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Dynamic behavior of a simply supported circular plate

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Abstract. *The paper presents a study regarding the dynamics behavior of thin circular plate simply supported with a relation obtained analytically and the graphic representation of the modal shapes. The modal shapes are obtained using Bessel functions and their graphic representation are compared with Finite Element Method (FEM) by using modal analysis. For the analyzed case, the first 70 eigenvalues and natural frequencies are calculated.*

Keywords: *Bessel functions, circular plate, mode shape, eigenvalue.*

1. Introduction

Thin plates are used in architecture, civil construction design, industrial plant and machinery, oil platforms, hydraulic structures, aerospace and mechanical engineering.

Study of plates and dynamic behavior has been of scientific interest since the 18th century. The first researchers to offer mathematical approaches to plate and free vibration analysis were L. Euler [1], J. Bernoulli [2], E. Chladni, C.L. Navier, G.R. Kirchhoff, Ventsel and Krauthammer [3], S.P. Timoshenko [4], J.V. Boussinesq, S.G. Lekhnitskii, A. Leissa [5].

The problem of vibration analysis was analyzed using different numerical methods: Boundary element method [6], Finite element method [7, 8], Finite difference method.

The paper is focused on the analytical approach by using Bessel function solution and the comparison with the numerical results of the dynamic behavior for a simply supported circular plate. The considered plate is homogeneous, of constant thickness and is subject to the action of its dead weight.

The frequency equation is deduced from which the first 70 eigenvalues and natural frequencies were calculated, the modal functions are presented and illustrated using the Excel software and compared with the modal shapes obtained from SolidWorks software.

2. Analytical approach

The origin 0 of a thin circular plate, in polar coordinates is defined by the radius r=a=0.4 m and the angle θ (fig. 1). The plate is made of structural steel with constant thickness h=0.002 m and having the modulus of elasticity $E=2.1\cdot10^{11}$ N/m^2, Poisson coefficient ν=0.28 and density ρ=7800 kg/m^3.

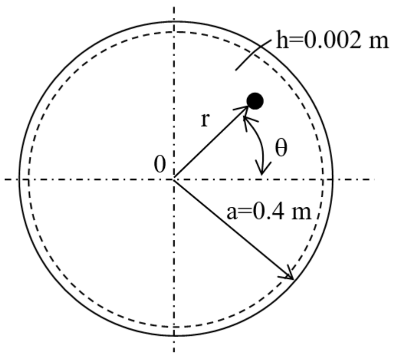

Figure 1. Simply supported circular plate defined by polar coordinates.

The flexural rigidity of the considered circular has the value:

$$D = \frac{Eh^3}{12(1-\nu^2)} = 151.91 \text{ Nm} \tag{1}$$

To obtain the analytical solution of the dynamic behavior of the simply supported circular plate, it starts from the general solution [5] of the equation in polar coordinates:

$$\begin{cases} W(r,\theta) = \sum_{n=0}^{\infty}[A_n J_n(kr) + B_n Y_n(kr) + C_n I_n(kr) + D_n K_n(kr)]\,cos(n\theta) + \\ \quad + \sum_{n=0}^{\infty}[A_n{}^* J_n(kr) + B_n{}^* Y_n(kr) + C_n{}^* I_n(kr) + D_n{}^* K_n(kr)]\,sin(n\theta) \end{cases} \tag{2}$$

where,

r, θ are polar coordinates;
a [m] is the circular plate radius;
h [m] is the thickness of the plate;
E [N/m^2] is the Young's modulus;
ν is the Poisson coefficient;
n = 0, 1, …, ∞ is the number of nodal diameters, or the order of the Bessel function;
$k = \sqrt[4]{\frac{\rho\omega^2}{D}}$ is a parameter of convenience.
$J_n(k\cdot r)$, $Y_n(k\cdot r)$ are the Bessel functions of the first and second kind;

$I_n(k\cdot r)$, $K_n(k\cdot r)$ are the modified Bessel functions of the first and second kind;

A_n, B_n, C_n, D_n, A_n^*, B_n^*, C_n^*, D_n^* are coefficients that are obtained from boundary conditions.

