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The effect of the signal initial phase on the amplitudes of a DFT spectrum

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Abstract. *Knowing the amplitudes in a spectrum calculated using the Discrete Fourier Transform (DFT) algorithm is essential for precisely estimating the frequencies of a real-valued signal. Since the amplitudes are affected by the signal's initial phase, it is crucial to consider this aspect when estimating the signal frequencies. This paper presents the initial signal phase's effect on the values calculated for the real and imaginary parts and the magnitudes in the DFT of a real-valued signal. Analyzing the evolution of the amplitudes with the initial phase for a complete cycle for the initial phase, we found that the values in the magnitude DFT are subject to minor alteration, while the values of the real and imaginary parts of the DFT are strongly affected by the initial phase.*

Keywords*: Discrete Fourier Transform, amplitude spectrum, frequency estimation, initial phase*

1. Introduction

The literature is rich with studies on accurately estimating the frequencies and amplitudes of signals [1]. Most often, the frequencies are located on an inter-beam position in the spectrum, thus requiring supplementary processing such as interpolation algorithms for finer frequency estimation. An in-depth approach to this topic is presented in [2]. The interpolation is made using two or three points in the real part of the DFT or the magnitude DFT. The main idea is to find a correction coefficient, which represents the distance between the actual frequency and the frequency found using a standard DFT. Several papers presenting these algorithms are [3]- [13]. Thus, knowing the amplitudes for a given context is crucial for the precision of frequency estimation algorithms. However, no papers have considered the initial phase of the signal in a special way when estimating the frequencies.

In our previous research [14]-[16], we developed a frequency estimation technique supported by Machine Learning (ML) to find the frequency at an inter-line position. This technique finds the correction coefficient using an Artificial Neural Network (ANN) that has as inputs the biggest values in the magnitude DFT.

By applying the ANN-supported algorithm, we observed that the estimation precision decreases if the signal has the initial phase differing from zero. The inaccuracy occurs because the amplitudes in the magnitude DFT present slight changes if the signal has an initial phase. To better train the network, we propose using as input in the ANN not just the biggest values in the magnitude DFT but also those real and the imaginary parts of the DFT. This paper presents the spectral values' dependency on the initial signal phase.

2. Theoretical background

The real Discrete Fourier Transform decomposes an input signal $x[n]$, having N discrete samples taken over time into two output signals that contain the amplitudes of the component sine and cosine waves. The output signals have $N/2+1$ points that represent the frequencies of the sine and cosine waves in which the input signal $x[n]$ is decomposed. The input signal is said to be in the time domain, while the output signals are representations in the frequency domain. A schematic of the process is presented in Figure 1.

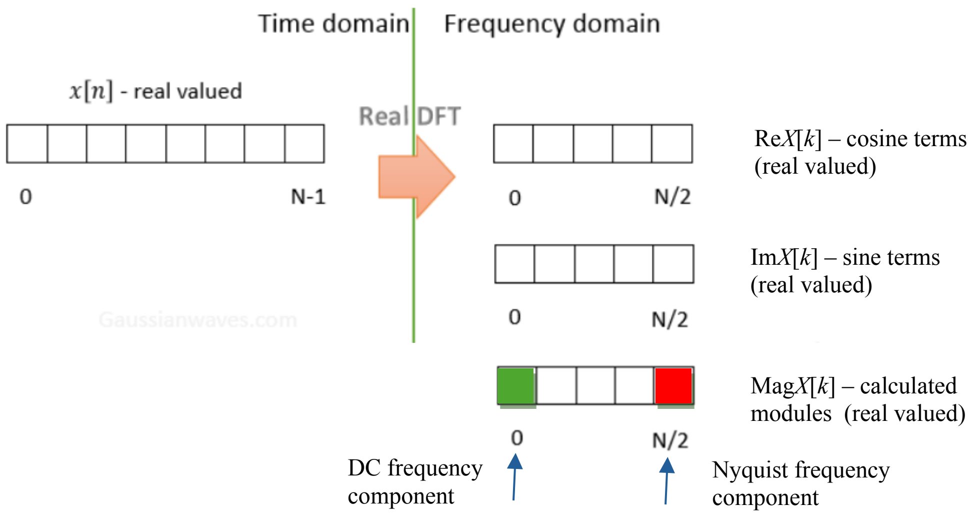

Figure 1. The real DFT

Let us consider the discrete-time sinusoid. It can be expressed

$$x[n] = A\sin(2\pi f_R \frac{n}{f_S} + \varphi_0) \tag{1}$$

where A is the signal amplitude, f_R is the actual signal frequency, f_S is the sampling rate, and φ_0 is the initial phase expressed in radians. In the above equation, n is the sample number, which takes values between 0 and N-1.

