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Simulating an improved algorithm for propagation of transverse oscillations through a string.

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In present paper we bring under discussion a mathematical algorithm for determining oscillations through an elastic string. We have noticed that the algorithm has a flaw which can be solved by introducing an approximation into the system. For improved algorithm we have developed a program in order to obtain a simulator for propagation of transverse oscillations through a string, which allow us to obtain graphical representations for n-order harmonics.

Keywords: *transverse oscillation, string, algorithm, simulator*

1. Theoretical background

A perfectly elastic string has the ends fixed in two points located at the same level and at a distance L from each other. Strain F is applied at ends of string in order to keep it taut. Mass of length unit is $\mu = \rho \cdot S$, where ρ is material density and S is string cross-section. At the initial moment, the string is pulled out of the equilibrium state, displacements of points being expressed by a given function, $u(x,0) = f(x)$. At any later time, displacements are solutions of the equation [1,2]:

$$\frac{\partial^2 u}{\partial t^2} = c_t^2 \frac{\partial^2 u}{\partial x^2} \ , \ 0 \le x \le L \ , \ t > 0 \tag{1}$$

which describes the propagation of transverse oscillations along the elastic string with speed $c_t = \sqrt{\frac{F}{\mu}}$. Relation (1) is a differential equation with second order partial derivatives, with boundary conditions:

$$u(0,t) = 0 \ , \ u(L,t) = 0 \ , \ t \ge 0 \tag{2}$$

and initial conditions:

$$u(x,0)=f(x)\ ,\ \frac{\partial u}{\partial t}|_{(x,0)}=0,\ 0\le x\le L \qquad (3)$$

which means that at time t = 0 the string is at rest, its initial shape being given by function *f(x),* with *f(0) = f(L) = 0.* Solution of such an equation is given by d'Alembert's formula, used to graphically illustrate the phenomenon of transverse oscillations propagation and phase changes with π at reflection on fixed ends:

$$u(x,t)=\frac{1}{2}\left[f(x-c_t t)+f(x+c_t t)\right] \qquad (4)$$

Using variable separation method, general solution of equation (1) can be easily deduced, under the established initial conditions and to the mentioned limits [3]:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}a_n\cos\omega_n t\cdot\sin\frac{n\pi}{L}x\ ,\quad 0\le x\le L\ ,\ t>0 \qquad (5)$$

where $\omega_n = c_t\frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$ are pulsations of eigenmodes of string. As it can be noticed, propagation occurs in an infinite field, from known initial conditions, on an open border. If:

$$f(x)=u_n(x,0)=a_n\sin\frac{n\pi}{L}x\ ,\ n=1,2,3,... \qquad (6)$$

then stationary waves corresponding to the n-order harmonics can be highlighted, having nodes in points $x_m=\frac{n}{m}L$ and loops in points $x_m=\frac{2m+1}{2n}L,\ (m=1,2,...,n-1).$ All string points, except the ends and those located in nodes, will perform oscillating movements having amplitude given by envelope (6) and period of:

$$T_n=\frac{1}{n}\frac{2L}{c_t}=\frac{1}{n}T_1\ ,\ n=1,2,3,... \qquad (7)$$

For numerical integration of (6) finite difference calculus can be used for rectangular type network nodes, step along the Ox axis being denoted by h, and the one along time axis by k. If, in resulted linear equation system, we introduce notation $r=c_t^2\frac{k^2}{h^2}$, we obtain [3-5]:

$$u_{i,j+1} = r\left(u_{i-1,j} + u_{i+1,j}\right) + 2(1-r)u_{i,j} - u_{i,j-1} \ ,$$
$$i = 1,2,3,\ldots,n_x - 1 \qquad (8)$$
$$j = 1,2,3,\ldots$$

Of course, initial conditions, as well as discrete boundary conditions will change, values of displacements in nodes located on the line immediately above the axis t = 0 being determined taking into account the new initial conditions.

2. Improving and implementing algorithm

It can be shown that algorithm (8) leads to a stable solution only for $r \le 1$. For r =1 we obtain discrete form of solution (4), however, as r value decreases, solution precision decreases. Using for r values close to 1 implies relatively small step values on time axis because, in this situation, algorithm (8) gives unsatisfactory results for large values of time due to accumulation of method and rounding errors.