The considered circular plate does not have a central hole and to avoid deformations and infinite stresses at r=0, the terms $Y_n(kr)$ and $K_n(kr)$ are not taken into account [5].

In addition, the boundary conditions present symmetry in relation to one or more diameters of the simply supported circular plate and in this case the terms involving the term sin(nθ) are not necessary.

The general solution of the equation in polar coordinates (2) becomes:

$$W(r,\theta) = [A_n J_n(kr) + C_n I_n(kr)]cos(n\theta) \tag{3}$$

The boundary condition for the simply supported plate are:

$$\begin{cases} W(a) = 0 = [A_n J_n(\lambda) + C_n I_n(\lambda)]cos(n\theta) \Rightarrow C_n = -A_n \frac{J_n(\lambda)}{I_n(\lambda)} \\ \frac{\partial^2 W(a,\theta)}{\partial r^2} = 0 = \lambda^2 A_n \left[\left(J''_n(\lambda) + \frac{v}{\lambda} J'_n(\lambda)\right) - \frac{J_n(\lambda)}{I_n(\lambda)}\left(I''_n(\lambda) + \frac{v}{\lambda} I'_n(\lambda)\right)\right] cos(n\theta) \end{cases} \tag{4}$$

where, λ=k·a is the eigenvalue.

By using the recurrence relations [9]:

$$\begin{cases} J'_n(\lambda) = -J_{n+1}(\lambda) + \frac{n}{\lambda} J_n(\lambda) = J_{n-1}(\lambda) - \frac{n}{\lambda} J_n(\lambda) \\ I'_n(\lambda) = I_{n+1}(\lambda) + \frac{n}{\lambda} I_n(\lambda) = I_{n-1}(\lambda) - \frac{n}{\lambda} I_n(\lambda) \end{cases} \tag{5}$$

and

$$\begin{cases} J_{n+2}(\lambda) = \frac{2}{\lambda}(n+1) J_{n+1}(\lambda) - J_n(\lambda) \\ I_{n+2}(\lambda) = -\frac{2}{\lambda}(n+1) I_{n+1}(\lambda) + I_n(\lambda) \end{cases} \tag{6}$$

it will be obtained:

$$\begin{cases} J''_n(\lambda) = \left(\frac{n^2}{\lambda^2} - 1\right) J_n(\lambda) + \frac{1}{\lambda} J_{n+1}(\lambda) \\ I''_n(\lambda) = \left(\frac{n^2}{\lambda^2} + 1\right) I_n(\lambda) - \frac{1}{\lambda} I_{n+1}(\lambda) \end{cases} \tag{7}$$

which replaced in the second relationship (4) give us the frequency equation:

$$\frac{J_{n+1}(\lambda_{n,s})}{J_n(\lambda_{n,s})} + \frac{I_{n+1}(\lambda_{n,s})}{I_n(\lambda_{n,s})} = \frac{2\lambda_{n,s}}{1-\nu} \tag{8}$$

The solutions of the frequency equation (8) give us the eigenvalues $\lambda_{n,s}$. It can be observed that in the case of the circular plate simply supported, the frequency equation takes into account Poisson's ratio ν.

The normalized mode shape function can be written as:

$$\begin{cases} W(r,\theta)_{n,s} = A_n \left[J_n\left(\lambda_{n,s}, \frac{r}{a}\right) - \frac{J_n(\lambda_{n,s})}{I_n(\lambda_{n,s})} I_n\left(\lambda_{n,s}, \frac{r}{a}\right) \right] \cos(n\theta) \\ W(r,\theta)_{n,s} = A_n \left[J_n\left(\lambda_{n,s}, \frac{r}{a}\right) - \frac{J_n(\lambda_{n,s})}{I_n(\lambda_{n,s})} I_n\left(\lambda_{n,s}, \frac{r}{a}\right) \right] \sin(n\theta) \end{cases} \quad (9)$$

and the natural frequencies are expressed with relationship:

$$f_{n,s} = \frac{{\lambda_{n,s}}^2}{2\pi \cdot a^2} \sqrt{\frac{D}{\rho \cdot h}} \quad (10)$$

where,

s is the number of nodal circles;

$\lambda_{n,s}$ is the eigenvalue as a function of n and s.

