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# Dimensionless wave numbers evolution of a three spans simply supported beam when the intermediate supports are moving along the whole beam

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***Abstract.*** *Continuous beams simply supported with several intermediate supports are very common in engineering achievements everywhere. The paper shows the evolution of the dimensionless wave number in 3D format, respectively of the eigenfrequencies for a continuous beam with three openings when the intermediate supports take any position inside the beam. The frequency equation for calculating the dimensionless wave number is presented and the modal function is given with an example for the case where the eigenfrequency has the maximum value at fist vibration mode.*

***Keywords****: natural frequency, dimensionless wave number, mode shape*

## 1. Introduction

Beams that have more than one span and there are continuous throughout their lengths are known as continuous beams. A continuous beam is a statically indeterminate multispan beam on hinged supports.

There are several methods to analyze the dynamic behavior of continuous beams, such as: transfer-matrix technique [1], wave-propagation approach, Rayleigh-Ritz procedure, iterative procedure, conventional method of solving the equation of motion directly, the finite element method.

Continuous structures such as beams, can be modeled by discrete mass and stiffness parameters and analyzed as multi-degree of freedom systems [3, 4]. For these types of structures, it is necessary to assume homogeneous and isotropic material that follows Hooke's law [2, 6].

For a continuous beam of constant section, the natural frequency (1) for each vibration mode can be determined if the dimensionless wave number is known.

$$f_n = \frac{{a_n}^2}{2\pi}\sqrt{\frac{E \cdot I}{m \cdot L^4}} \tag{1}$$

where,

$f_n$ [Hz] is the natural frequency;

$a_n$ - dimensionless wave number;
$E$ [N/m$^2$] – Young's modulus;
$I$ [m$^4$] - moment of inertia;
$m$ [kg] – beam mass;
$L = l_1 + l_2 + l_3$ [m] – beam length;
$n = 1, 2, \ldots, \infty$ – number of vibration mode.

The dimensionless wave number is important both for calculating the natural frequencies and for the relationship of the mode shapes for each vibration mode. The precision of evaluating the natural frequencies is crucial, because small structural changes lead to reduced modal parameter changes [10].

## 2. Boundary conditions and frequency equation

In this paper it is considered a continuous beam supported at 4 (four) hinges, that means three spans (fig. 1). It is known that the deflection and the bending moment is zero for the end hinges. Since the beam is continuous, the slope and bending moment to the left and to the right of the intermediate supports are the same. Also, the deflection is zero for the intermediate supports [7, 8, 9].

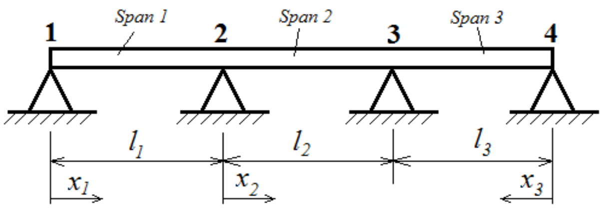


**Figure 1.** Continuous beam with three spans

Based on the above statements, for each support (1, 2, 3, 4), it can be written:

$$
1.\ \begin{cases} W_1(0)=0 \\ \dfrac{d^2W_1(0)}{dx^2}=0 \end{cases}
\qquad
2.\ \begin{cases} W_1(l_1)=0 \\ W_2(0)=0 \\ \dfrac{dW_1(l_1)}{dx}=\dfrac{dW_2(0)}{dx} \\ \dfrac{d^2W_1(l_1)}{dx^2}=\dfrac{d^2W_2(0)}{dx^2} \end{cases}
\qquad
3.\ \begin{cases} W_2(l_2)=0 \\ W_3(l_3)=0 \\ \dfrac{dW_2(l_2)}{dx}=-\dfrac{dW_3(l_3)}{dx} \\ \dfrac{d^2W_2(l_2)}{dx^2}=\dfrac{d^2W_3(l_3)}{dx^2} \end{cases}
\qquad
4.\ \begin{cases} W_3(0)=0 \\ \dfrac{d^2W_3(0)}{dx^2}=0 \end{cases}
\tag{2}
$$

where, the characteristic function or normal mode of span can be expressed:

$$W_i(x_i) = A_i \sin(a_n x_i) + B_i \cos(a_n x_i) + C_i \sinh(a_n x_i) + D_i \cosh(a_n x_i) \tag{3}$$

with $i$ = 1, 2, 3 represents the number of spans.