An improvement of the initial method can be obtained if we introduce the aproximation [3-6]:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(i,j)} \cong \frac{1}{2}\left[\left.\frac{\partial^2 u}{\partial x^2}\right|_{(i,j+1)} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{(i,j-1)}\right] \cong \frac{1}{2}\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} + \frac{u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1}}{h^2}\right] \quad (9)$$

The resulted algorithm is:

$$-\frac{r}{2}u_{i-1,j+1} + (1+r)u_{i,j+1} - \frac{r}{2}u_{i+1,j+1} = p_i \ ,$$
$$i = 1,2,3,\ldots,n_x - 1 \qquad (10)$$
$$j = 1,2,3,\ldots,$$

where:

$$p_i = \frac{2}{r}u_{i-1,j-1} - (1+r)u_{i,j-1} + \frac{r}{2}u_{i+1,j-1} + 2u_{i,j} \ , \ i = 1,2,3,\ldots,n_x - 1 \qquad (11)$$

solution being stable for any r value greater or equal to 0.

For a given j, system of linear algebraic equations (10) has a tridiagonal matrix, which can easily be solved using the Gaussian elimination method. For evaluating the solution on a given line, results calculated on two previous lines are used. Case “j=0“constitutes an exception to the rule.

The program developed for improved algorithm calculates and displays displacements $n_x = 200$ uniform points, for $n_t = 400$ succesive time values. If the value set with the scroll bar is zero, the initial displacements are given by a

Gaussian function which ensures a maximum in the middle of the string and a width at half height of one sixth of the total length. For the other values, forming of stationary waves corresponding to the first six harmonics is studied.

Following code lines are used for determining r values:

```
n = Harmonic.Value
    If n = 0 Then
      r = (2 * nx / nt) ^ 2     ' r value for Gaussian straining
    Else
      r = (2 * nx / nt / n) ^ 2   ' r value for n harmonic
    End If
```

After calculating r values we can determine elements of system matrix:

```
    r1 = 1 + r
    r2 = r / 2
    For i = 1 To nx1
      a(i) = -r2 ‘non-diagonal elements of system matrix
      b(i) = r1   ' diagonal elements of system matrix
    Next i
```

3. Tests and results

After compiling the program, we obtain the Main screen from figure 1. By pressing START button the program will calculate and generate a graphical representation of displacements in the string. There are also available “HELP” button, if there is need of information about program functioning, and “EXIT” button which interrupts program execution.

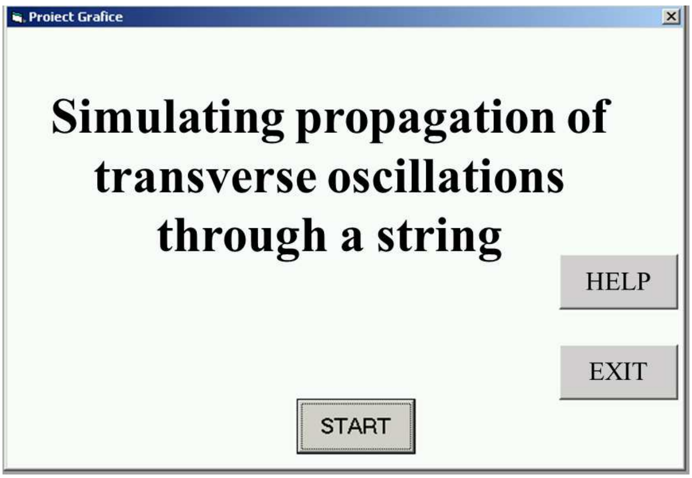

Figure 1. Program Main window

After pressing START button from figure 1 a new window will open, containing graphical representation which displays, as earlier mentioned, displacements in the string, as it can be noticed in figure 2. We again have a START button that allows us to simulate for different harmonic orders, specific number being introduced in the “Valoare” field as a number or by moving the progress bar visible on the right of window from figure 2.

