STUDIA UNIVERSITATIS BABEȘ-BOLYAI
Engineering 68(1) 2023 DOI: 10.24193/subbeng.2023.1.4

The strain energy in loosening the clamped end of a beam (part I)

Patric-Timotei Stan, Zeno-Iosif Praisach*, Gilbert-Rainer Gillich, Tiberiu Mănescu, Cristian Tufiși

Abstract. *Using analytical equations, the paper aims to solve the dynamic behavior of beams where a clamped end of the beam does not respect the ideal boundary conditions by introducing a weakening coefficient. In the paper, the characteristic equation for determining the eigenvalues and the relationship of the modal function and strain energy are derived. The results show the first six vibration modes for different values of the weakening coefficient which is considered in the clamped end and the evolution of the strain energy.*

Keywords: *weak clamped end, mode shapes, strain energy*

1. Introduction

Different types of failures it can be occurred in mechanical structures and they can be caused by a lots of factors, such as: loosening of joints due to excessive vibrations and shocks, degradation caused by environmental conditions, material fatigue and exceeding the expected operating demands, improper manufacturing conditions [1].

From dynamic behavior, the analysis of beams with fixed ends involves the consideration of displacements and slopes perfect boundary conditions. Most researchers use a measurement of natural frequencies to characterize imperfect boundary conditions, while others consider modal shapes to detect deviation from ideal conditions [2]. The precise calculation of the natural frequencies is significantly influenced by the correct positioning of the supports, respectively by the correct choice of the boundary conditions [3, 4].

The loss of integrity of structures can be attributed not only to the presence of cracks but also to joint failure, especially for beam-type structures. Methods used of modal parameters prove reliable for the detection and evaluation of damage in beams by applying several techniques like flexibility coefficients, derived stiffness matrix, the frequency response function FRF [5 – 7].

©2023 Studia UBB Engineering. Published by Babeş-Bolyai University.
This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License

Because real beams have non-ideal boundary conditions, it is necessary to use advanced models to determine the real modal parameters [8].

In this paper, the authors present an analytical solution regarding the dynamic behavior of a beam for which the clamped end is defined by a weakening coefficient $k_1 \in [0, \ldots, 1]$, which allows us to have both bending moment and slope at this support [9]. So, that for the zero value of k_1 the support is considered to be a hinge, and for the 1 value of k_1, the support becomes clamped.

2. Analytical approach

It is analyzed the dynamic behavior of a normalized beam (L=1) under the action of its dead weight (q) with constant cross-section. It is considered the case of a simply supported beam (Fig. 1) where a bending moment equal to the moment is introduced on the left hinge bending moment of a clamped end multiplied by a stiffness coefficient k_1 which can have values between [0, …, 1]. The right support has the stiffness coefficient k_2=0, so, at this point we have a hinge.

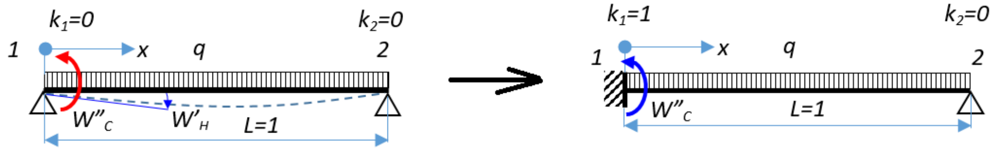

Figure 1. A schematic diagram with left end hinge is subjected to a bending moment (left) equal to that in the clamp end (right)

It is known from the strength of materials that for a hinge support (H) located at x=0 the slope has the expression:

$$W'_H(0) = \frac{q \cdot L^3}{24E \cdot I} \tag{1}$$

and for the clamped end (C) at x=0 the bending moment can be written as:

$$W''_C(0) = -\frac{q \cdot L^2}{8E \cdot I} \tag{2}$$

where,

q [N/m] – is the load per unit of length (dead load);
L [m] – is the beam length;
E [N/m^2] – is the Young's modulus;
I [m^4] – is the moment of the inertia of the cross section of the beam.

If the bending moment from relation (2) is applied to the hinged at x=0 (Fig. 1 – left), it becomes a clamped end (Fig. 1 – right), and the slope from relation (1) becomes negative. The connection between the slope and the bending moment from (2) can be written:

$$W_H'(0) = -\frac{q \cdot L^3}{24E \cdot I} = -\frac{L}{3}\left(\frac{q \cdot L^2}{8E \cdot I}\right) = -\frac{L}{3}\left(-W_C''(0)\right) = \frac{L}{3} W_C''(0) \tag{3}$$

or, expressing the bending moment from (3) and taking into account the stiffness k_1:

$$k_1 W_C''(0) = k_1 \frac{3}{L} W_H'(0) \tag{4}$$

Under these conditions and taking into considerations the weakened coefficient, the bending moment from the clamped end must be equal to the bending moment applied to the hinge:

$$k_1 W_C''(0) = (1 - k_1) W_H''(0) \tag{5}$$

also taking into account (4), in the left support we get:

$$(1 - k_1) W_H''(0) - k_1 W_C''(0) = 0 = (1 - k_1) W_H''(0) - k_1 \frac{3}{L} W_H'(0) \tag{6}$$

From relation (6) it can be seen that for k_1=0, the bending moment is zero ($W''_H(0)=0$), so in x=0 we have a hinge, and for k_1=1, the slope is zero ($W'_H(0)=0$), so we have a clamped end. For any other values of $k_1 \in [0, ..., 1]$, in the left support we will find both bending moment and slope.

