised.

Furthermore, items (2) and (3) follows directly from Lemma 2.9 and item (1). \square

Our next assertion is a helpful consequence of Lemma 2.9 quoted above.

Corollary 2.11. *If $G = \mathbb{Q} \oplus B$ is a group, where \mathbb{Q} is the rational group, then in G each subgroup is completely inert if, and only if, B is finite.*

Proof. Necessity. An appeal to Lemma 2.9 reaches us that B is torsion, almost all p -components of B are cyclic and the non-cyclic p -components are finite. If, in a way of contradiction, we assume for a moment that B is infinite (i.e., an infinite number of its p -components are non-zero), then in \mathbb{Q} there will exist a subgroup, X say, with an infinite image of the existing homomorphism $X \rightarrow B$, that this can not be happen, contrary to our assumption.

Sufficiency. It is obvious, so the arguments are removed. \square

Given a reduced torsion-free group G , we shall denote by $R(G)$ the *maximal subring* of the field of rational numbers \mathbb{Q} contained in $E(G)$, which subring is generated by the 1 and the inverses of the prime numbers p for which $G = pG$.

Lemma 2.12. *Let G be a torsion-free group such all endomorphism are injections. Then, in G each subgroup is completely inert if, and only if, $\text{Aut } G = U(R(G))$ and, moreover, if $R(G) \neq \mathbb{Z}$, then the rank of G is finite.*

Proof. Necessity. Suppose that H is a pure subgroup of rank 1, and $\varphi \in \text{Aut } G$. Since $H \sim \varphi(H)$, we derive $H = \varphi(H)$, so $\varphi \upharpoonright H \in R(H)$ and hence $\varphi \upharpoonright H = m/n$, where m and n are mutually simple integers. However, as $(n\varphi - m)H = \{0\}$, we deduce $n\varphi = m$. Therefore, $nG = G$ whence $\varphi = m/n \in R(G)$.

Sufficiency. If $R(G) = \mathbb{Z}$, then one sees that each subgroup is totally inert. If, however, $pG = G$ for some prime p , then under assumption G possesses finite rank, and so each $H \leq G$ will too have finite rank, whence $H \sim \varphi(H)$ for every $\varphi \in R(G)$. \square

Let us now recollect the following technicality necessary for our further presentation.

Lemma 2.13. ([11, §8, Exercise 5]) *If H is a subgroup of $G = B \oplus C$, then H is the sub-direct sum of the groups $B \cap (H + C)$ and $(B + H) \cap C$.*

We now intend to prove the following statement.

Proposition 2.14. *In the splitting group $G = T \oplus R$, where $T = T(G)$, each subgroup is completely inert if, and only if, in both T and R each subgroup is completely inert and, moreover, the image of every homomorphism $f : S \rightarrow T$ is finite for every subgroup $S \leq R$.*

Proof. Necessity. Assume that $f(S)$ is infinite for some $S \leq R$. Letting $H := \{f(x) + x \mid x \in S\}$, then for $\varphi := 1_T \oplus (-1_R)$ we have

$$T' = 2f(S) = \{2f(x) \mid x \in S\} \leq H + \varphi(H) \sim H.$$

So, T' is finite, because $H \cap T = \{0\}$. However, since $2T_q(f(S)) = T_q(f(S))$ for any prime $q > 2$, $T_2(f(S))$ has to be infinite, and so it is non-reduced. But, in this case, $2(f(S))$ must be infinite, and this contradiction guarantees that $f(S)$ is finite.

Sufficiency. Assume $H \leq G$. Then, Lemma 2.13 informs us that H is a sub-direct sum of the groups T_1 and R_1 , where $T_1 := T \cap (H + R)$ and $R_1 := (T + H) \cap R$. Consequently, if $T_0 := T_1 \cap H$ and $R_0 := H \cap R_1$, then we know with help of the property of sub-direct sums that $T_1/T_0 \cong R_1/R_0$.

If, for a moment, the factor-group T_1/T_0 is non-reduced, then given the structure of the group T , we obtain that non-reduced must be the group T as well; thus, there is a homomorphism $R_1 \rightarrow T$ with infinite image, that is manifestly wrong.

If, however, T_1/T_0 is reduced, then each of its p -components is a factor-group of either a cyclic group or of a finite group, and thus there exists a homomorphism $T_1/T_0 \rightarrow T_1$ with non-zero image of each non-zero p -component of T_1/T_0 . Hence, if T_1/T_0 is infinite, then there will exist a homomorphism $R_1 \rightarrow T$ with infinite image, that is manifestly untrue.