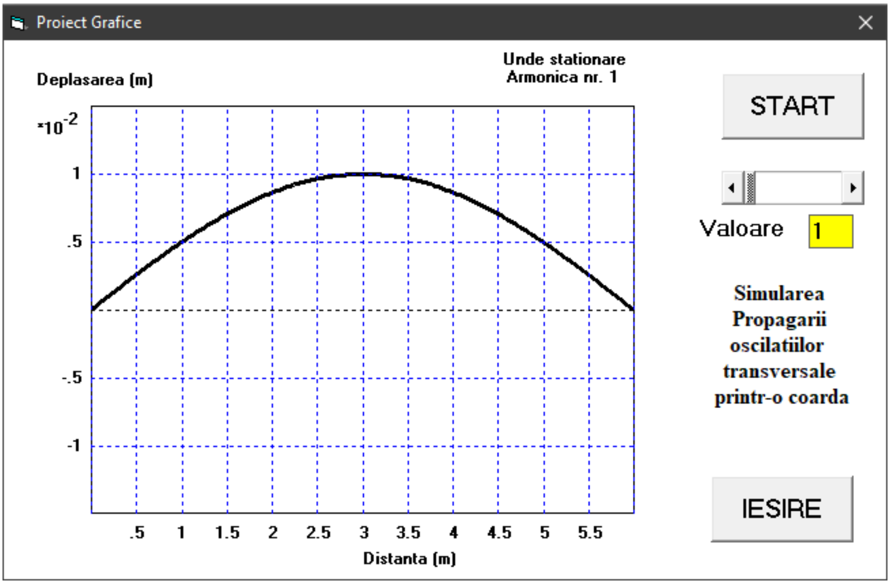

Figure 2. Graphical representation for first order harmonic

In cases corresponding to formation of stationary waves, individual oscillations of points on string can be observed.

For n = 1 we obtain the natural frequency, ν1, which corresponds to the natural vibration mode (fundamental harmonic), and for other values of n higher harmonics are obtained which are equal to an integer multiple of the natural frequency, as it results from figures 3 to 6.

The frequencies values for which the string vibrates in stationary state constitutes the discrete spectrum as eigenvalue of vibration, or resonances, of the string.

Further on, we present in figures 3 to 6 graphical representations of higher harmonics of 3, 5, 10 and 20 order, with the purpose in demonstrating program functionality. By means of simulator we build up, we can obtain graphical representation of n - order harmonics.