3. Modal analysis and the strain energy

It can be started from the spatial solution of the differential equation of bending vibrations, free and undamped using Euler-Bernoulli model:

$$W(x) = \mathrm{A}\sin(\alpha x) + \mathrm{B}\cos(\alpha x) + \mathrm{C}\sinh(\alpha x) + \mathrm{D}\cosh(\alpha x) \tag{7}$$

where,

W(x) – is the modal motion function;

A, B, C, D – are integration constants that are obtained from the boundary conditions;

α – is the eigenvalue;

x – is the variable length of the normalized beam.

Boundary condition for clamped end (C) and hinged end (H), at x=0:

$$\begin{cases} W_H(0) = W_C(0) = 0 = \mathrm{B} + \mathrm{D} \Rightarrow D = -B \\ W'_C(0) = 0 = \alpha(\mathrm{A} + \mathrm{C}) \\ W_H''(0) = -2\alpha^2 \mathrm{B} \end{cases} \tag{8}$$

Boundary condition for hinged end (H), at x=L=1:

$$\begin{cases} W(L) = W(1) = 0 = \mathrm{A}\sin\alpha + \mathrm{B}(\cos\alpha - \cosh\alpha) + \mathrm{C}\sinh\alpha \\ W''(L) = W''(1) = 0 = \alpha^2[-\mathrm{A}\sin\alpha - \mathrm{B}(\cos\alpha + \cosh\alpha) + \mathrm{C}\sinh\alpha] \end{cases} \quad (9)$$

and the integration constants *B* and *C* are obtained from system (9) are:

$$\begin{cases} B = -\mathrm{A}\frac{\sin\alpha}{\cos\alpha} \\ C = -\mathrm{A}\frac{\sin\alpha \cdot \cosh\alpha}{\cos\alpha \cdot \sinh\alpha} \end{cases} \quad (10)$$

By introducing the constants B, C and D in relation (6), the characteristic equation (11) is obtained. Solutions (11) give us the eigenvalues for each vibration mode.

$$2\alpha(1 - \mathrm{k}_1)\sin\alpha \cdot \sinh\alpha + k_1\frac{3}{L}(\sin\alpha \cdot \cosh\alpha - \cos\alpha \cdot \sinh\alpha) = 0 \quad (11)$$

The eigenvalues α_n for the first six vibration modes (n=6) and different values of k_1, solutions of relationship (11) for L = 1, can be found in table 1.

Table 1. Eigenvalues for the n=6 vibration modes and different values of stiffness coefficient k_1

k_1	Vibration mode (n)					
	1	2	3	4	5	6
0.0	π	2·π	3·π	4·π	5·π	6·π
0.1	3.191179	6.308917	9.442121	12.57945	15.71845	18.85832
0.2	3.244789	6.338981	9.462877	12.59527	15.73124	18.86904
0.3	3.303022	6.374539	9.488147	12.61482	15.74717	18.88247
0.4	3.366603	6.417199	9.519554	12.63956	15.76753	18.89977
0.5	3.436416	6.469232	9.559584	12.67182	15.79448	18.92287
0.6	3.513548	6.533943	9.612229	12.71560	15.83176	18.95527
0.7	3.599335	6.616272	9.684261	12.77816	15.88658	19.00385
0.8	3.695415	6.723820	9.787957	12.87416	15.97451	19.08434
0.9	3.803753	6.868497	9.947185	13.03697	16.13541	19.24065
1.0	3.926602	7.068583	10.21018	13.35177	16.49336	19.63495

The expression of the modal function for the bending vibration modes is represented in relationship (12):

$$W(x) = \mathrm{A}\left[\sin(\alpha x) - \frac{\sin\alpha}{\cos\alpha}(\cos(\alpha x) - \cosh(\alpha x)) - \frac{\sin\alpha\cdot\cosh\alpha}{\cos\alpha\cdot\sinh\alpha}\sinh(\alpha x)\right] \quad (12)$$

The strain energy, or potential energy, is a function of the square of the curvature of the modal function, and is defined as:

$$E_{p_n} = \frac{1}{2}\int_0^L E \cdot I \cdot W''(x)^2\, dx \quad (13)$$

From relation (12) we obtain the curvature expression, which its squared is directly proportional to the normalized strain energy:

$$W''(x)^2 = \left\{\mathrm{A}\alpha^2\left[-\sin(\alpha x) + \frac{\sin\alpha}{\cos\alpha}(\cos(\alpha x) + \cosh(\alpha x)) - \frac{\sin\alpha\cdot\cosh\alpha}{\cos\alpha\cdot\sinh\alpha}\sinh(\alpha x)\right]\right\}^2 \quad (14)$$

4. Results

Below are presented the first n=6 (six) normalized vibration modes and the normalized strain energy for the following values of k_1=0.0, 0.25, 0.50, 0.75, 0.85, 0.95 and 1, 00 (k_2=0). The obtained results are illustrated in figures 2-8.