The mathematical relation to calculate the values for the amplitude on the k-th spectral bin of the real part is

$$\operatorname{Re} X[k] = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\cos(2\pi k \frac{n}{N}) \tag{2}$$

while the amplitude on the k-th spectral bin of the imaginary part is calculated with the mathematical relation

$$\operatorname{Im} X[k] = -\frac{2}{N}\sum_{n=0}^{N-1} x[n]\sin(2\pi k \frac{n}{N}) \tag{3}$$

With these two sets of values, the N/2+1 values of the magnitude DFT for the individual spectral bins are calculated with the mathematical relation

$$\operatorname{Mag} X[k] = \sqrt{\left(\operatorname{Re} X[k]\right)^2 + \left(\operatorname{Im} X[k]\right)^2} \tag{4}$$

We denote the biggest amplitude in the magnitude DFT as A_{max}, the value found at its left as $A_{pre\text{-}max}$, and the value found at its right as $A_{post\text{-}max}$. Similarly, for the amplitudes in the real part, we use $R_{pre\text{-}max}$, R_{max}, and $R_{post\text{-}max}$; for the imaginary part, we use $I_{pre\text{-}max}$, I_{max}, and $I_{post\text{-}max}$.

3. Numerical simulation

To find the effect of the initial phase on the amplitudes, we first consider a sinusoidal signal with a given frequency and length and with the initial phase set to zero. For this signal, we calculate, with equations (2)-(4), the values of the real part Re$X[k]$, the imaginary part Im$X[k]$, and the magnitudes Mag$X[k]$. Note that we normalized the values of Re$X[k]$ and Im$X[k]$ by multiplication with 2/N in order to obtain all amplitudes related to the signal.

Next, we iteratively increase the initial phase with a small step until the initial phase achieves 2π. For each step, we identify the picks and its neighbors in the real, imaginary, and magnitude DFT. We associate these values with the frequency f_E found on the spectral line of A_{max} and the actual frequency f_R.

The identified values are stored in a database with the structure presented in Table 2. The database aims to provide data for training an ANN to estimate the correction coefficients and the signal's frequencies accurately. Numerous initial phases are considered in this analysis to complete the database.

Table 1. The structure of the database realized for training the ANN

Role	Parameter	Symbol	Initial phase [rad]		
			0	…	2π
ANN Input	Amplitude [m/s^2]	$R_{pre\text{-}max}$			
		R_{max},			
		$R_{post\text{-}max}$			
		$A_{pre\text{-}max}$			
		A_{max}			
		$A_{post\text{-}max}$			
		$I_{pre\text{-}max}$			
		I_{max}			
		$I_{post\text{-}max}$			
Support	Actual frequency [Hz]	f_R			
	Estimated frequency [Hz]	f_E			
ANN Target	Correction coefficient [Hz]	δ			

First, we exemplify the evolution of the amplitudes in several cases. The first signal contains $N = 207$ samples by a sampling rate $f_S = 167$ Hz, resulting in a signal time length $t = 1.23353$ sec, a time resolution $\Delta t = 0.00599$ sec, and a frequency resolution $\Delta f = 0.81068$ Hz. The signal amplitude is $A = 1$ m/s^2, and the frequency is $f_R = 5$ Hz. As mentioned, we set the initial phase $\varphi_0 = 0$ rad for the first simulation. Afterward, we increase the value of the initial phase with 0.1π until the initial phase achieves 2π.

To have an image of the amplitude/initial phase dependency, we consider it sufficient to present the evolution of the picks in the real part, imaginary part, and magnitude DFT, namely R_{max}, I_{max}, and A_{max}. Note that while the values of R_{max} and I_{max} can be real positives or negatives, the values of A_{max} are always real positives. Figure 2 presents this evolution.

We observe in Figure 2 that the evolution of the picks is sinusoidal/cosinusoidal, which is justified by the mathematical relations that express the real and imaginary part terms.