Therefore, T_1/T_0 has to be finite, and so finite is R_1/R_0 too. Furthermore, since $\theta(H) = R_1 \sim R_0$, where $\theta : G \rightarrow R$ is the standard projection, we similarly perceive that

$$(1 - \theta)(H) = T_1 \sim T_0.$$

But, because

$$T_0 \oplus R_0 \leq H \leq T_1 \oplus R_1,$$

we then can conclude $H \sim T_1 \oplus R_1$.

Now, letting $\varphi \in \text{Aut } G$, by assumption we have $T_1 \sim \varphi(T_1)$, and thus the action of φ on R_1 can be represented as the sum of some $\psi \in \text{Aut } R$ and $\alpha \in \text{Hom}(R, T)$. Consequently, $\psi(R_1) \sim R_1$ and $\alpha(R_1)$ is finite, so that $\varphi(R_1) \sim R_1$ means $\varphi(H) \sim H$, as needed. \square

Example 2.15. The next two statements are valid:

(1) Write $G = D \oplus R$, where R is a torsion-free group, and D is a torsion group such that each its subgroup is completely inert and, for each prime p , its p -component is a non-reduced non-zero group. Then, each subgroup in G is completely inert if, and only if, $R \cong \mathbb{Z}$.

(2) If $T = \bigoplus_{p \in \Pi} T_p$ is a torsion group such that $T_p \cong \mathbb{Z}(p)$ for each $p \in \Pi$ and $|\Pi| = \aleph_0$. Then, every direct summand of $G = \prod_{p \in \Pi} T_p$ is fully invariant, and so completely inert. Besides, in G there is a subgroup which is *not* completely inert.

Proof. (1) **Necessity.** Suppose $H \leq R$ is a pure subgroup such that the rank of R/H is exactly 1. If $R/H \not\cong \mathbb{Z}$, then there exists a homomorphism $R \rightarrow D$ with infinite image, which is an absurd. So, $R/H \cong \mathbb{Z}$ and hence R is decomposable, which is again absurdly.

Notice that in our construction the group D possesses an infinite number of non-zero primary components: in fact, if the factor-group R/H is isomorphic to the subgroup of \mathbb{Q} which is generated by all rational numbers of the form $\frac{1}{p}$ with p a prime number and D has a finite number of p -primary components, then our arguments do *not* work properly.

Sufficiency. It follows at once from Proposition 2.14.

(2) Writing $G = A \oplus B$, we then arrive at $T(G) = T(A) \oplus T(B)$, and $f(T(A)) = \{0\}$ for every $f \in \text{Hom}(A, B)$; so, $f(A) = \{0\}$ as the quotient-group $A/T(A)$ is divisible.

Setting $x = (\dots, x_p, \dots)$, where $x_p \neq 0$, if, and only if, $p \in \Pi$ and $X \leq G$ is such a torsion-free rank 1 subgroup that $x \in X$.

Furthermore, putting $y \in G$, $y := (\dots, x'_p, \dots)$ with $x'_p \neq 0$ for every $p \in \Pi$ such that the coset $y + T(G)$ is independent of $x + T(G)$ over \mathbb{Q} in $G/T(G)$, one finds that there exists $\varphi \in \text{Aut } G$ with $\varphi(x) = y$. That is why, a simple check shows that the subgroup X is not completely inert in G , as stated. \square

An extremely difficult question is that of globally characterizing those reduced (p -)groups for which every their fully inert subgroup is commensurable with a fully invariant subgroup (see, e.g., [17, Problem 2.1]). The best achievements in this directions are that the class of such groups contains both the classes of totally projective groups and torsion-complete groups (see [12], [15] and [17, Theorem 2.2], respectively).

That is why, it is quite logical to ask what happens in the case of characteristically inert subgroups (compare also with [9]). Concretely, one may ask to characterize

those reduced (p -)groups whose totally (resp., completely) inert subgroups are commensurable with some fully invariant (resp., characteristic) subgroups.

In this vein, we are now prepared to attack the following helpful observation.

Proposition 2.16. *The subgroup H of a group G is commensurable with a characteristic subgroup of G if, and only if, H is uniformly completely inert in G .*

Proof. Necessity. It is almost evident since Proposition 2.22 reaches us that the subgroup is uniformly completely inert if, and only if, it is uniformly characteristically inert, and each subgroup commensurable with a characteristic subgroup is uniformly characteristically inert (see [10]).

Sufficiency. If we assume H is uniformly completely inert, then it is uniformly characteristically inert and so [10, Corollary 1.9] works to deduce that H is commensurable with a characteristic subgroup of G , as required. \square

We, thus, now come to a major assertion which summarizes a part of the above assertions and, thereby, stimulates our further writing.