5. Conclusions

The paper presents the eigenvalues, modal shapes and strain energy for bending vibration modes for the case where the left clamped end of the beam is weakened by the coefficient k ϵ[0, …, 1] and the right end of the beam is a hinge.

For the extreme cases: k_1=0, the eigenvalues (Table 1) were obtained for the simply supported beam (hinged at both ends); respectively for k_1=1, we find the eigenvalues for the beam clamped at one end and hinged at the other.

From the analysis of the figures from figures 2 – 4, it can be observed that for stiffness values of $k_1<0.5$, from the point of view of the mode shapes, the soft clamped end has a behaviour very close to that of a hinged support.

Instead, the normalized strain energy in the left support, which in the hinge has zero value (k_1=0), with the increase of k_1, its value increases, reaching ~7% of the maximum value for k_1=0.25 and ~37% of the maximum value for k_1=0.5, at first vibration mode. For the second vibration mode, the normalized strain energy has smaller values in this point, respectively, ~2% for k_1=0.25 and ~14% for k_1=0.5 and the trend continues for the other vibration modes.

For stiffness values $k_1>0.75$ (figures 5 – 8), the mode shapes of the beam are significantly affected and although the relation (6) that describes the soft clamped end of the beam is a linear expression of k_1, the effect of k_1 in the modal function does not have a linear behavior. At the point x=0, the normalized strain energy becomes maximum for k_1=0.74 at the first vibration mode, k_1=0.86 at the 2nd vibration mode, k_1=0.9 at the 3rd vibration mode, k_1=0, 925 at vibration mode 4, k_1=0, 94 at vibration mode 5 and k_1=0, 95 at vibration mode 6.

