STUDIA UNIVERSITATIS BABEȘ-BOLYAI
Engineering 68(1) 2023 DOI: 10.24193/subbeng.2023.1.5

The strain energy in loosening the clamped end of a beam (part II)

Dan-Alexandru Pîrșan, Zeno-Iosif Praisach*, Gilbert-Rainer Gillich, Cornel Hațiegan

Abstract. *In the second part of the paper, the dynamic behavior of a doubly clamped beam is presented, where the right clamped end of the beam is weakened by introducing a weakening coefficient. The analytical calculation is based on the determination of the bending moment from the weakened clamped end expressed as a function of slope, after which the modal function, strain energy, and the characteristic equation are determined to obtain the eigenvalues of the first six vibration modes depending on the weakened coefficient of the clamped end. The obtained mode shapes and strain energies are determined for seven values of the weakened coefficient.*

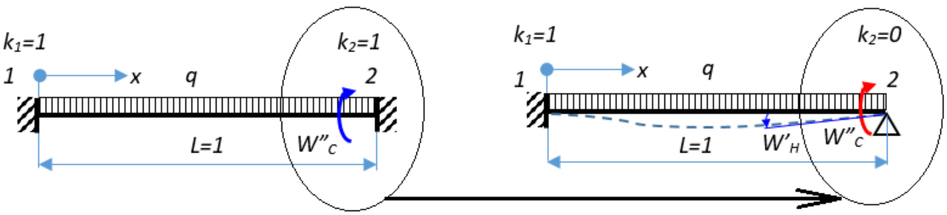
Keywords*: weak clamped end, mode shapes, strain energy*

1. Introduction

The second part of the paper is a continuation of part 1 and aims to analytically solve a weakened clamped end for a doubly clamped beam in terms of its dynamic behavior. For these reasons, the bibliographic citations are presented in the introduction chapter of first part. For this analysis case, it is considered that the weakened coefficient $k_1=1$ (left clamped support) and the dynamic behavior of the beam is analyzed for the right support with the weakened coefficient $k_2 \epsilon [0, \ldots, 1]$ which allows us to have both bending moment and slope at this support. So, that for the zero value of k_2 the support is considered to be a hinge, and for the 1 value of k_2, the support becomes clamped.

2. Analytical approach

For the normalized beam (L=1) of constant cross-section and loaded with dead weight (q), in accordance with the relations provided by the strength of the materials, for the right clamped (C) end (fig. 1 left), the bending moment can be expressed:



$k_2 \in [0, ..., 1]$

Figure 1. A schematic diagram with right end hinge is subjected to a bending moment (right) equal to that in the clamp end (left)

$$W_C''(L) = -\frac{q \cdot L^2}{12E \cdot I} \tag{1}$$

and for the hinge end (H) at x=L the slope can be written as:

$$W_H'(\mathrm{L}) = -\frac{q \cdot L^3}{48E \cdot I} \tag{2}$$

where,

q [N/m] – is the load per unit of length (dead load);

L [m] – is the beam length;

E [N/m^2] – is the modulus of elasticity of the beam material;

I [m^4] – is the moment of the inertia of the cross section of the beam.

If the bending moment (1) is applied to the hinge, the slope becomes positive, and the relationship (2) depending on the bending moment (1) becomes:

$$W_H'(\mathrm{L}) = \frac{q \cdot L^3}{48E \cdot I} = -\frac{L}{4}\left(-\frac{q \cdot L^2}{8E \cdot I}\right) = -\frac{L}{4} W_C''(\mathrm{L}) \tag{3}$$

or, expressing the bending moment from (3) and taking into account the stiffness k_2:

$$k_2 W_C''(\mathrm{L}) = -k_2 \frac{4}{L} W_H'(\mathrm{L}) \tag{4}$$

Taking into considerations the weakened coefficient k_2, the bending moment from the clamped end must be equal to the bending moment applied to the hinge:

$$k_2 W_C''(\mathrm{L}) = (1 - k_2) W_H''(\mathrm{L}) \tag{5}$$

also taking into account (4), in the right support it will obtained:

$$(1 - k_2) W_H''(\mathrm{L}) - k_2 W_C''(\mathrm{L}) = 0 = (1 - k_2) W_H''(\mathrm{L}) + k_2 \frac{4}{L} W_H'(\mathrm{L}) \tag{6}$$

From relation (6) if $k_2=0$, the bending moment is zero ($W''_H(L)=0$), that means for x=L we have a hinge, and if $k_2=1$, the slope is zero ($W'_H(L)=0$), so we have a clamped end. For any other values of $k_2 \in [0, ..., 1]$, the right support becomes a weakened clamped end and in this point we will find both bending moment and slope.

3. Eigenvalues, modal function and strain energy function

Using the procedure presented in the first part of the paper, the spatial solution of the differential equation of bending vibrations, free and undamped using Euler-Bernoulli model:

$$W(x) = \mathrm{A}\sin(\alpha x) + \mathrm{B}\cos(\alpha x) + \mathrm{C}\sinh(\alpha x) + \mathrm{D}\cosh(\alpha x) \tag{7}$$

where,

W(x) – is the modal motion function;

A, B, C, D – are integration constants that are obtained from the boundary conditions;

α – is the eigenvalue;

x – is the variable length of the normalized beam.