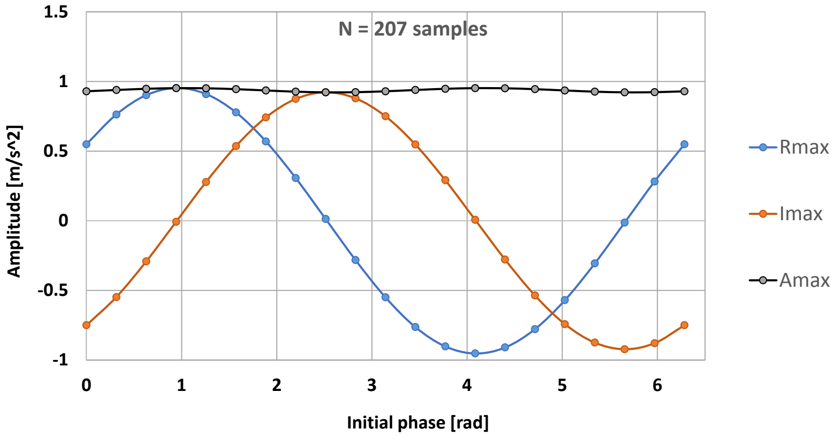

Figure 2. The evolution of the picks with the initial phase for a signal generated with $N = 207$ samples

We now consider a signal generated with $N = 201$ samples to get a more complete image of the phenomenon. This signal length is chosen because it comprises six sinusoids, so its DFT gives a good frequency estimate. The signal amplitude A = 1 m/s^2, the frequency f_R = 5 Hz, the sampling rate f_S = 167 Hz, and the time resolution Δt = 0.00599 sec remain unchanged. For these parameters, the length of the second signal becomes t = 1.1976 sec, which results in the new frequency resolution Δf = 0.835 Hz. As in the first case, we set the initial phase φ_0 = 0 rad for the first simulation and increase it with 0.1π until the initial phase achieves 2π.

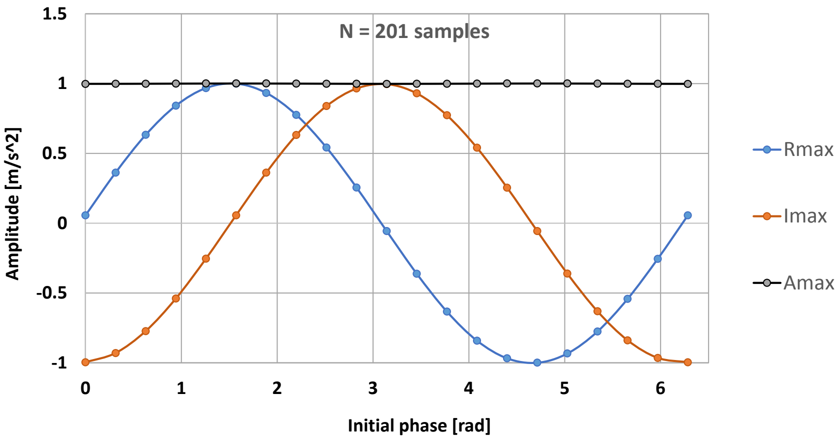

Figure 3. The evolution of the picks with the initial phase for a signal generated with $N = 201$ samples (signal contains six entire cycles)

Figure 3 depicts the pick evolution for the real part, imaginary part, and magnitude DFT, namely R_{max}, I_{max}, and A_{max}, for the signal generated with 201 samples. It can be observed that the A_{max} is much closer to the actual amplitude, and it has a more minor variation.

To have consistent results, we analyze a third signal generated with $N = 204$ samples by maintaining the amplitude, the frequency, the sampling rate, and the time resolution. For these parameters, the signal length becomes $t = 1.21557$ sec and the frequency resolution $\Delta f = 0.82266$ Hz. As in the first case, we set the initial phase $\varphi_0 = 0$ rad for the first simulation and increase it with 0.1π until the initial phase achieves 2π. The obtained results are presented in Figure 4.