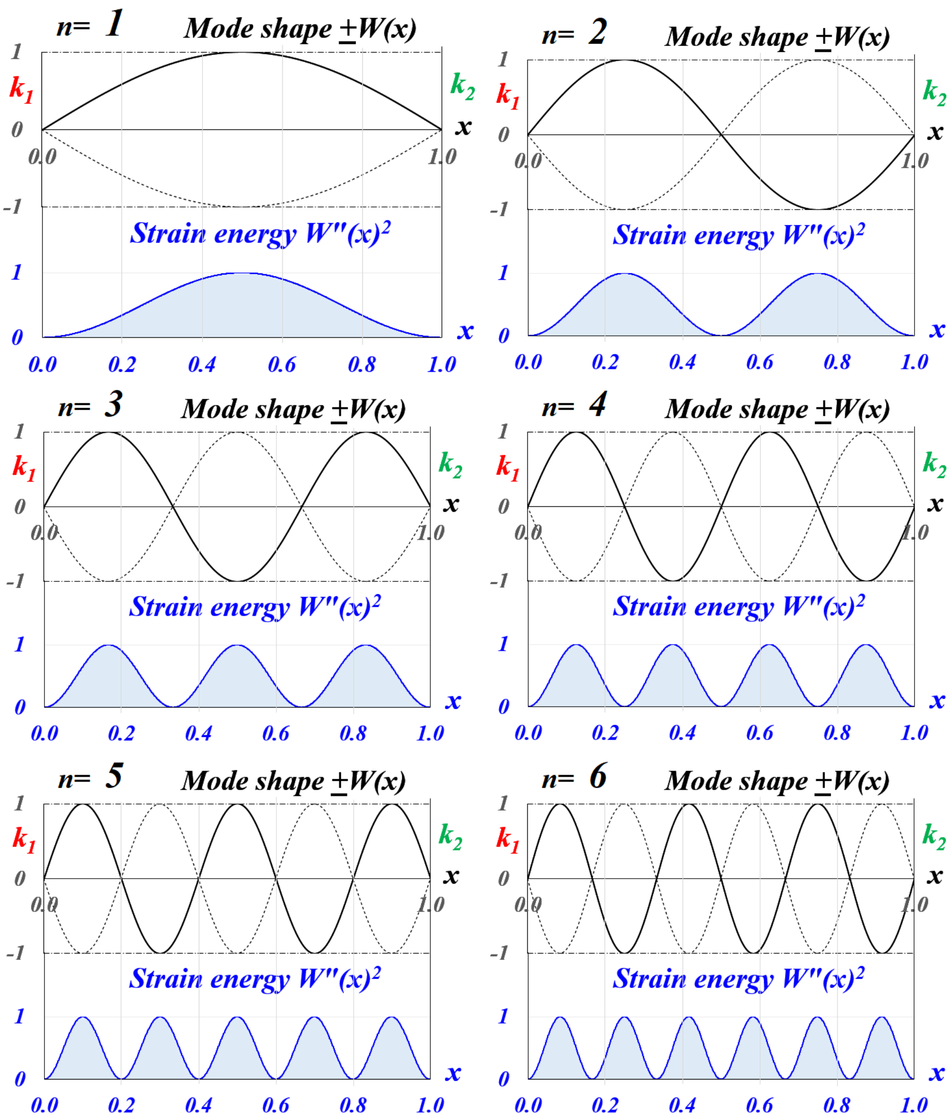

Figure 2. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_1=0.0

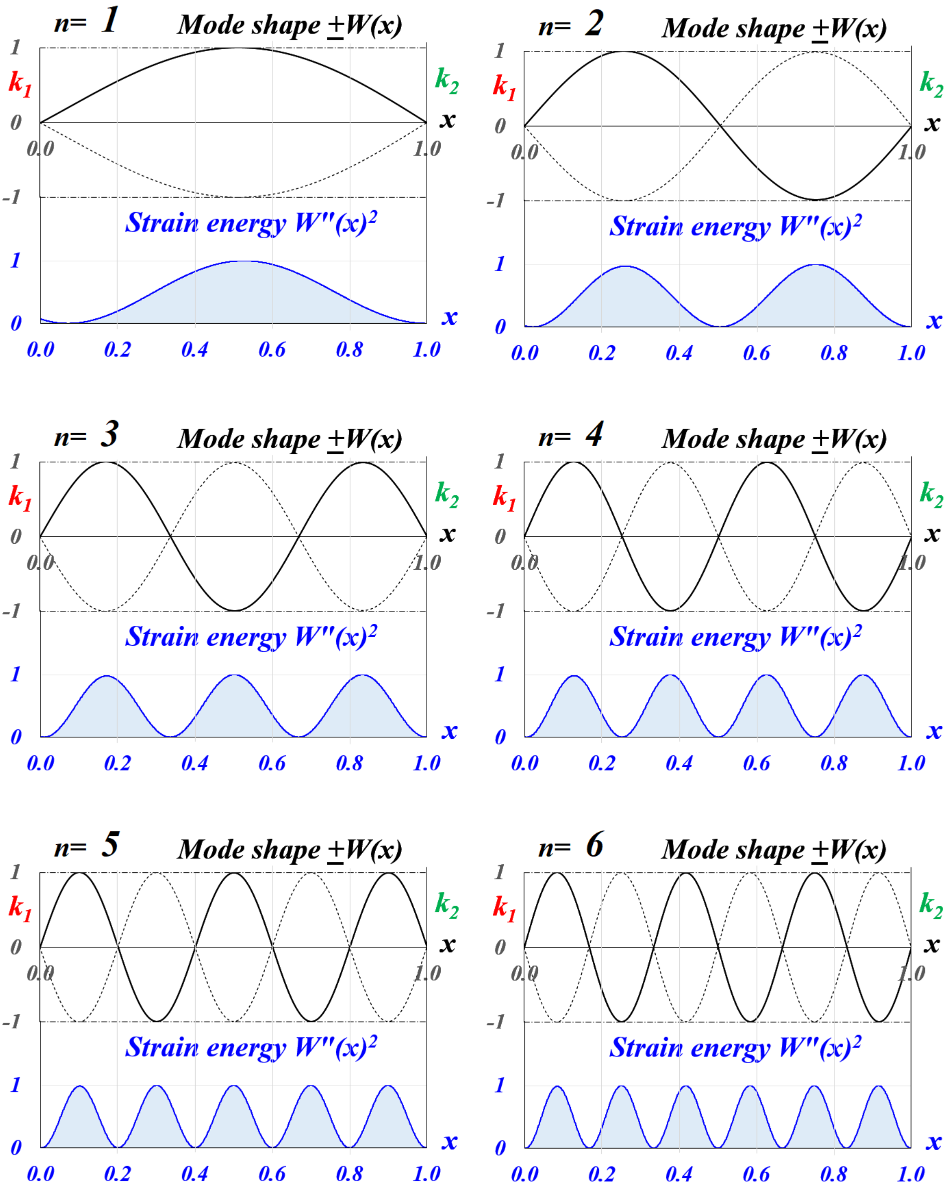

Figure 3. Normalized mode shapes (±W(x)) and normalized strain energy (W"(x)^2) for the first n=6 vibration modes and k_1=0.25