Boundary condition for clamped end (C), at x=0:

$$\begin{cases} W_C(0) = 0 = \mathrm{B} + \mathrm{D} \Rightarrow D = -B \\ W'_C(0) = 0 = \alpha(\mathrm{A} + \mathrm{C}) \Rightarrow C = -A \end{cases} \tag{8}$$

In the right support, the deflection is zero both for the hinge end and for the clamped end. Taking into account the results obtained in (8), introduced in (7) for x=L=1, the integration constant B is obtained:

$$\begin{cases} W(1) = 0 = \mathrm{A}(\sin\alpha - \sinh\alpha) + \mathrm{B}(\cos\alpha - \cosh\alpha) \\ \qquad \Rightarrow \mathrm{B} = -\mathrm{A}\frac{\sin\alpha - \sinh\alpha}{\cos\alpha - \cosh\alpha} \end{cases} \tag{9}$$

By introducing the constants B, C and D in relation (6), the characteristic equation (10) is obtained. Solutions (10) give us the eigenvalues for each vibration mode.

$$\alpha(1 - \mathrm{k}_2)(\sin\alpha \cdot \cosh\alpha - \cos\alpha \cdot \sinh\alpha) + k_2\frac{4}{L}(1 - \cos\alpha \cdot \cosh\alpha) = 0 \tag{10}$$

The expression of the modal function for the bending vibration modes is represented in relationship (11):

$$W(x) = \mathrm{A}\left[\sin(\alpha x) - \sinh(\alpha x) - \frac{\sin\alpha - \sinh\alpha}{\cos\alpha - \cosh\alpha}(\cos(\alpha x) - \cosh(\alpha x))\right] \tag{11}$$

The strain energy depends of the square of the curvature of the modal function, and is defined as:

$$E_{p_n} = \frac{1}{2}\int_0^L E \cdot I \cdot W"(x)^2\, dx \quad (12)$$

From relation (11) we obtain the curvature expression, which its squared is directly proportional to the normalized strain energy:

$$W"(x)^2 = \left\{A\alpha^2 \left[\frac{\sin\alpha - \sinh\alpha}{\cos\alpha - \cosh\alpha}(\cos(\alpha x) + \cosh(\alpha x)) - \sin(\alpha x) - \sinh(\alpha x)\right]\right\}^2 \quad (13)$$

4. Results

The eigenvalues α_n for the first six vibration modes (n=6) and different values of k_2, solutions of relationship (10) for L = 1, can be found in table 1.

Table 1. Eigenvalues for the n=6 vibration modes and different values of stiffness coefficient k_2

k_2	Vibration mode (n)					
	1	2	3	4	5	6
0.0	3.926602	7.068583	10.21018	13.35177	16.49336	19.63495
0.1	3.982267	7.098856	10.23143	13.36812	16.50664	19.64614
0.2	4.041832	7.133840	10.25663	13.38776	16.52273	19.65975
0.3	4.105765	7.174669	10.28695	13.41177	16.54258	19.67667
0.4	4.174604	7.222855	10.32407	13.44177	16.56771	19.69827
0.5	4.248967	7.280433	10.37048	13.48025	16.60047	19.72675
0.6	4.329552	7.350196	10.42997	13.53129	16.64490	19.76597
0.7	4.417137	7.436006	10.50854	13.60187	16.70831	19.82325
0.8	4.512551	7.543231	10.61608	13.70490	16.80534	19.91407
0.9	4.616622	7.679200	10.76929	13.86595	16.96874	20.07655
1.0	4.730041	7.853205	10.99561	14.13717	17.27876	20.42035

The first n=6 (six) normalized vibration modes ($\pm$W(x) – relationship (11)) and the normalized strain energy (W"$(x)^2$ – relationship (13)) for the following values of k_2=0.0, 0.25, 0.50, 0.75, 0.85, 0.95 and 1, 00 are illustrated in the figures 2 - 8.

For this analyzed cases the weakened coefficient k_1 is considered to be 1.