3. Modal analysis by using FEM

The modal analysis used for Finite Element Method (FEM) simulation was performed using the SolidWorks software. The boundary conditions applied to the 3D circular plate model for simply supported conditions are presented in figure 2.

The dimensions of the elements (mesh) and the details provided by SolidWorks can be seen in figure 3. The dimensions of the circular plate and the physical characteristics and mechanical properties are the same as presented in Chapter 2.

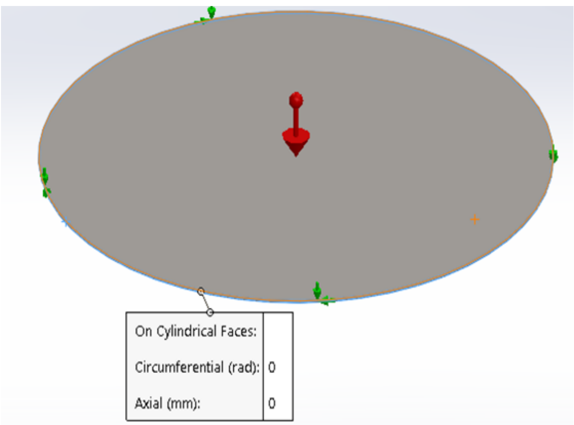

Figure 2. Boundary condition for simply supported circular plate.

Mesh Details

Study name	Frequency 1 (-Default-)
DetailsMesh type	Solid Mesh
Mesher Used	Blended curvature-based mesh
Jacobian points for High quality mesh	16 points
Max Element Size	5.03826 mm
Min Element Size	5.03826 mm
Mesh quality	High
Total nodes	310301
Total elements	154448
Maximum Aspect Ratio	5.9104
Percentage of elements with Aspect Ratio < 3	99.3
Percentage of elements with Aspect Ratio > 10	0
Percentage of distorted elements	0
Number of distorted elements	0
Time to complete mesh(hh:mm:ss)	00:00:08
Computer name	

Figure 3. Mesh details.

4. Results

The eigenvalues $\lambda_{n,s}$ depending on the number of nodal diameters n, respectively the number of nodal circles s, solutions of relation (8), calculated for 70 vibration modes are presented in the table 1 and the natural frequencies calculated with relation (10) are shown in table 2.

Table 1. Eigenvalues $\lambda_{n,s}$.

s	Nodal diameters n						
	0	1	2	3	4	5	6
0	2.215	3.725	5.059	6.319	7.538	8.728	9.898
1	5.449	6.961	8.372	9.723	11.031	12.308	13.562
2	8.610	10.137	11.588	12.987	14.347	15.677	16.982
3	11.760	13.296	14.771	16.201	17.595	18.961	20.303
4	14.906	16.448	17.939	19.390	20.809	22.201	23.571
5	18.051	19.597	21.100	22.567	24.004	25.416	26.807
6	21.194	22.744	24.255	25.734	27.186	28.614	30.022
7	24.337	25.890	27.407	28.896	30.359	31.800	33.222
8	27.480	29.034	30.557	32.053	33.526	34.978	36.412
9	30.623	32.178	33.706	35.208	36.689	38.150	39.594

Table 2. Natural frequencies $f_{n,s}$ [Hz].