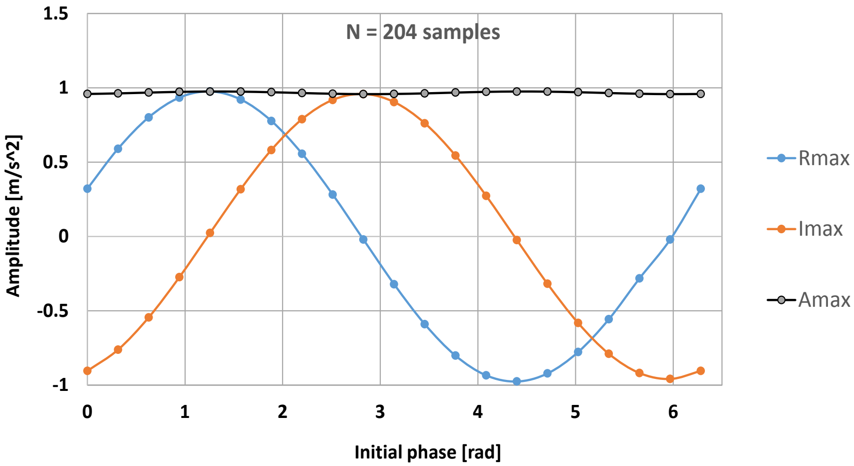

Figure 4. The evolution of the picks with the initial phase for a signal generated with $N = 204$ samples

In Figure 4, we observe that A_{max} is closer to the actual amplitude when compared to the signal containing 207 samples but not so close as those for the signal generated with 201 samples. The amplitude variation is also between the two previously studied cases. This demonstrates that the closer the signal length is to a multiple of the period T, the better the amplitude is found.

A more explicit evolution of A_{max} is shown in Figure 5. It can be observed that, for all signals, the biggest amplitude was not obtained for the initial phase set at zero. It is also remarked that the amplitudes for the signal generated with 207 samples vary between 0.92205 and 0.95195, so the absolute difference is less than 0.015, which means a difference of 1.5%. This difference is relevant for frequency estimation so that the training of the ANN must contain the sets of values derived for all initial phases.

For the shorter signal generated with 204 samples, the amplitudes A_{max} vary between 0.95821 and 0.97535, so the absolute difference is 0.00857, which means a difference of less than 0.857%.

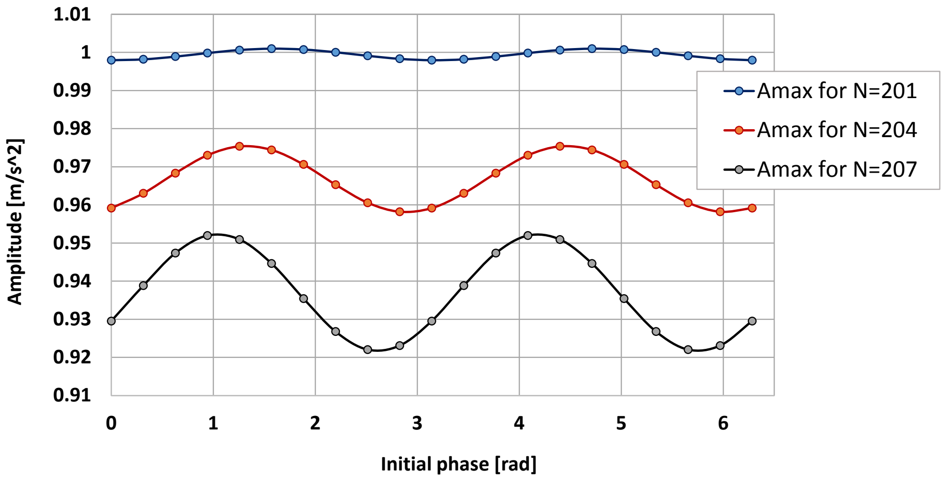

Figure 5. The evolution of A_{max} with the initial phase for different signal lengths

The signal containing six cycles provides the most accurate amplitudes A_{max}, varying between 0.99797 and 1.00096. The absolute difference is 0.0015, which can be neglected in practical applications. However, because the initial phase is not known *a priori*, the input data for the training of the ANN must comprise scenarios for different signal lengths and initial phases.

4. Conclusion

The initial phase influences the amplitudes shown in the magnitude spectrum and the spectra of the real and imaginary parts. For a signal that is a multiple of the signal period *T*, all amplitudes calculated in the DFT magnitude spectrum, regardless of the initial phase, vary very little around the actual signal amplitude. We found a variation of less than ±0.15% for the studied signal.

When the signal is not a multiple of the signal period *T*, all the calculated amplitudes are smaller than the actual signal amplitude, and the amplitude variation is more significant than in the particular case when the signal comprises an entire number of cycles. For such signals, we found an amplitude variation due to the initial phase change of up to ±1.5% for the analyzed signals, but we presume it can be more extensive than that when the signal length is $t = T(m+0.5)$ cycles.

In the following study, we will approach more signal lengths to cover an entire cycle of length T representing a signal period. This study, in correlation with the phase shift analysis, will permit the creation of a robust database for ANN training and accurate frequency estimation.

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