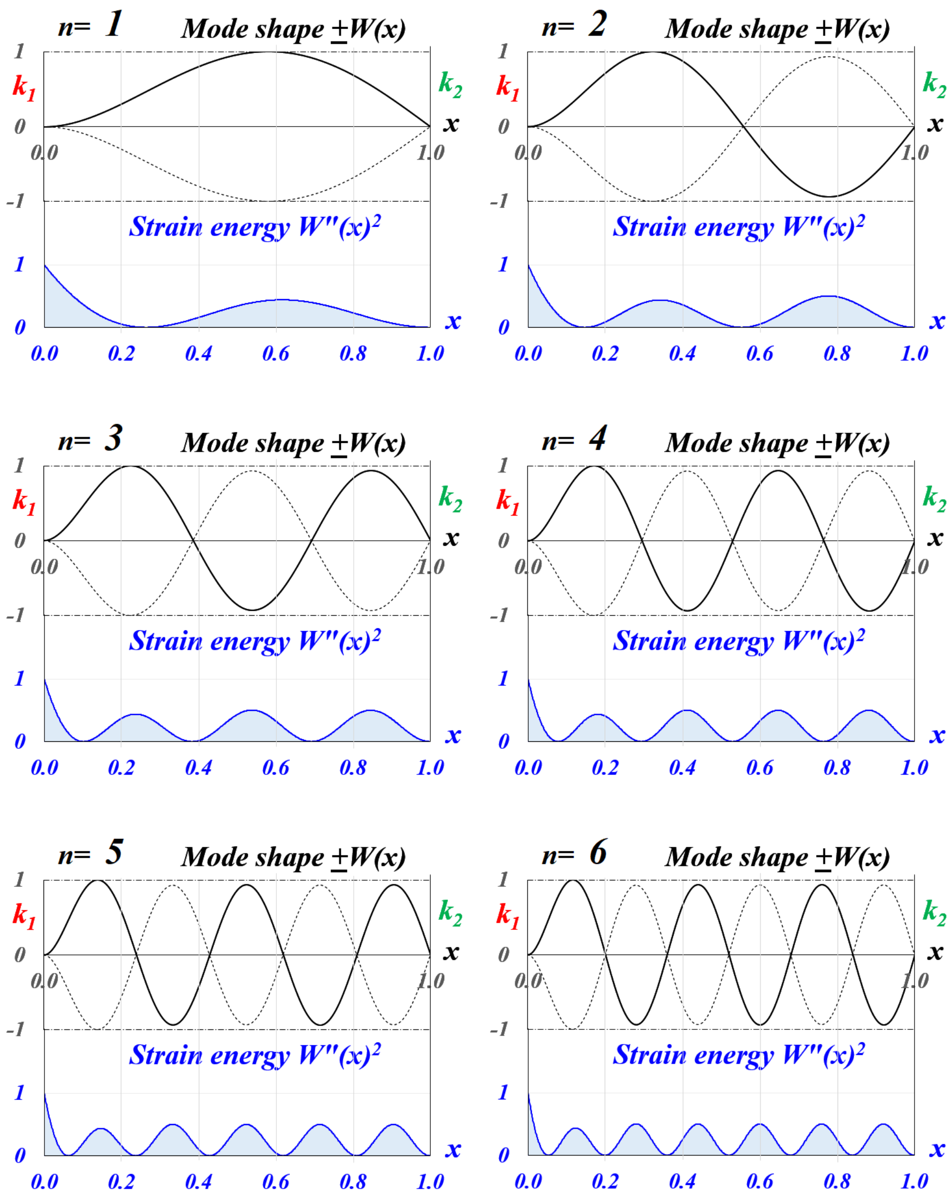


Figure 2. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_2=0.0

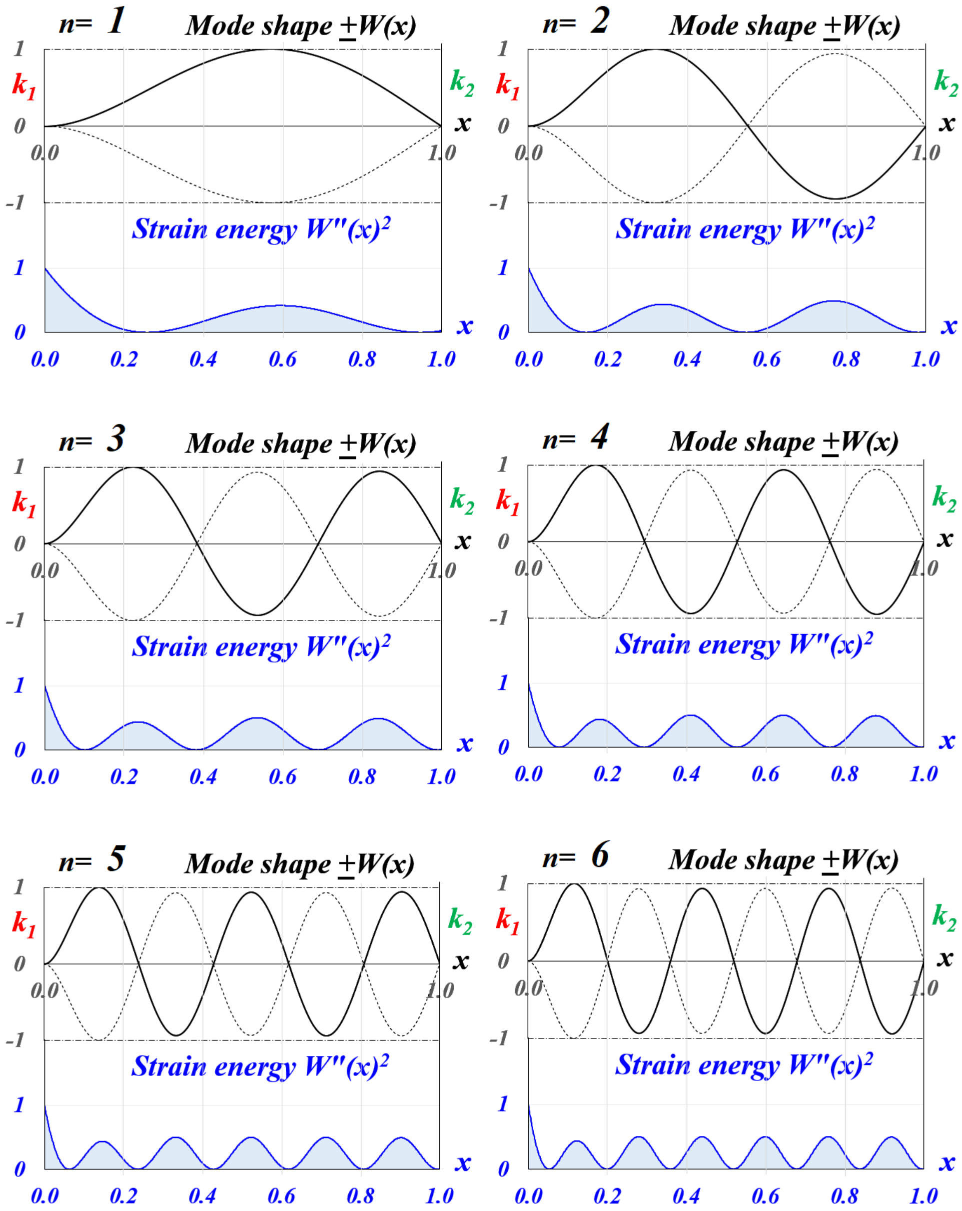


Figure 3. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_2=0.25