s	Nodal diameters n						
	0	1	2	3	4	5	6
0	15.23	43.06	79.43	123.96	176.37	236.47	304.11
1	92.18	150.41	217.58	293.42	377.71	470.26	570.92
2	230.12	318.95	416.80	523.51	638.91	762.85	895.19
3	429.28	548.74	677.25	814.70	960.98	1115.94	1279.50
4	689.70	839.79	998.95	1167.10	1344.15	1529.99	1724.53
5	1011.39	1192.11	1381.91	1580.74	1788.51	2005.14	2230.55
6	1394.35	1605.70	1826.14	2055.63	2294.09	2541.46	2797.68
7	1838.57	2080.56	2331.64	2591.78	2860.92	3139.01	3425.98
8	2344.07	2616.69	2898.41	3189.20	3489.00	3797.78	4115.48
9	2910.84	3214.09	3526.45	3847.88	4178.35	4517.81	4866.21

The $\lambda_{n,s}$ values were obtained by iterative calculation. For example, for n=0, the first s=0, ..., 9 solutions of the frequency equation (8) resulted, respectively the values: $\lambda_{0,0}$, ..., $\lambda_{0,9}$, after which the number of n was increased from 1 to 6.

In the numerical modal analysis, the first 30 natural frequencies and vibration modes were obtained and they are shown in figure 4.

The natural frequencies obtained from FEM analysis, with identical values that appear at different vibration modes in figure 4, are due to the terms cos and sin from the analytically derived modal function (9). The natural frequency deviations ($\varepsilon_{n,s}$) obtained by the two calculation methods are very small and can be seen in table 3.

Table 3. Natural frequencies deviations $\varepsilon_{n,s} = f_{FEM}/f_{n,s} - 1$ [%].

s	Nodal diameters n						
	0	1	2	3	4	5	6
0	-0.0197	0.0046	-0.0013	0.0081	-0.0113	-0.0127	-0.0164
1	-0.0011	0.0000	-0.0094	-0.0102	-0.0132		
2	-0.0043	-0.0094	-0.0096				
3	-0.0093						

Study name:Frequency 1

Mode No.	Frequency(Rad/sec)	Frequency(Hertz)	Period(Seconds)
1	95.675	15.227	0.065672
2	270.57	43.062	0.023222
3	270.57	43.062	0.023222
4	499.07	79.429	0.01259
5	499.07	79.429	0.01259
6	579.18	92.179	0.010848
7	778.81	123.95	0.0080677
8	778.81	123.95	0.0080677
9	945.03	150.41	0.0066486
10	945.03	150.41	0.0066486
11	1,108.1	176.35	0.0056704
12	1,108.1	176.35	0.0056704
13	1,367	217.57	0.0045963
14	1,367	217.57	0.0045963
15	1,445.8	230.11	0.0043458
16	1,485.6	236.44	0.0042294
17	1,485.6	236.44	0.0042294
18	1,843.4	293.39	0.0034084
19	1,843.4	293.39	0.0034084
20	1,910.5	304.06	0.0032888
21	1,910.5	304.06	0.0032888
22	2,003.9	318.92	0.0031355
23	2,003.9	318.92	0.0031355
24	2,372.9	377.66	0.0026479
25	2,372.9	377.66	0.0026479
26	2,382	379.1	0.0026378
27	2,382	379.11	0.0026378
28	2,618.6	416.76	0.0023995
29	2,618.6	416.76	0.0023995
30	2,697	429.24	0.0023297

Figure 4. Natural frequencies f_{FEM} obtained by FEM.

Examples of modal shapes obtained analytically (left) and FEM (right) are presented in the figures 5 – 12. Figures 9 – 12 show the modal shapes of the circular plate simply supported that take into account both the cos function and the sin function, according to (9).