After solving the system of equations (2) for calculating the integration coefficients: $A_i$, $B_i$, $C_i$, $D_i$ by using the following notation (4) for the terms with constant value for a certain configuration of the continuous beam between two consecutive supports:

$$\begin{aligned}
&(1-2)\ \begin{cases} Z_{11} = \cos(a_n l_1) - \dfrac{\sin(a_n l_1)}{\sinh(a_n l_1)} \cosh(a_n l_1) \\ Z_{12} = 2\sin(a_n l_1) \end{cases} \\
&(2-3)\ \begin{cases} Z_{21} = 1 - \cos(a_n l_2)\cosh(a_n l_2) \\ Z_{22} = \cos(a_n l_2)\sinh(a_n l_2) - \sin(a_n l_2)\cosh(a_n l_2) \\ Z_{23} = 2\sin(a_n l_2)\sinh(a_n l_2) \end{cases} \\
&(3-4)\ \begin{cases} Z_{31} = \cos(a_n l_3) - \dfrac{\sin(a_n l_3)}{\sinh(a_n l_3)} \cosh(a_n l_3) \\ Z_{32} = 2\sin(a_n l_3) \end{cases}
\end{aligned} \tag{4}$$

the frequency equation of the system (2), which allows us to obtain the dimensionless wave number $a_n$, for a continuous beam with three spans, becomes:

$$(Z_{12} \cdot Z_{21} + Z_{11} \cdot Z_{22}) \cdot Z_{32} + (Z_{12} \cdot Z_{22} + Z_{11} \cdot Z_{23}) \cdot Z_{31} = 0 \tag{5}$$

The solutions of equation (5) represent the dimensionless wave numbers for the $n$ vibration modes with which we can calculate the natural frequencies (1) and also, we can plot the normalized mode shapes (3) for the continuous beam with three spans.

### 3. Mode shape equation and integration constants

The normalized mode shape equation for the continuous beam with three spans are obtained by solving the system (2) with the results replaced in (3).

Thus, for each span, the functions result:

$$
h\begin{cases}
W_1(x_1) = A_1\left[\sin(a_n x_1) - \dfrac{\sin(a_n l_1)}{\sinh(a_n l_1)}\sinh(a_n x_1)\right] \\
W_2(x_2) = A_2\sin(a_n x_2) + B_2\left[\cos(a_n x_2) - \cosh(a_n x_2)\right] + C_2\sinh(a_n x_2) \\
W_3(x_3) = A_3\left[\sin(a_n x_3) - \dfrac{\sin(a_n l_3)}{\sinh(a_n l_3)}\sinh(a_n x_3)\right]
\end{cases} \tag{6}
$$

with $x_1 \in \lfloor 0,\ l_1 \rfloor,\ x_2 \in \lfloor 0,\ l_2 \rfloor,\ x_3 \in \lfloor 0,\ l_3 \rfloor$.

The integration constants from (6) are:

$$
\begin{cases}
A_2 = -\dfrac{A_1}{\sin(a_n l_2) - \sinh(a_n l_2)}\left[Z_{12}\dfrac{\cos(a_n l_2) - \cosh(a_n l_2)}{2} + Z_{11}\sinh(a_n l_2)\right] \\
B_2 = A_1\dfrac{Z_{12}}{2} \\
C_2 = A_1 \cdot Z_{11} - A_2 \\
A_3 = -\dfrac{A_1}{\sin(a_n l_2) - \sinh(a_n l_2)} \cdot \dfrac{Z_{12} \cdot Z_{21} + Z_{11} \cdot Z_{22}}{Z_{32}}
\end{cases} \tag{7}
$$

and the constant $A_1$ has the value so that the mode shape function (6) to be normalized by the entire length of the continuous beam for each vibration mode separately.

## Dimensionless wave numbers evolution when the intermediate supports are moving along the whole beam

We will take into account the fact that the intermediate supports (fig. 1) can be in any position along the normalized continuous beam ($l_1 + l_2 + l_3 = 1$, or $l_3 = 1 - l_1 - l_2$). This means that, for each location of the intermediate supports, from relation (5), values of the dimensionless wave number are obtained for each vibration mode.

By plotting all the obtained values of the dimensionless wave number in a 3D diagram, for each vibration mode separately, we obtain a surface that gives us a general image on the evolution of the wave number depending on the position that the intermediate supports can have.