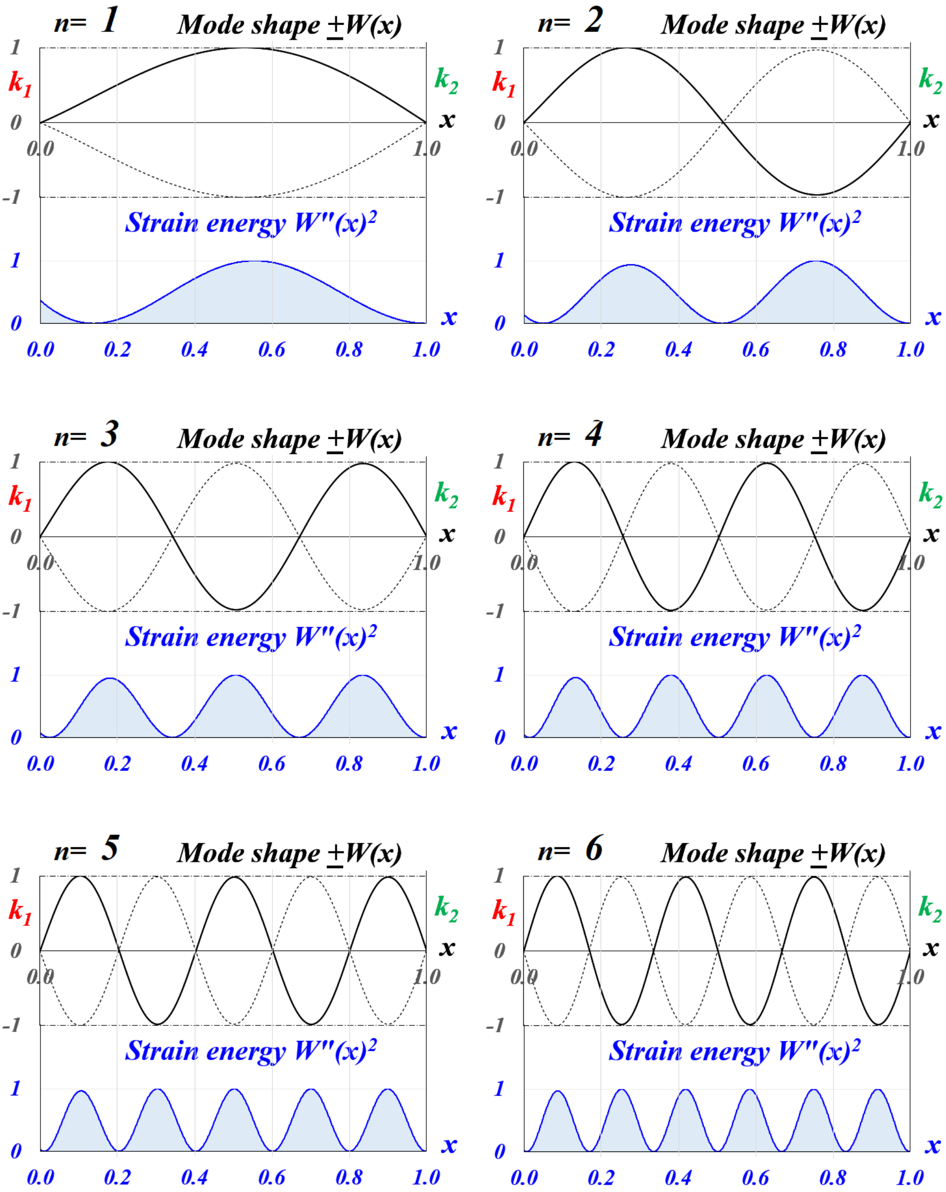

Figure 4. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_1=0.50

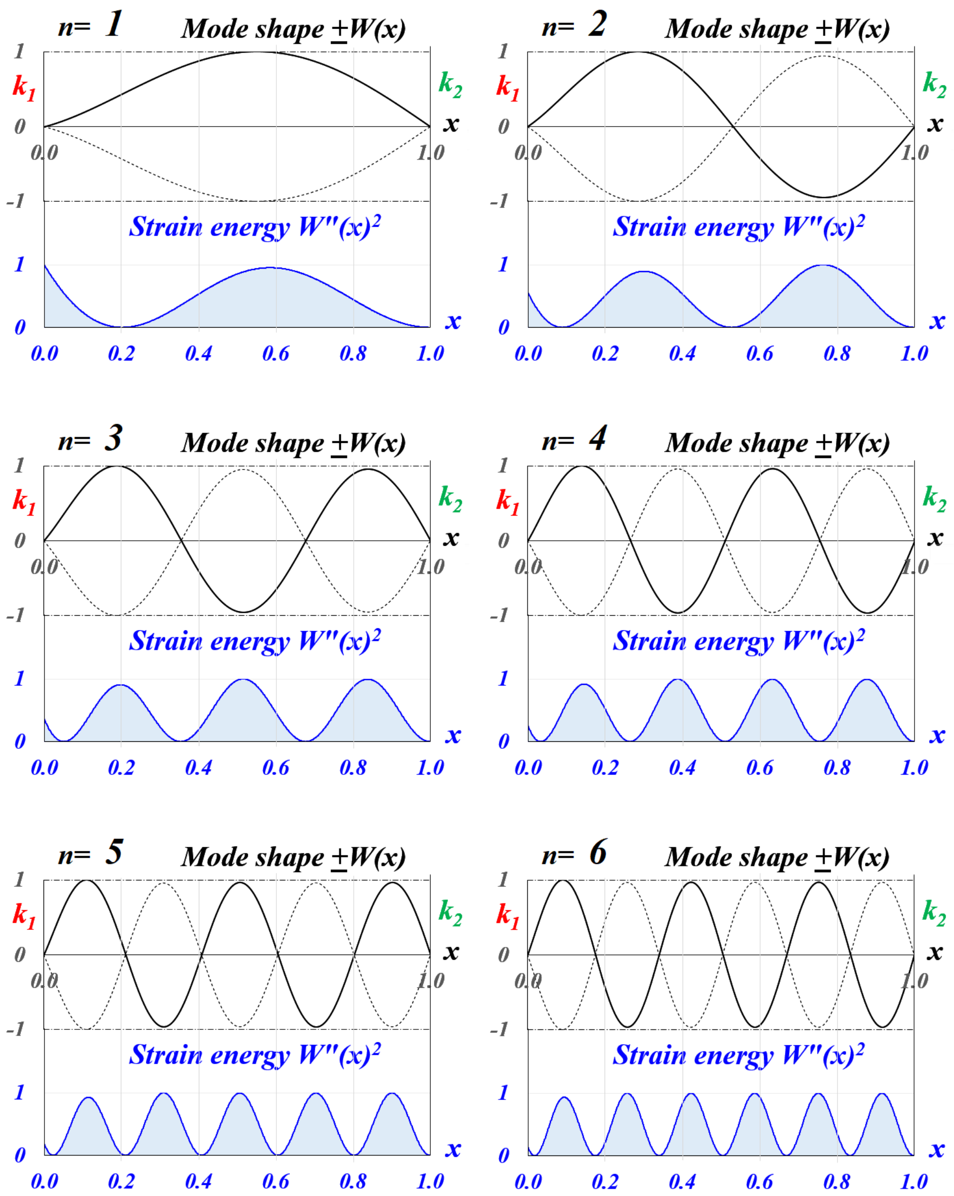

Figure 5. Normalized mode shapes ($\pm$W(x)) and normalized