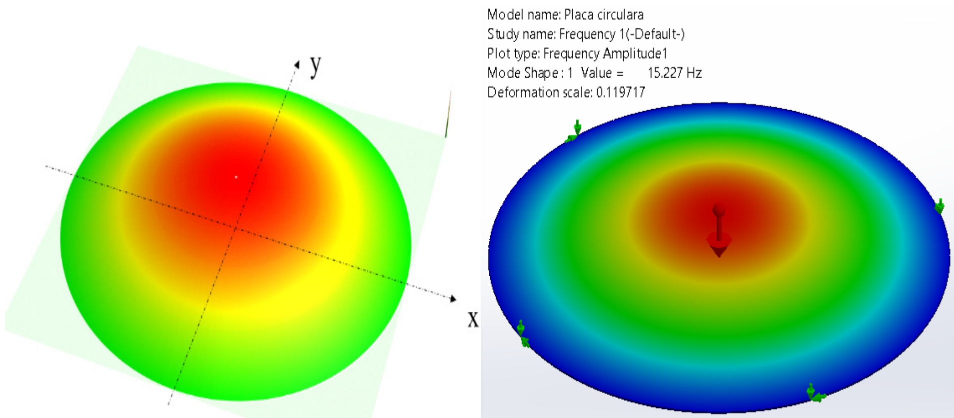

Figure 5. Simply supported circular plate. Mode shape for n=0 and s=0.

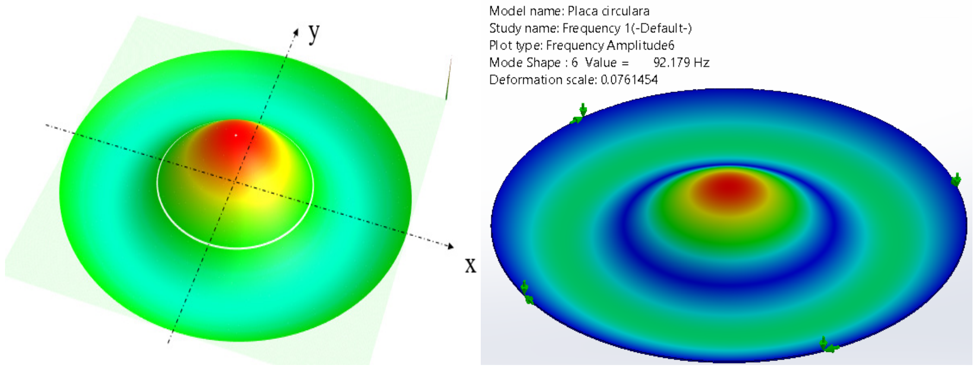

Figure 6. Simply supported circular plate. Mode shape for n=0 and s=1.

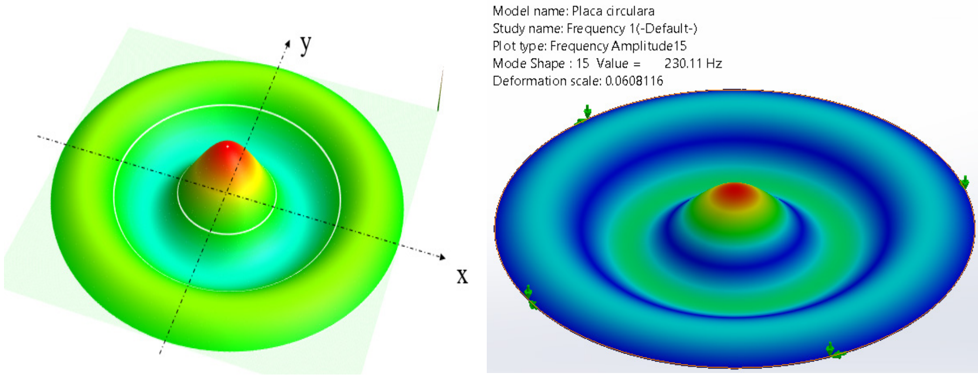

Figure 7. Simply supported circular plate. Mode shape for n=0 and s=2.