Theorem 2.17. *Suppose G is a group such that each its subgroup is commensurable with a characteristic subgroup of G . Then, G can completely be characterized.*

Proof. A consultation with Proposition 2.16 is a guarantor that every subgroup of G is uniformly completely inert. Besides, each uniformly completely inert subgroup is manifestly completely inert. That is why, we can subsequently apply all statements starting from Lemma 2.9 to Example 2.15 to get the desired complete characterization after all. \square

Note that it cannot be derived any useful information when G is a reduced group such that each its (totally inert) subgroup is commensurable with a fully invariant subgroup of G . Indeed, it was noted in [13] that a fully invariant subgroup H of a group G is totally inert in G if, and only if, $f(H)$ has finite index in H for all endomorphisms $f : G \rightarrow G$.

Notice also that both Proposition 2.4 and Lemma 2.6 are common generalizations of [17, Theorem 2.3], and the same theorem appears in [13, Theorem 3.5].

The following extra comments are, hopefully, worthwhile.

Remark 2.18. The class of groups considered in Theorem 2.17 forms a much smaller class than the classes of groups in which every completely/characteristically inert subgroup is commensurable with a characteristic subgroup. In fact, the proper inclusion follows from the simple facts alluded to above that there are too many subgroups that are definitely *not* either completely nor characteristically inert. Thus, it cannot be happen that we will succeed to obtain their comprehensive descriptions at all.

It was pointed out in [17, Theorem 2.3] that, if $X = \bigoplus_{i \in I} G_i$ is a group such that each direct summand G_i is isomorphic to a fixed unbounded fully transitive p -group in which every fully inert subgroup is commensurable with a fully invariant subgroup, then X has the same property.

We now will expand this to the case of characteristic subgroups as follows.

Theorem 2.19. *Suppose that $G = \oplus_{i \in I} G_i$ is a group such that each direct summand G_i is isomorphic to a fixed unbounded transitive p -group in which every characteristically inert subgroup is commensurable with a characteristic subgroup. Then, G possesses the same property.*

Proof. It entirely relies on the same arguments as in Proposition 2.4 and Lemma 2.6, so the drop off the complete details. \square

Some additional things of the subject are these:

Lemma 2.20. *If H is an essential subgroup in G and $\phi \in \text{Aut } G$, then $\phi(H)$ is also essential in G . In particular, if H does not contain its proper essential subgroups, then H is characteristic in G .*

Proof. If we assume $A \cap \phi(H) = \{0\}$ for some $A \leq G$, then one verifies that

$$\phi^{-1}(A \cap \phi(H)) = \phi^{-1}(A) \cap H = \{0\},$$

so both $\phi^{-1}(A)$ and A are zero, as required. \square

We terminate this section with the following.

Proposition 2.21. *Let G be a torsion-free group of finite rank, and suppose $H \leq G$ is a free essential subgroup (i.e., $r(H) = r(G)$). Then, H is completely inert in G .*

Proof. Letting $\phi \in \text{Aut } G$, then Lemma 2.20 employs to get that $\phi(H)$ is essential in G , so that $H \cap \phi(H)$ is essential in both H and $\phi(H)$; in particular, one discovers that $H \cap \phi(H) \cong H$ whence $H \cap \phi(H)$ has finite index simultaneously in H and $\phi(H)$, thus substantiating our claim. \square

In particular, if G is decomposable, then with no too many efforts we can receive additional examples of completely inert *not* totally inert subgroups.

2.2. Uniformly completely inert subgroups

Our instrument here is the following.

Proposition 2.22. *A subgroup U of a group G is uniformly completely inert if, and only if, it is uniformly characteristically inert.*

Proof. The same idea as in Proposition 2.2 is workable since, by assumption, there is $n \in \mathbb{N}$ such that

$$|\psi^{-1}(C)/(C \cap \psi^{-1}(C))| \leq n$$

for each $\psi \in \text{Aut}(G)$, and so we claim that

$$|C/(C \cap \psi(C))| \leq n$$

as being an isomorphic image of $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ acting by ψ , as demonstrated above. \square

Recall now that a group G is said to have *unit sum number* $\text{usn}(G) = n \in \mathbb{N}$ if each endomorphism of G is a sum of $\leq n$ automorphisms of G .

As the next lemma illustrates, if $\text{usn}(G) = n$, then every uniformly characteristically inert subgroup of G is uniformly fully inert in G , hopefully bounded by another fixed positive integer.

