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COMPUTATION OF THE FIRST EDGE WIENER INDEX OF A COMPOSITION OF GRAPHS

MAHDIEH AZARI[a]*, ALI IRANMANESH[b], ABOLFAZL TEHRANIAN[a]

ABSTRACT The edge versions of Wiener index, based on distance between two edges in a connected graph G, were introduced by Iranmanesh et al. in 2009. In this paper, we find the first edge Wiener index of the composition of graphs.

Keywords: *Wiener index, finite graphs*

INTRODUCTION

Within this paper, we consider only simple, undirected, connected and finite graphs. A simple graph is a graph, without any loops or multiple bonds. Denote by $G=(V(G),E(G))$ a graph G with the set of vertices/ atoms $V(G)$ and the set of edges/bonds $E(G)$. For a (molecular) graph G, the degree of a vertex u is the number of edges incident to u and denoted by $\deg(u|G)$ and the distance between the vertices u and v of G, is denoted by $d(u,v|G)$ and it is defined as the number of edges in a shortest path, connecting u and v. In this paper, we denote by $[u,v]$, the edge connecting the vertices u, v of G.

A topological index is a real number related to the structural graph of a molecule. It dose not depend on the labeling or pictorial representation of a graph.

The ordinary (vertex) version of the Wiener index (or Wiener number) of G, is the sum of distances between all pairs of vertices of G, that is:

$$W(G)=W_v(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v|G).$$

This index was introduced by the Chemist, Harold Wiener [1], within the study of relations between the structure of organic compounds and their properties. This index is the first and most important topological index in Chemistry. So many interesting works have been done on it, in both Chemistry and Mathematics [2-13].

[a] *Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), P. O. Box: 14515-1775, Tehran, Iran, E-mail: azari@kau.ac.ir, tehranian@srbiau.ac.ir*

[b] *Department of Mathematics, Tarbiat Modares University,P. O. Box: 14115-137, Tehran, Iran, E-mail: iranmanesh@modares.ac.ir*

The Zagreb indices have been defined more than thirty years ago by Gutman and Trinajestic, [14].

Definition 1. [14] The first Zagreb index of G is defined as:

$$M_1(G) = \sum_{u\in V(G)} \deg(u|G)^2 \cdot$$

The edge versions of Wiener index of G, which were based on the distance between all pairs of edges of G, were introduced by Iranmanesh et al. in 2009 [15]. We encourage the reader to consult [16-20], for computational techniques and mathematical properties of the edge Wiener indices. The first edge Wiener index of G, is defined as follows:

Definition 2. [15] The first edge Wiener index of G, is denoted by $W_{e_0}(G)$. That is:

$W_{e_0}(G) = \sum_{\{e,f\}\subseteq E(G)} d_0(e,f|G)$, where $d_0(e,f|G) = \begin{cases} d_1(e,f|G)+1 & if\ e\neq f \\ 0 & if\ e=f \end{cases}$ and $d_1(e,f|G) = \min\{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\}$, such that $e=[u,v]$, $f=[z,t]$. This index satisfies the relation $W_{e_0}(G) = W_v(L(G))$, where $L(G)$ is the line graph of G.

In this paper, we want to find the first edge Wiener index of the composition of graphs.

Recall definition of the composition of two graphs.

Definition 3. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two connected graphs. We denote the composition of G_1 and G_2 by $G_1[G_2]$, that is a graph with the vertex set $V(G_1[G_2]) = V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) of $G_1[G_2]$ are adjacent if and only if: $[u_1 = v_1\ and\ [u_2,v_2]\in E(G_2)]$ or $[u_1,v_1]\in E(G_1)$.

By definition of the composition, the distance between every pair of distinct vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ of $G_1[G_2]$, is equal to

$$d(u,v|G_1[G_2]) = \begin{cases} d(u_1,v_1|G_1) & if\ u_1\neq v_1 \\ 1 & if\ u_1=v_1,\ [u_2,v_2]\in E(G_2) \\ 2 & if\ u_1=v_1,\ v_2\ is\ not\ adjacent\ to\ u_2\ in\ G_2 \end{cases}$$

COMPUTATION OF THE FIRST EDGE WIENER INDEX OF THE COMPOSITION OF GRAPHS

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. Consider the sets E_1 and E_2 as follows:

$$E_1 = \{[(u_1,u_2),(u_1,v_2)] \in E(G_1[G_2]) : u_1 \in V(G_1),\ [u_2,v_2] \in E(G_2)\}$$
$$E_2 = \{[(u_1,u_2),(v_1,v_2)] \in E(G_1[G_2]) : [u_1,v_1] \in E(G_1),\ u_2,v_2 \in V(G_2)\}$$

By definition of the composition, $E_1 \bigcup E_2 = E(G_1[G_2])$ and obviously,

$$E_1 \bigcap E_2 = \phi,\ |E_1| = |V(G_1)|\ |E(G_2)| \text{ and } |E_2| = |V(G_2)|^2 |E(G_1)|.$$

Set:

$$A = \{\{e,f\} \subseteq E(G_1[G_2]) : e \neq f,\ e,f \in E_1\}$$
$$B = \{\{e,f\} \subseteq E(G_1[G_2]) : e \neq f,\ e,f \in E_2\}$$
$$C = \{\{e,f\} \subseteq E(G_1[G_2]) : e \in E_1,\ f \in E_2\}$$