Figures 2 - 7 show the evolution of the dimensionless wave number for the first six vibration modes ($n = 6$) for the normalized continuous beam with three spans.

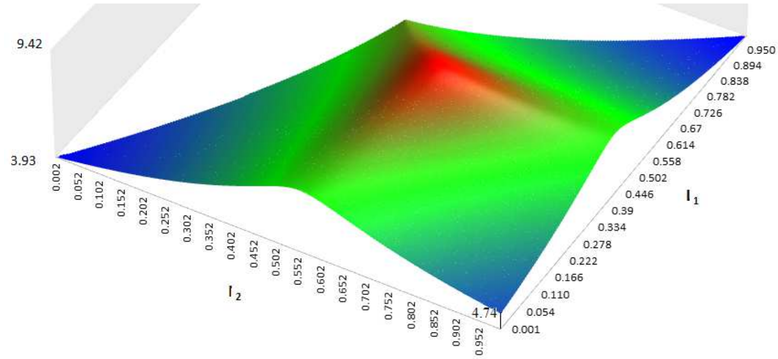


**Figure 2.** Dimensionless wave number evolution for the 1st vibration mode.

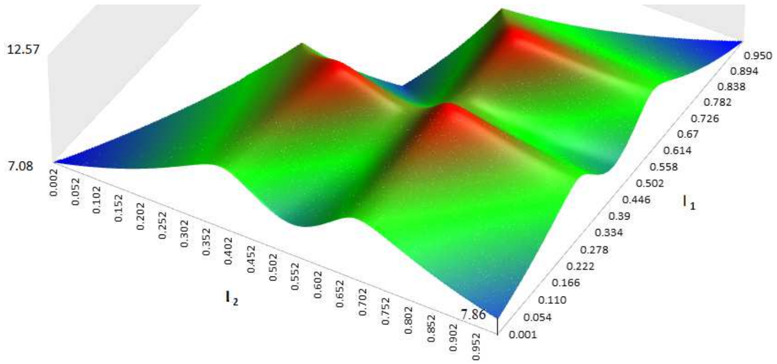


**Figure 3.** Dimensionless wave number evolution for the 2nd vibration mode.

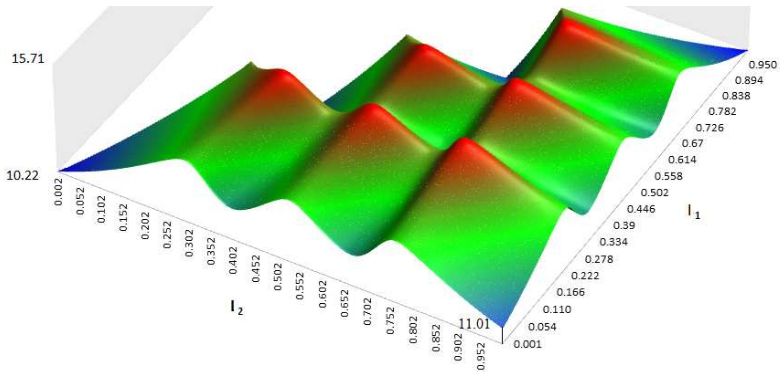


**Figure 4.** Dimensionless wave number evolution for the 3rd vibration mode.

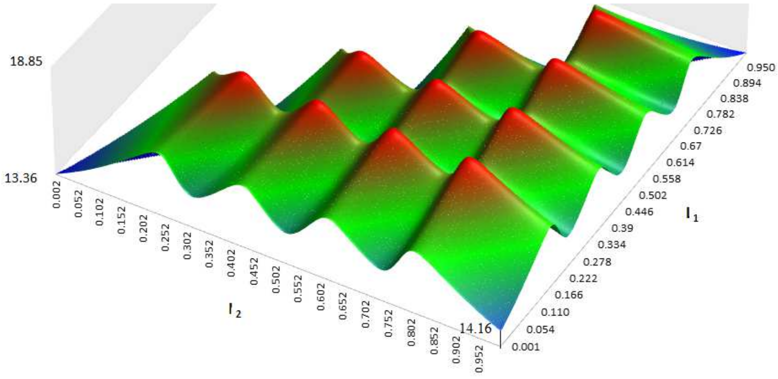


**Figure 5.** Dimensionless wave number evolution for the 4$^{th}$ vibration mode.

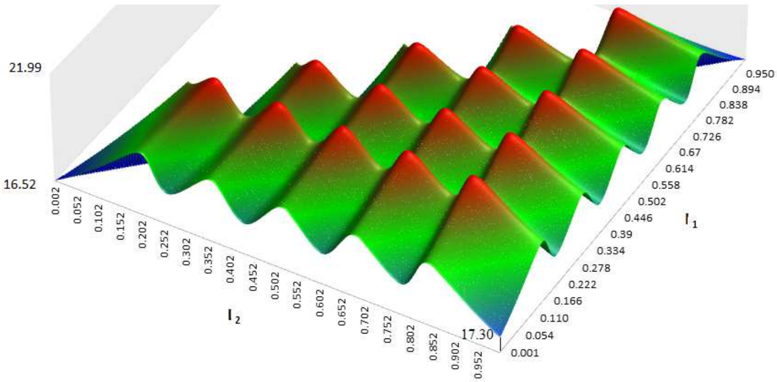


**Figure 6.** Dimensionless wave number evolution for the 5$^{th}$ vibration mode.

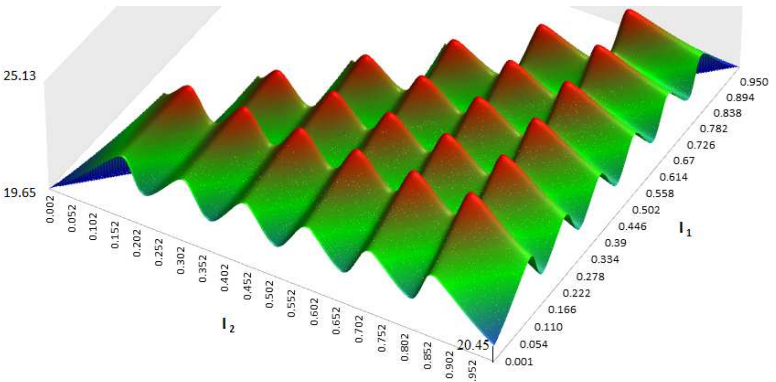


**Figure 7.** Dimensionless wave number evolution for the 6$^{th}$ vibration mode.

## 4. Conclusion

By analyzing figures 2 - 7, the following conclusions can be drawn:

1. for first vibration mode, the maximum dimensionless wave number is obtained when the intermediate supports are placed equidistantly (fig. 8), $l_1 = l_2 = l_3$. Taking into account the relation (1), for this configuration of the continuous beam we also have the maximum eigenfrequency;
2. when $l_1 \to 0$ and $l_2 \to 0$, the continuous beam behaves like a beam clamped at the left end and hinged at the right end;
3. when $l_1 \to 1$ and $l_2 \to 1$, the continuous beam behaves like a beam hinged at the left end and clamped at the right end;
4. when $l_1 \to 0$ and $l_2 \to 1$, the continuous beam behaves like a beam clamped at the both ends;

Knowing the analytical expression of the modal function, it is easy to obtained the mode shape curvature function, on which depends the establishment of the location of a damage on the beam [5], in case of its appearance.

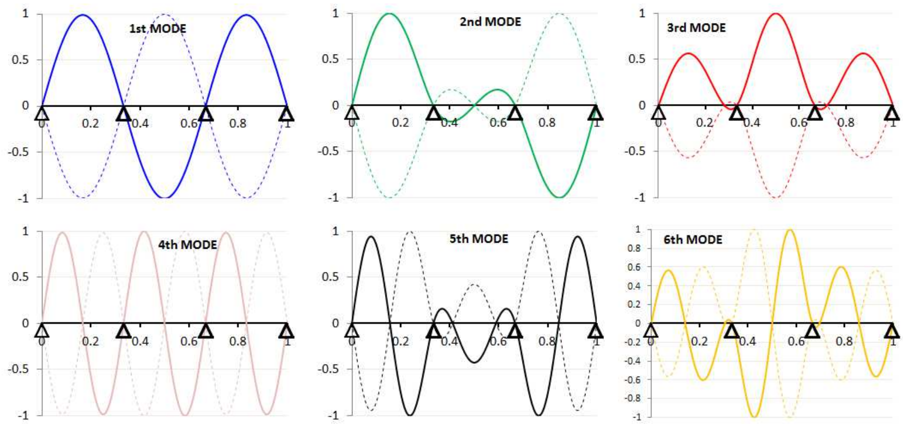


**Figure 8.** Continuous beam with intermediate supports placed equidistantly – the first six normalized mode shapes.

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