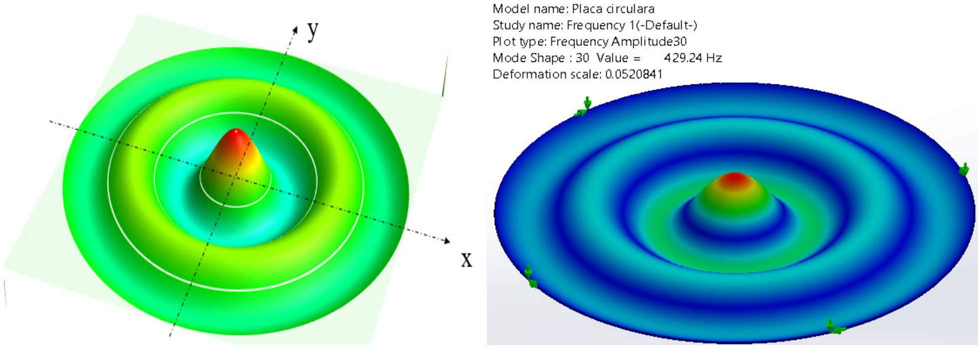

Figure 8. Simply supported circular plate. Mode shape for n=0 and s=3.

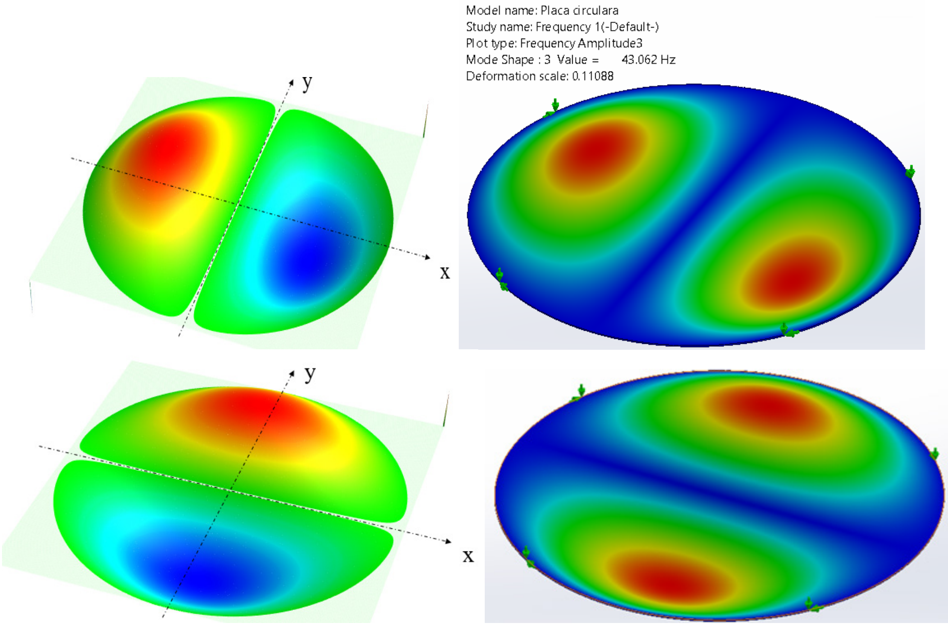

Figure 9. Simply supported circular plate. Mode shape for n=1 and s=0.