strain energy (W"(x)^2) for the first n=6 vibration modes and k_1=0.75

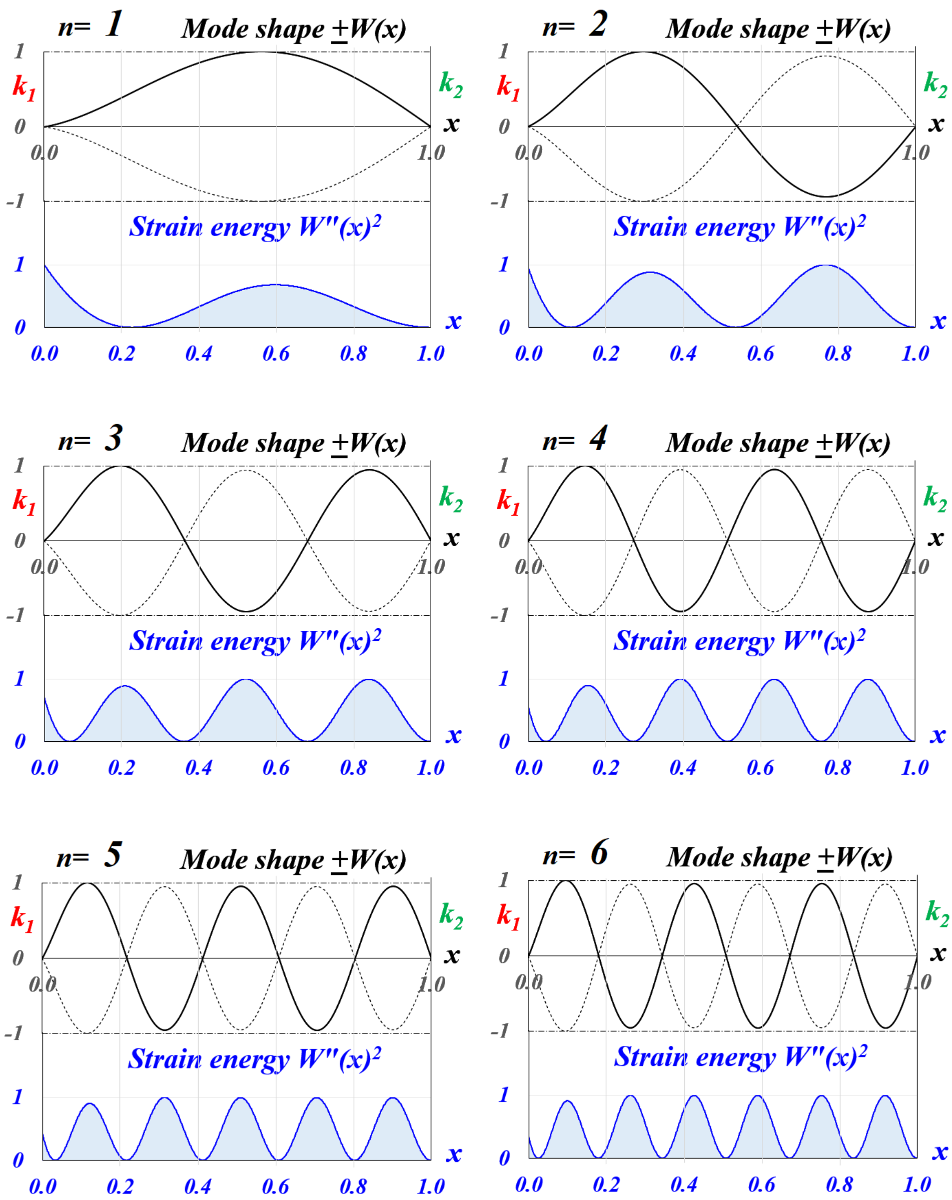

Figure 6. Normalized mode shapes (±W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_1=0.85