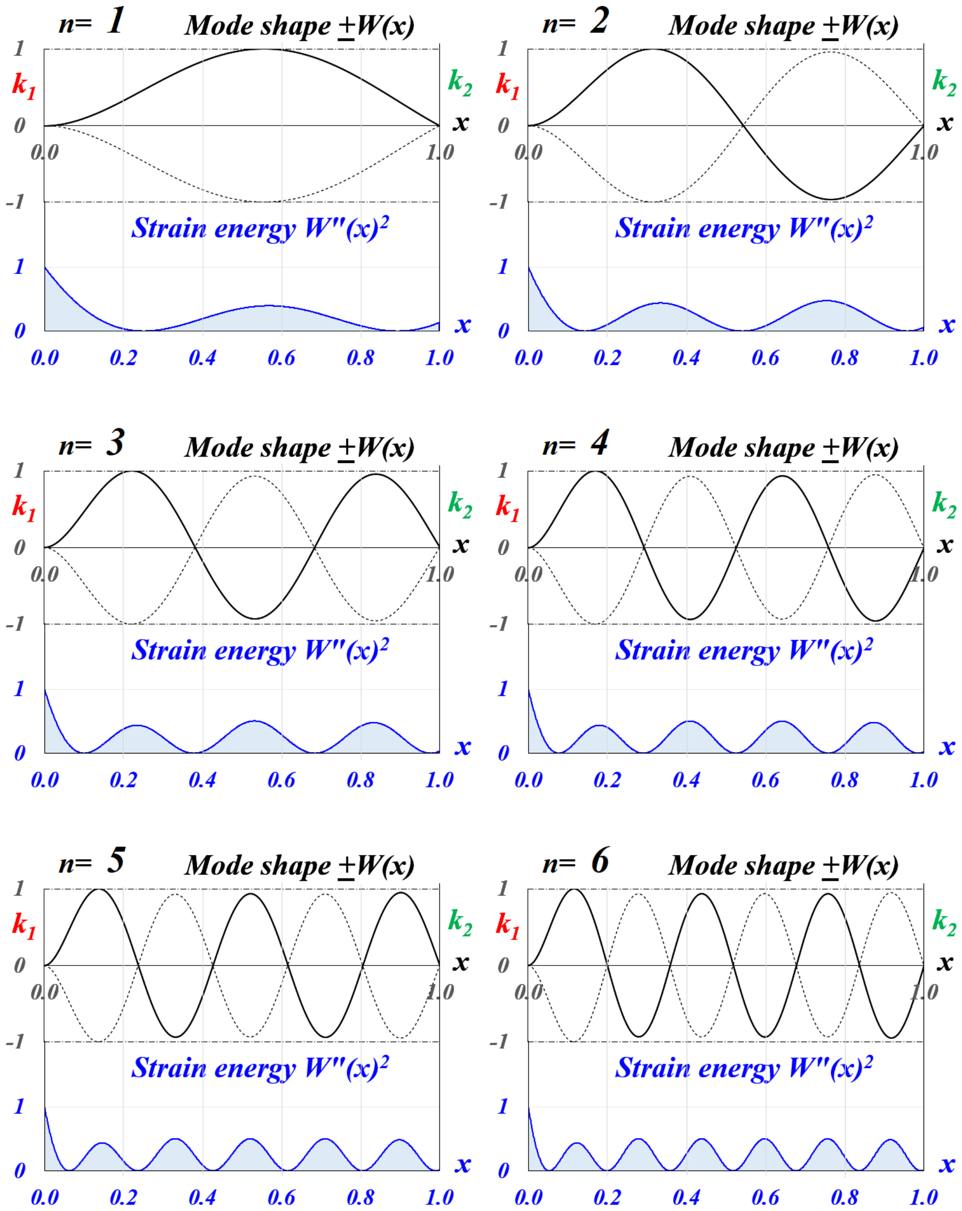


Figure 4. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_2=0.50

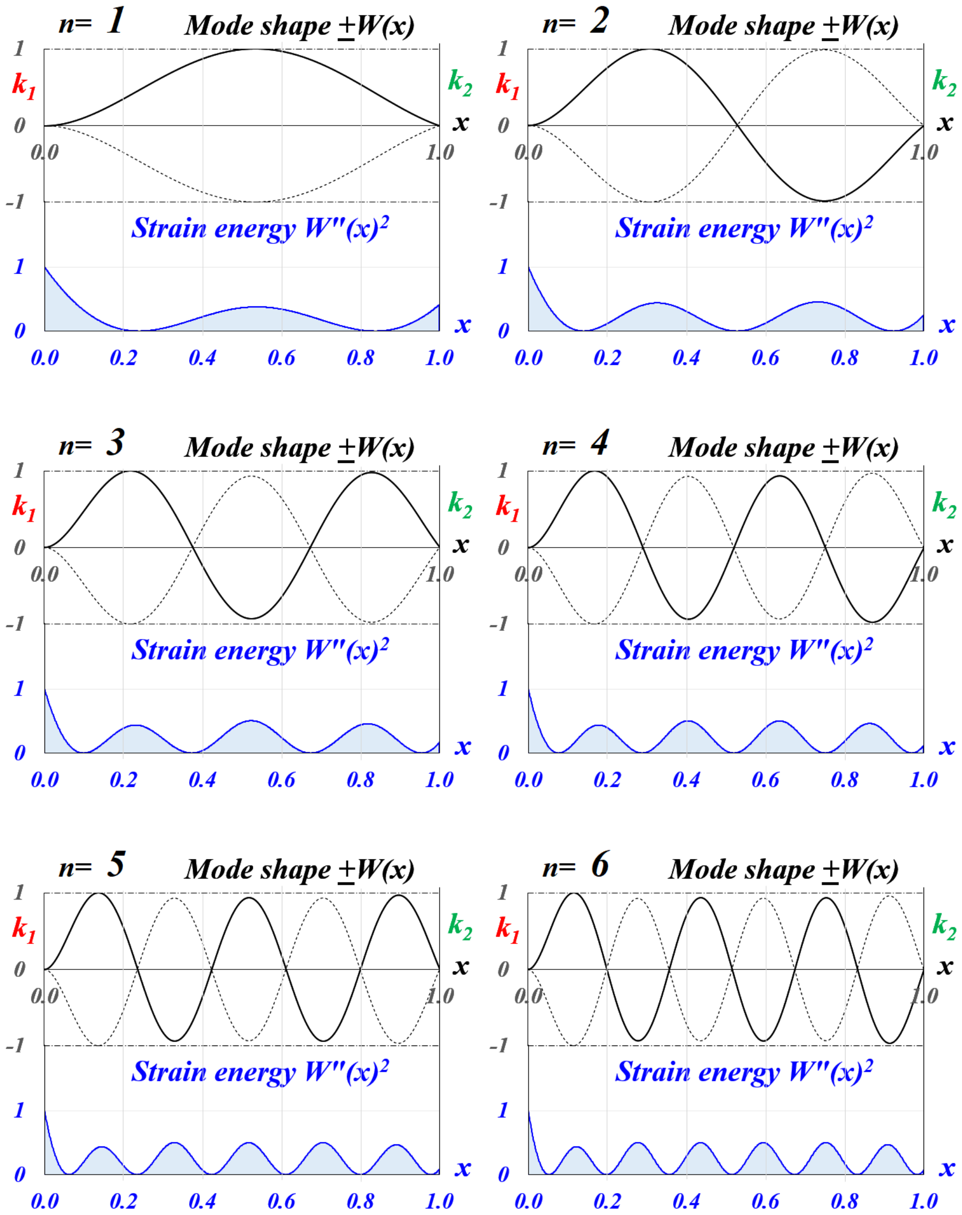


Figure 5. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_2=0.75

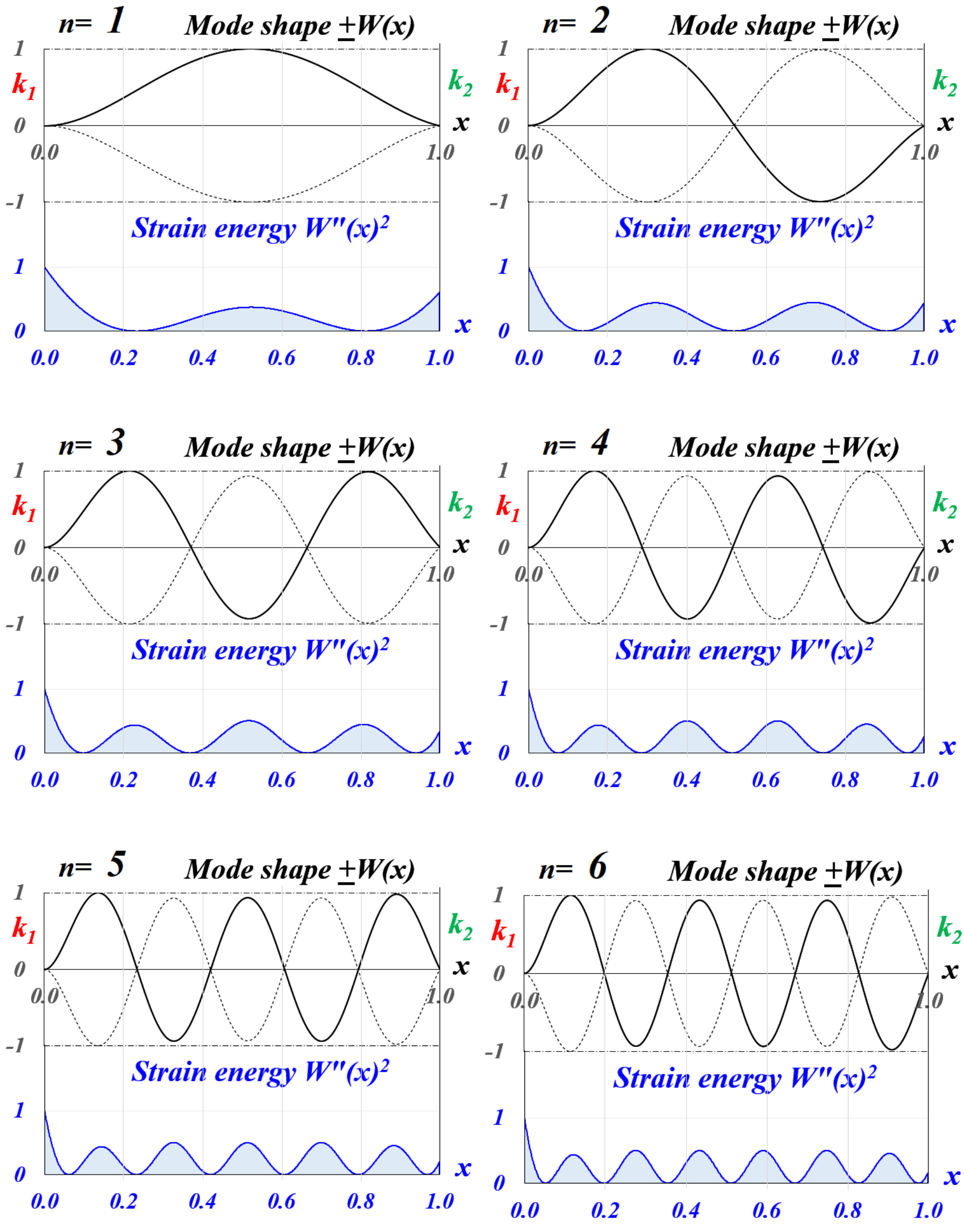


Figure 6. Normalized mode shapes (±W(x)) and normalized strain energy (W"(x)^2) for the first n=6 vibration modes and k_2=0.85