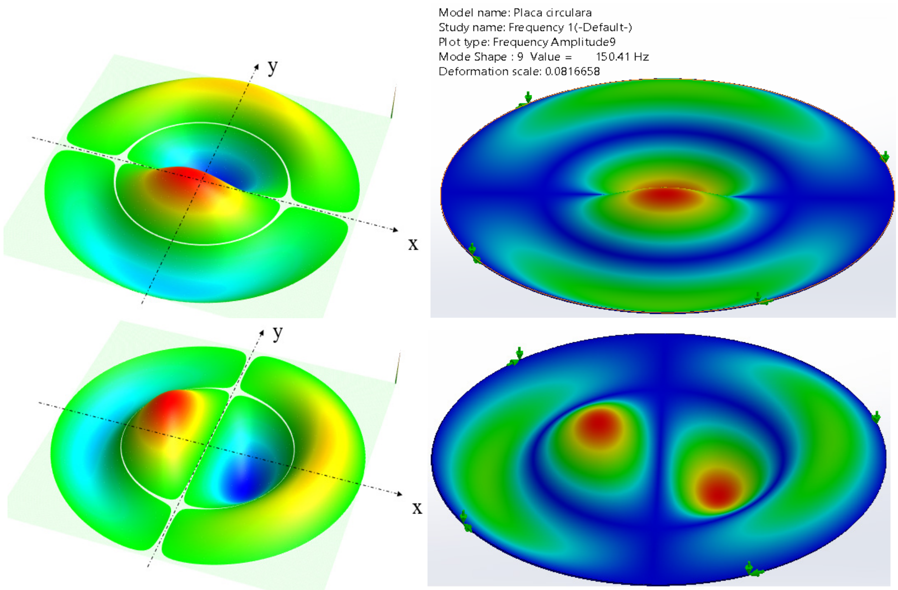

Figure 10. Simply supported circular plate. Mode shape for n=1 and s=1.

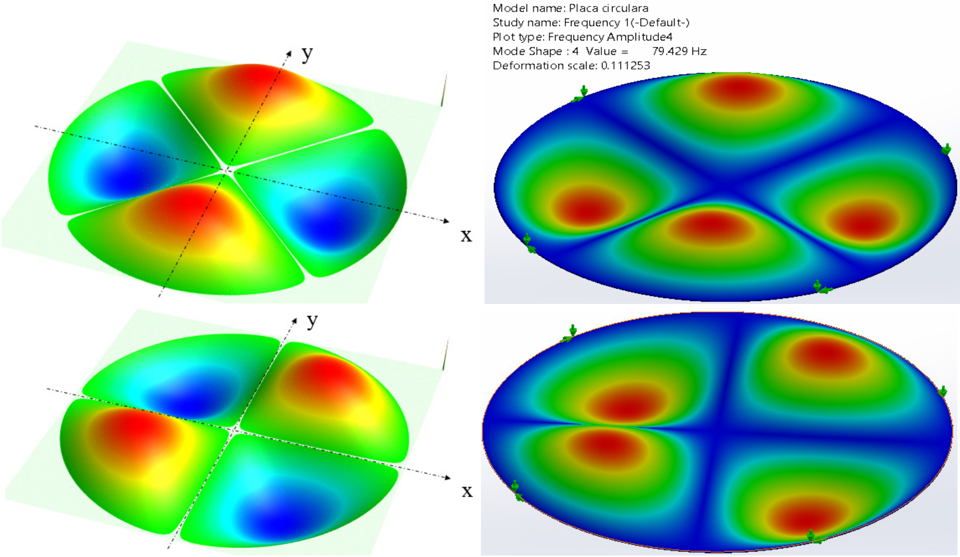

Figure 11. Simply supported circular plate. Mode shape for n=2 and s=0.