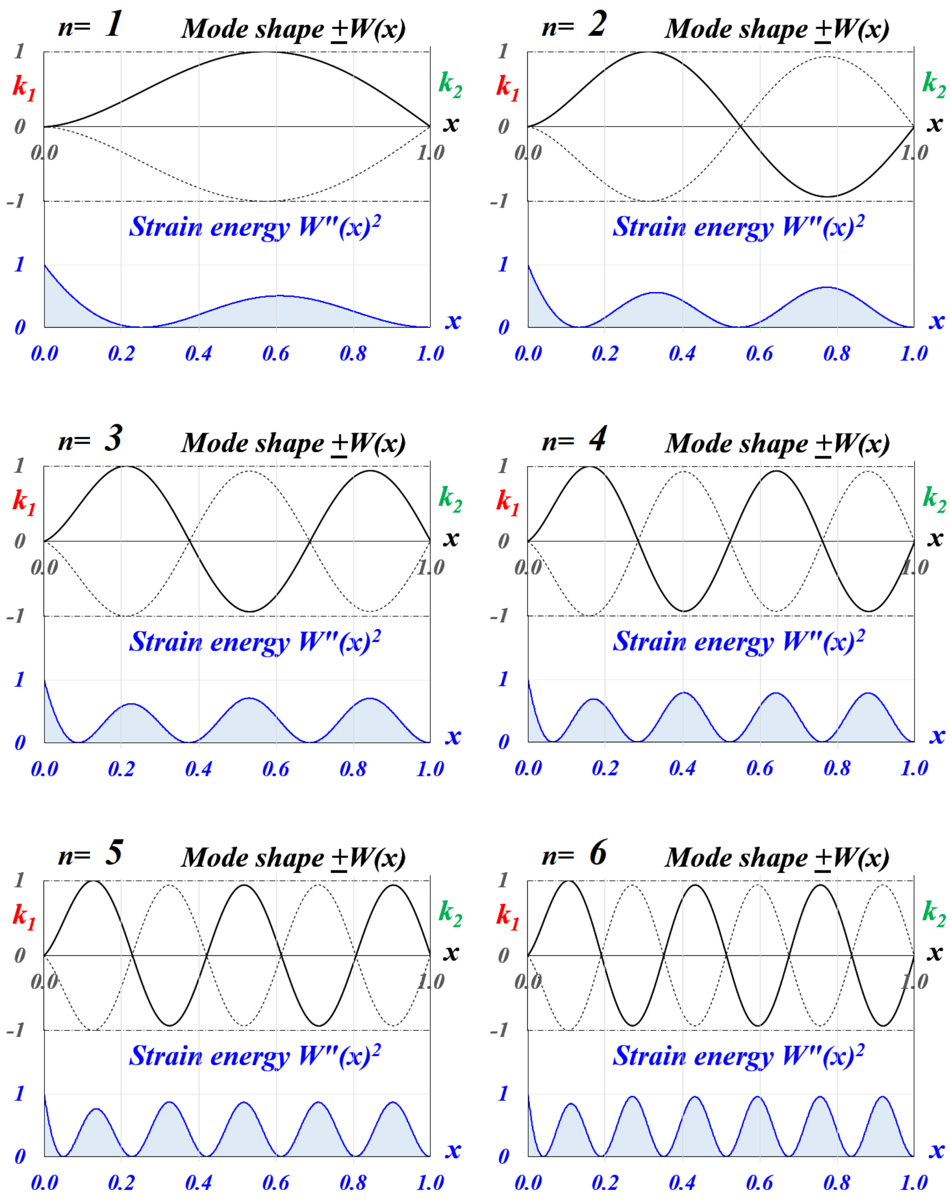

Figure 7. Normalized mode shapes (±W(x)) and normalized strain energy (W''(x)^2) for the first n=6 vibration modes and k_1=0.95