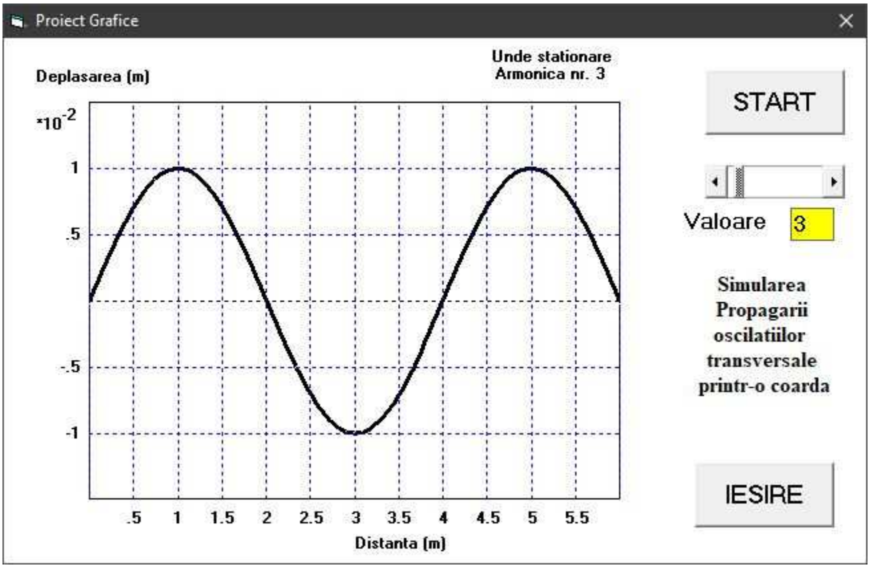

Figure 3. Graphical representation for third order harmonic

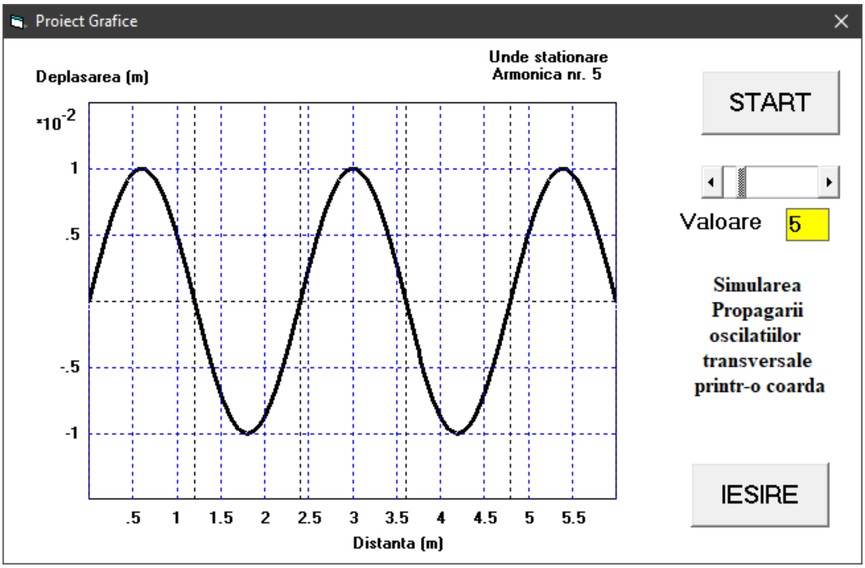

Figure 4. Graphical representation for fifth order harmonic

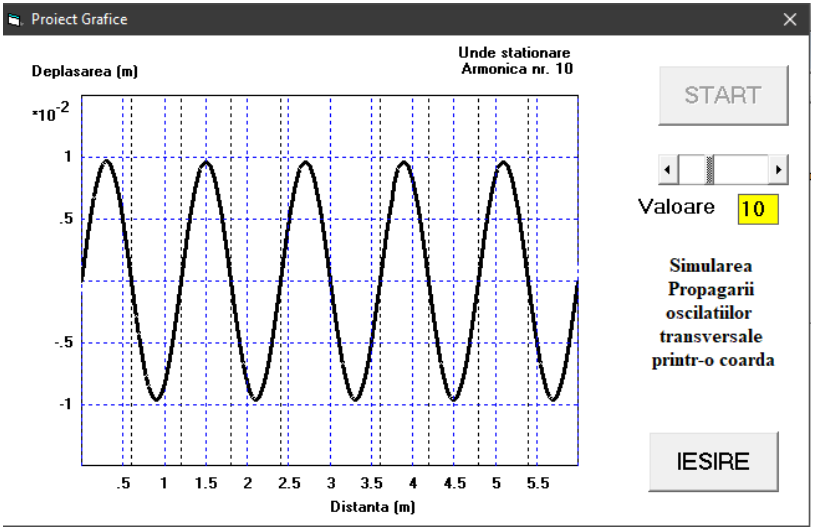

Figure 5. Graphical representation for tenth order harmonic

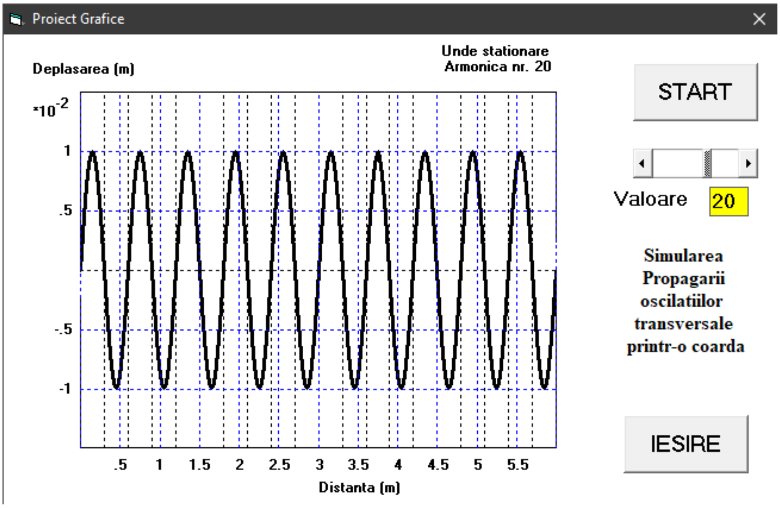

Figure 6. Graphical representation for twentieth order harmonic

4. Conclusion

By means of numerical programing, using improved mathematical algorithm, we implemented a simulator using Visual Basic software. The purpose of program we have developed for improved, briefly described in this article was to show a solution for simulating eigenmodes of a string, in arbitrary units (1, 3, 5, 10, 20 harmonics). By means of simulator we build up, we can obtain graphical representation of n order harmonics.

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