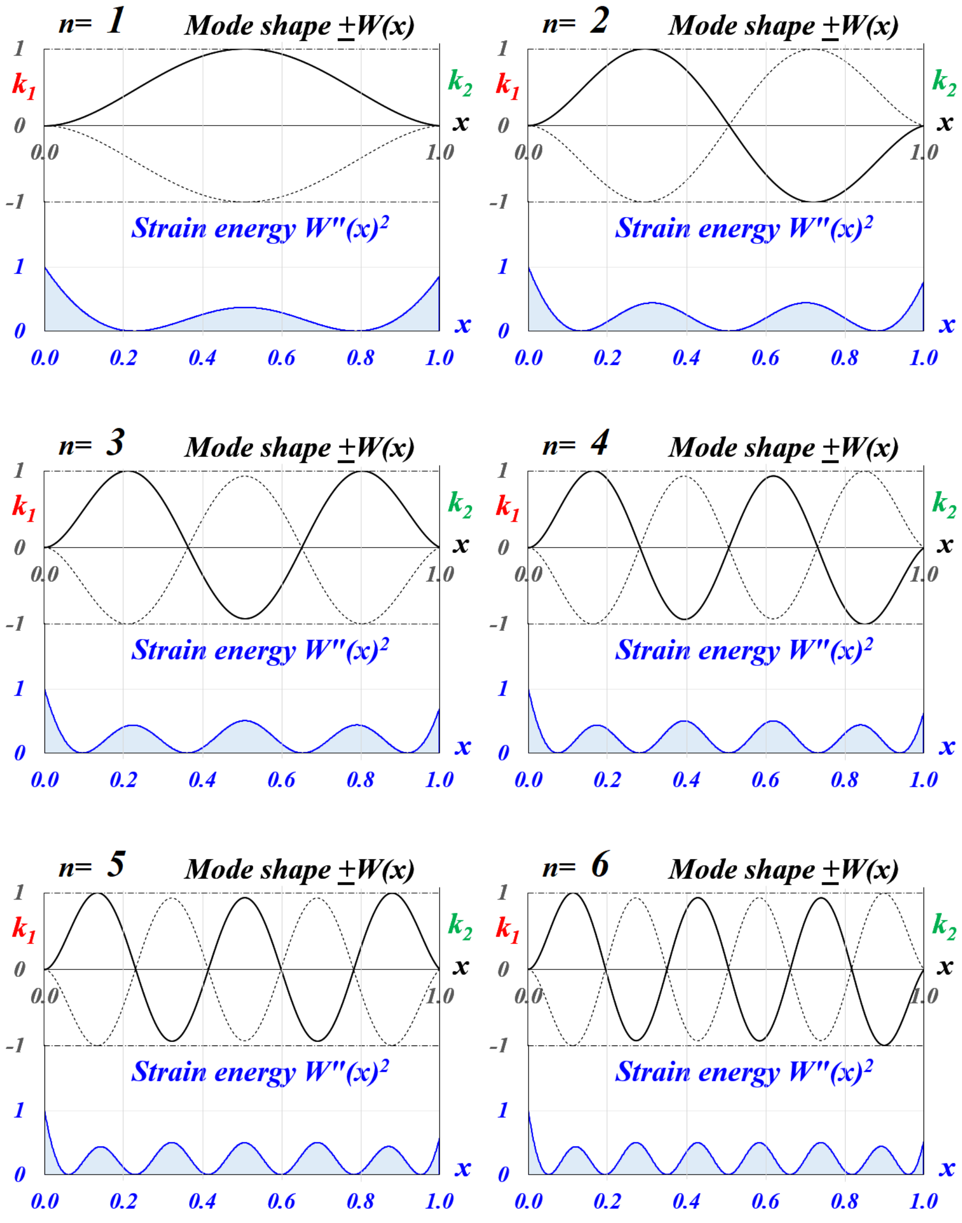


Figure 7. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_2=0.95

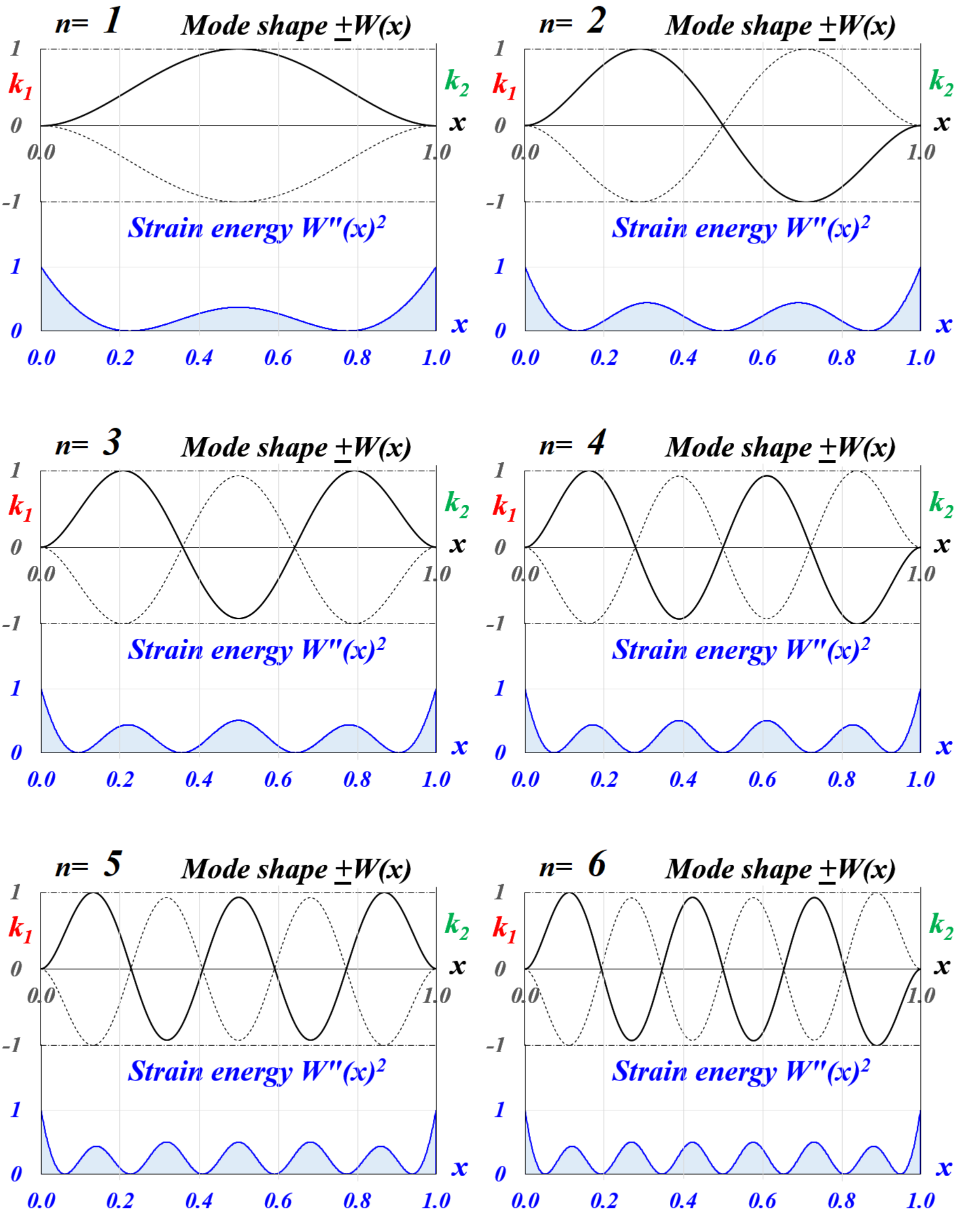


Figure 8. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_2=1.0

5. Conclusions

The paper presents the eigenvalues, modal shapes and strain energy for the six bending vibration modes for the case where the right clamped end of the beam is weakened by the coefficient $k_2 \epsilon [0, \ldots, 1]$ and the left end of the beam is clamped.

For the extreme cases: $k_2=0$, the eigenvalues (Table 1) were obtained for the beam clamped at one end and hinged at the other; respectively for $k_2=1$, we find the eigenvalues for the double clamped beam.

From the analysis of the figures from figures 2 – 5, it can be observed that for stiffness values of $k_2<0.75$, from the point of view of the mode shapes, the soft clamped end has a behavior very close to that of a hinged support.

Instead, the normalized strain energy in the right support, which in the hinge has zero value ($k_2=0$), with the increase of k_2, its value increases, reaching ~2.5% of the maximum value for $k_2=0.25$, ~13.4% of the maximum value for $k_2=0.5$ and ~41% of the maximum value for $k_2=0.75$, at first vibration mode.

For the second vibration mode, the normalized strain energy has smaller values in this point, respectively, ~0.8% for $k_2=0.25$, and ~5.7% for $k_2=0.5$ and ~25% for $k_2=0.75$ and the trend continues for the other vibration modes.

For stiffness values $k_2>0.85$ (figures 6 – 8), the mode shapes of the beam are significantly affected and although the relation (6) that describes the soft clamped end of the beam is a linear expression of k_2, the effect of k_2 in the modal function does not have a linear behavior. At the point x=L, the normalized strain energy becomes maximum for $k_2=1.0$ for all vibration modes.