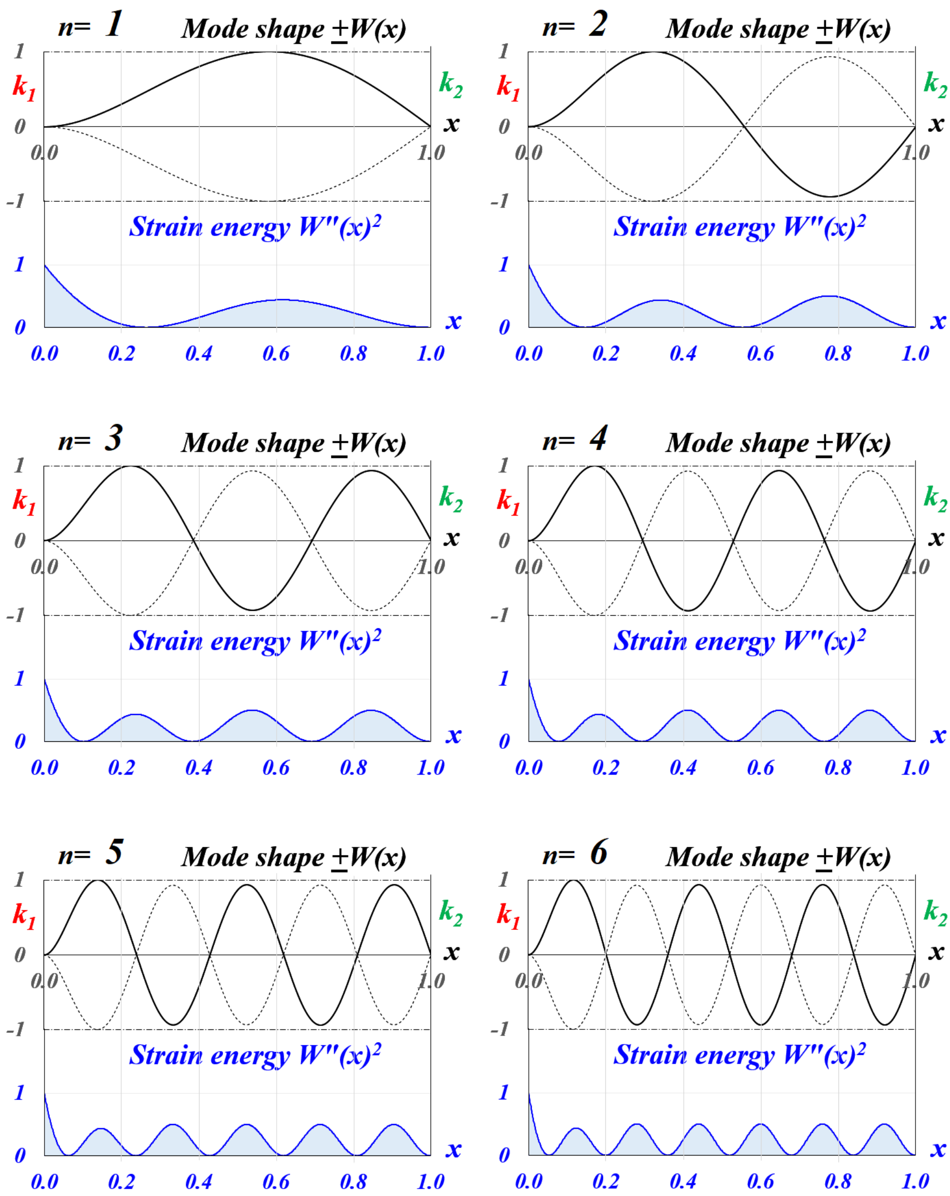

Figure 8. Normalized mode shapes (±W(x)) and normalized strain energy (W"(x)^2) for the first n=6 vibration modes and k_1=1.0

References

1 Lupu D., Tufişi C., Gillich G.R., Ardeljan M., Detection of transverse cracks in prismatic cantilever beams affected by weak clamping using a machine learning method, *Analecta Technica Szegedinenesia*, 16(01), 2022, pp. 122-128.

2 Lupu D., Gillich G.R., Nedelcu D., Gillich N., Mănescu T., *A method to detect cracks in the beams with imperfect boundary conditions,* International Conference on Applied Science (ICAS 2020), Journal of Physics: Conference Series 1781(2021), 012012, IOP Publishing, pp. 1-13.

3 Praisach Z.I., Ardeljan D., Pîrșan D.A, Gillich G.R., A new approach for imperfect boundary conditions on the dynamic behavior, *Analecta Technica Szegedinenesia*, 16(01), 2022, pp. 56-61.

4 Gillich G.R., Praisach Z.I., Exact solution for the natural frequencies of slender beams with an abrupt stiffness decrease, *Journal of Engineering Sciences and Innovation*, 2(1), 2017, A. Mechanics, Mechanical and Industrial Engineering, Mechatronics, pp. 13-21.

5 Karthikeyan M., Tiwari R., Talukdar S., *Identification of crack model parameters in a beam from modal parameters*, in 12th National Conference on Machines and Mechanisms (NaCoMM-2005), 2005.

6 Nahvi H., Jabbari M., Crack detection in beams using experimental modal data and finite element model, *International Journal of Mechanical Sciences*, 47(10), 2005, pp. 1477–1497.

7 Dems K., Turant J., Structural damage identification using frequency and modal changes, *Bulletin of the Polish Academy of Sciences Technical Sciences*, 59(1), 2011, pp. 23–32.

8 Gillich G.R., Nedelcu D., Wahab M.A., Pop M.V., Hamat C.O., A new mathematical model for cracked beams with uncertain boundary conditions, *International Conference on Noise and Vibration Engineering (ISMA 2020),* Leuven, Belgium, pp. 3871-3883.

9 Shi D., Tian Y., Choe K.N., Wang Q., A weak solution for free vibration of multi-span beams with general elastic boundary and coupling condition, *JVE International Ltd. Vibroengineering PROCEDIA*, 2016, vol. 10, pp. 298–303.

Addresses:

- Phd. stud. Eng. Patric-Timotei Stan, Doctoral School of Engineering, Babeș-Bolyai University, Cluj-Napoca, Romania, Faculty of Engineering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa
  patric.stan@stud.ubbcluj.ro

- Lect. PhD. Habil. Zeno-Iosif Praisach, Department of Engineering Science, Faculty of Engineering, Babeș-Bolyai University, Cluj-Napoca, Romania, Faculty of Engineering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa,
  zeno.praisach@ubbcluj.ro
  (*corresponding author)
- Prof. PhD. Gilbert-Rainer Gillich, Department of Engineering Science, Faculty of Engineering, Babeș-Bolyai University Cluj-Napoca, Romania, Faculty of Engineering, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa,
  gilbert.gillich@ubbcluj.ro
- Assoc. Prof. PhD Habil. Tiberiu Mănescu, Aurel Vlaicu University, Bd. Revoluției, nr. 77, Arad, Romania
  manescu.tiberiu@gmail.com
- Lect. PhD. Cristian Tufişi, Department of Engineering Science, Faculty of Engineering, Babeș-Bolyai University Cluj-Napoca, Romania, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa,
  cristian.tufisi@ubbcluj.ro