Lemma 2.23. (1) If $\text{usn}(G) = n$, then every uniformly characteristically inert subgroup C of G (bounded by a fixed integer $k > 0$) is uniformly fully inert in G (bounded by an integer $\leq nk$).

(2) If C is uniformly characteristically inert in G and $H \sim C$, then H is uniformly characteristically inert in G .

Proof. (1) Letting φ be an arbitrary endomorphism of G , then there are $\alpha_1, \dots, \alpha_n \in \text{Aut } G$ with $\varphi = \alpha_1 + \dots + \alpha_n$. But then we obtain

$$(C + \varphi(C))/C \leq (\alpha_1(C) + C)/C + \dots + (\alpha_n(C) + C)/C,$$

and so

$$|(C + \varphi(C))/C| \leq |(\alpha_1(C) + C)/C| + \dots + |(\alpha_n(C) + C)/C| \leq nk,$$

as claimed.

(2) It can be verified in two different ways:

(a) It must be shown that the order of the factor-group $\alpha(H)/(\alpha(H) \cap H)$ is finite and the same for all $\alpha \in \text{Aut } G$. To that end, since $|H/(H \cap C)|, |C/(H \cap C)| \leq n$ for some n , one inspects that

$$|\alpha(H)/(\alpha(H) \cap \alpha(C))|, |\alpha(C)/(\alpha(H) \cap \alpha(C))| \leq n$$

for all $\alpha \in \text{Aut } G$. So, $\alpha(C) \sim \alpha(H)$, but as $\alpha(C) \sim C$, in view of $C \sim H$ we can get $\alpha(H) \sim H$ for any $\alpha \in \text{Aut } G$.

(b) Knowing that each uniformly characteristically inert subgroup is commensurable with some characteristic subgroup (see [10, Corollary 1.9]), we may deduce that H is commensurable with some characteristic subgroup, and consequently is uniformly characteristically inert, as asserted. \square

As a useful consequence of the previous lemma, we record.

Corollary 2.24. Let H be an infinite uniformly characteristically inert subgroup with bounded index $k \geq 1$ of the group $G = \bigoplus_{i \in I} G_i$, where $G_i \cong G_j$ for all $i, j \in I$ and the index set I is infinite. If π_i denotes the canonical projection of G onto G_i , then $H \leq \bigoplus_{i \in I} \pi_i(H)$ with bounded index $3k$.

Proof. Recall that $\text{usn}(G) = 3$ (see, e.g., [7]). Now, Lemma 2.23 tells us that H is, actually, uniformly fully inert. If, however, we assume the contrary that

$$\left| \left(\bigoplus_{i \in I} \pi_i(H) \right) / H \right| > 3k,$$

then there will exist $i_1, \dots, i_n \in I$ such that, for the sum $\psi = \pi_{i_1} + \dots + \pi_{i_n}$, we have

$$|(\psi(H) + H)/H| = |(\pi_{i_1}(H) + \dots + \pi_{i_n}(H))/H| > 3k,$$

contradicting to Lemma 2.23, as suspected. \square

Proposition 2.25. If G is a group in which each fully inert subgroup is uniformly fully inert and A is a direct summand in G with a finitely generated additional direct summand, then in A each fully inert subgroup is uniformly fully inert.

Proof. Write $G = A \oplus B$, $H \leq A$ is fully inert in A and set $F := \text{Hom}(A, B)H$. Since F is fully invariant in B , then it is not too hard to establish that the subgroup $H \oplus F$ is fully inert in G . Indeed, since F also is finitely generated, it follows that F is, actually, the image of a finite number of homomorphisms $A \rightarrow B$. Thus, the action of each endomorphism of G on $H \oplus F$ is the action of some endomorphism $\alpha \in E(A)$ on H , of some homomorphism $\beta : A \rightarrow B$ on H , of some endomorphism $\gamma \in E(B)$ on F , and of some homomorphism $\delta : B \rightarrow A$ on F . We, furthermore, have that $\gamma(F) \leq F$, $\beta(H) \leq F$ and so $\delta(F)$ is contained in the image of a finite number of some endomorphisms of A . Hence, by assumption, $H \oplus F$ is uniformly fully inert in G . However, every endomorphism of A can be viewed as an endomorphism of G and, therefore, H is uniformly fully inert in A , as pursued. \square

We now end our entire work with the following question which, hopefully, will stimulate a further investigation on the subject.

Problem 2.26. Explore those subgroups K of a group G such that the intersection $K \cap f(K)$ has finite index in both K and $f(K)$ for all homomorphisms $f : K \rightarrow G$.

It is apparent that these subgroups are always totally inert and strongly inert simultaneously.

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