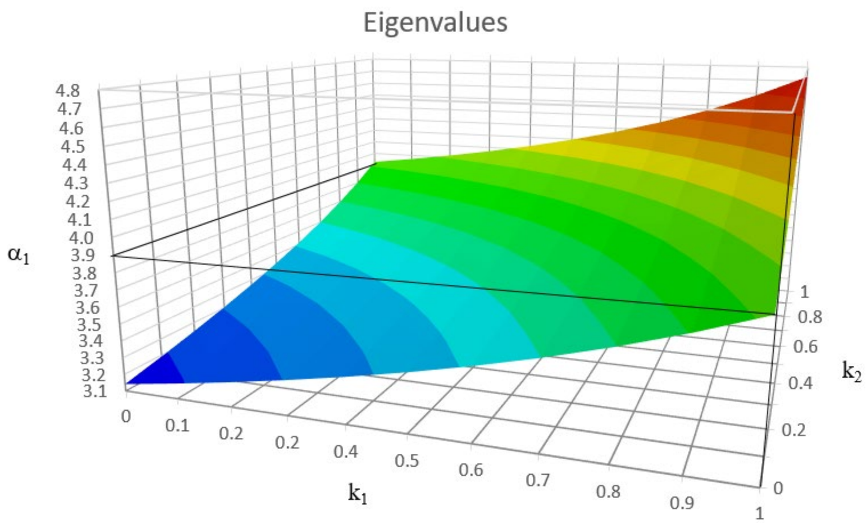


Figure 9. First vibration mode eigenvalues for $k_1 \epsilon [0, \ldots, 1]$ and $k_2 \epsilon [0, \ldots, 1]$

Taking into account the results presented in the first and second part of the paper, in figure 9 it can be seen in 3D representation, the eigenvalues for the first mode of vibration for all the evolution of the weakened coefficients k_1 and k_2.

The minimum value $\alpha_1=\pi$ corresponds to the simply supported case, the peaks for $k_1=1$, $k_2=0$, respectively $k_1=0$, $k_2=1$, correspond to the clamp-hinge, respectively hinge-clamp cases, with the eigenvalue of $\alpha_1=3.926602$, and the maximum value $\alpha_1=4.730041$ corresponds to the double clamped beam.

References

1 Lupu D., Tufiși C., Gillich G.R., Ardeljan M., Detection of transverse cracks in prismatic cantilever beams affected by weak clamping using a machine learning method, *Analecta Technica Szegedinenesia*, 16(1), 2022, pp. 122-128.

2 Lupu D., Gillich G.R., Nedelcu D., Gillich N., Mănescu T., *A method to detect cracks in the beams with imperfect boundary conditions,* International Conference on Applied Science (ICAS 2020), Journal of Physics: Conference Series 1781(2021) 012012, IOP Publishing, pp. 1-13.

3 Praisach Z.I., Ardeljan D., Pîrșan D.A, Gillich G.R., A new approach for imperfect boundary conditions on the dynamic behavior, *Analecta Technica Szegedinenesia*, 16(1), 2022, pp. 56-61.

4 Gillich G.R., Praisach Z.I., Exact solution for the natural frequencies of slender beams with an abrupt stiffness decrease, *Journal of Engineering Sciences and Innovation*, 2(1) 2017, *A. Mechanics, Mechanical and Industrial Engineering, Mechatronics,* pp. 13-21.

5 Karthikeyan M., Tiwari R., Talukdar S., *Identification of crack model parameters in a beam from modal parameters*, in 12th National Conference on Machines and Mechanisms (NaCoMM-2005), 2005.

6 Nahvi H., Jabbari M., Crack detection in beams using experimental modal data and finite element model, *International Journal of Mechanical Sciences*, 47(10), 2005, pp. 1477–1497.

7 Dems K., Turant J., Structural damage identification using frequency and modal changes, *Bulletin of the Polish Academy of Sciences Technical Sciences*, 59(1), 2011, pp. 23–32.

8 Gillich G.R., Nedelcu D., Wahab M.A., Pop M.V., Hamat C.O., *A new mathematical model for cracked beams with uncertain boundary conditions*, International Conference on Noise and Vibration Engineering (ISMA 2020), Leuven, Belgium, pp. 3871-3883.

9 Shi D., Tian Y., Choe K.N., Wang Q., A weak solution for free vibration of multi-span beams with general elastic boundary and coupling condition, *JVE International Ltd. Vibroengineering PROCEDIA*, vol. 10, 2016, pp. 298–303.

Addresses:

- Phd stud. Eng. Dan-Alexandru Pîrșan, Doctoral School of Engineering, Babeș-Bolyai University, Cluj-Napoca, Romania, Faculty of Engi-neering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, Romania
  dan.pirsan@ubbcluj.ro
- Lect. PhD. Habil. Zeno-Iosif Praisach, Department of Engineering Science, Babeș-Bolyai University. Cluj-Napoca, Romania, Faculty of Engineering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, Romania
  zeno.praisach@ubbcluj.ro
  (*[*]corresponding author*)
- Prof. PhD. Gilbert-Rainer Gillich, Department of Engineering Science, Babeș-Bolyai University, Cluj-Napoca, Romania, Faculty of Engineering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, Romania
  gilbert.gillich@ubbcluj.ro
- Lect. PhD. Cornel Hațiegan, Department of Engineering Science, Babeș-Bolyai University, Cluj-Napoca, Romania, Faculty of Engineering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, Romania
  cornel.hatiegan@ubbcluj.ro