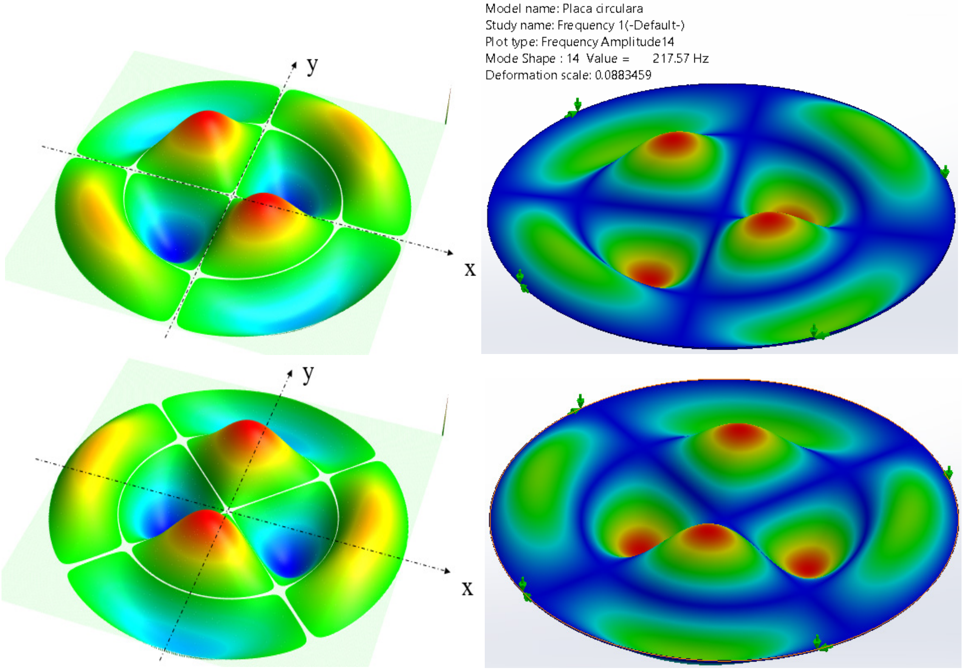

Figure 12. Simply supported circular plate. Mode shape for n=2 and s=1.

5. Conclusions

The paper presents the vibration modes for a simply supported circular plate in a 3D representation using MS Excel software and the comparison with modal shapes obtained from the modal analysis obtained using SolidWorks software.

The eigenvalues $\lambda_{n,s}$ were calculated for seven values of the nodal diameters n=0, 1, .., 6 and ten values of the nodal circles s=0, 1, ..., 9. For these values the natural frequencies were calculated.

The large number of natural frequencies (70) was chosen to illustrate the ease with which the natural frequencies, respectively the vibration modes, can be calculated by the analytical method compared to the modal numerical method. For example, out of the 30 results obtained by the numerical modal analysis, only the first 16 natural frequencies can be compared with the analytically determined natural frequencies which are presented in Table 3.

It should be taken into account that the results obtained from the modal numerical analysis give us the vibration modes for both variants of the modal function (9): $\cos(n\theta)$ and $\sin(n,\theta)$, vibration modes for which the natural frequency has the same value, respectively the same eigenvalue $\lambda_{n,s}$, according to (10).

The numerical results by FEM for the first 30 vibration modes of the circular plate have highlighted the fact that the natural frequency deviations obtained by the two methods are less than 0.02%.

From the analysis of figures 5 – 12, a very good correlation of the modal shapes obtained both analytically and through FEM can be observed.

References

1. L. Euler, De motu vibratorio tympanorum. Vol. 10, *Novi Commentarii academiae scientiarum Petropolitanae,* 1766, pp. 243-260.
2. Diarmud O. Mathuna, (1994), *Jacques II Bernoulli and The Problem of the Vibrating Plate.* Dublin Institute for Advanced Studies. Ireland.
3. E. Ventsel, T. Krauthammer, *Thin Plates and Shells Theory, Analysis, and Applications*. 1st Edition. The Pennsylvania State University Park, Pennsylvania, Copyright 2001 by Marcel Dekker, Inc. U.S.A.
4. S.P. Timoshenko, S. Woinowsky-Krieger, *Theory of Plates and Shells*, Copyright by McGraw-Hill Book Company, second edition, 1989, U.S.A.
5. A.W. Leissa, *Vibration of Plates,* NASA SP-160. Washington, DC U.S. Government Printing Office. Special Publication 2013.

6. J.R. Hutchinson, *Analysis of plates and shells by boundary collocations. In: Boundary Element Analysis of Plates and Shells*, Ed. D.E. Beskos, Springer-Verlag, Berlin, 1991, pp. 341–368.
7. I. Fried, *Linear and Nonlinear Finite Elements*, Boston University Department of Mathematics Boston, U.S.A, 1984.
8. T.Y. Yang, *Finite Element Structural Analysis.* Pearson College Div, New Jersey. U.S.A. 1985.
9. Bessel Functions of the First and Second Kind, https://pub.me755_web.tex (uwaterloo.ca), (downloaded at July 20rd, 2018)

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