It is easy to see that each pair of the above sets is disjoint and the union of them is the set of all two element subsets of $E(G_1[G_2])$. Also we have:

$$|A| = \binom{|E_1|}{2} = \binom{|V(G_1)|\ |E(G_2)|}{2},$$
$$|B| = \binom{|E_2|}{2} = \binom{|V(G_2)|^2 |E(G_1)|}{2},$$
$$|C| = |E_1|\ |E_2| = |V(G_1)|\ |V(G_2)|^2 |E(G_1)|\ |E(G_2)|$$

Consider four subsets A_1, A_2, A_3 and A_4 of the set A as follows:

$$A_1 = \{\{e,f\} \in A : e = [(u_1,u_2),(u_1,v_2)], f = [(u_1,u_2),(u_1,z_2)],\ u_1 \in V(G_1),\ u_2,v_2,z_2 \in V(G_2)\}$$
$$A_2 = \{\{e,f\} \in A : e = [(u_1,u_2),(u_1,v_2)], f = [(u_1,z_2),(u_1,t_2)],\ u_1 \in V(G_1),\ u_2,v_2,z_2,t_2 \in V(G_2),\ \textit{both } z_2 \textit{ and } t_2 \textit{ are adjacent neither to } u_2 \textit{ nor to } v_2 \textit{ in } G_2\}$$
$$A_3 = \{\{e,f\} \in A : e = [(u_1,u_2),(u_1,v_2)], f = [(u_1,z_2),(u_1,t_2)],\ u_1 \in V(G_1),\ u_2,v_2 \in V(G_2),\ z_2,t_2 \in V(G_2) - \{u_2,v_2\}\} - A_2$$
$$A_4 = \{\{e,f\} \in A : e = [(u_1,u_2),(u_1,v_2)], f = [(v_1,z_2),(v_1,t_2)],\ u_1,v_1 \in V(G_1),\ v_1 \neq u_1,\ u_2,v_2,z_2,t_2 \in V(G_2)\}$$

It is clear that, every pair of the above sets is disjoint and $A = \bigcup_{i=1}^{4} A_i$.

In the next Proposition, we characterize $d_0(e,f|G_1[G_2])$ for all $\{e,f\} \in A$.

Proposition 1. Let $\{e,f\} \in A$.

(i) If $\{e,f\}\in A_1$, then $d_0(e,f|G_1[G_2])=1$

(ii) If $\{e,f\}\in A_2$, then $d_0(e,f|G_1[G_2])=3$

(iii) If $\{e,f\}\in A_3$, then $d_0(e,f|G_1[G_2])=2$

(iv) If $\{e,f\}\in A_4$, then $d_0(e,f|G_1[G_2])=1+d(u_1,v_1|G_1)$,

where $e=[(u_1,u_2),(u_1,v_2)], f=[(v_1,z_2),(v_1,t_2)]$

Proof. (i) Let $\{e,f\}\in A_1$ and $e=[(u_1,u_2),(u_1,v_2)], f=[(u_1,u_2),(u_1,z_2)]$. Due to distance between two vertices in $G_1[G_2]$ and by definition of $d_0(e,f)$, we have:

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,u_2)|G_1[G_2]),d((u_1,u_2),(u_1,z_2)|G_1[G_2]),$$
$$d((u_1,v_2),(u_1,u_2)|G_1[G_2]),d((u_1,v_2),(u_1,z_2)|G_1[G_2])\}=1+\min\{0,1,1,d(v_2,z_2|G_2)\}=1+0=1$$

(ii) Let $\{e,f\}\in A_2$ and $e=[(u_1,u_2),(u_1,v_2)], f=[(u_1,z_2),(u_1,t_2)]$. By definition of the set A_2, z_2 is adjacent neither to u_2 nor to v_2 in G_2 and this is also true for t_2. Therefore,

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,z_2)|G_1[G_2]),d((u_1,u_2),(u_1,t_2)|G_1[G_2]),$$
$$d((u_1,v_2),(u_1,z_2)|G_1[G_2]),d((u_1,v_2),(u_1,t_2)|G_1[G_2])\}=1+\min\{2,2,2,2\}=3.$$

(iii) Let $\{e,f\}\in A_3$ and $e=[(u_1,u_2),(u_1,v_2)], f=[(u_1,z_2),(u_1,t_2)]$. By definition of the set A_3, $z_2\notin\{u_2,v_2\}$, $t_2\notin\{u_2,v_2\}$.

On the other hand $\{e,f\}\notin A_2$, so at least one of the following situations occurs:

$[u_2,z_2]\in E(G_2)$, $[u_2,t_2]\in E(G_2)$, $[v_2,z_2]\in E(G_2)$ or $[v_2,t_2]\in E(G_2)$.

This means that, at least one of the distances $d((u_1,u_2),(u_1,z_2)|G_1[G_2])$, $d((u_1,u_2),(u_1,t_2)|G_1[G_2])$, $d((u_1,v_2),(u_1,z_2)|G_1[G_2])$ or $d((u_1,v_2),(u_1,t_2)|G_1[G_2])$ is equal to 1. Therefore,

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,z_2)|G_1[G_2]),d((u_1,u_2),(u_1,t_2)|G_1[G_2]),$$
$$d((u_1,v_2),(u_1,z_2)|G_1[G_2]),d((u_1,v_2),(u_1,t_2)|G_1[G_2])\}=1+1=2.$$

(iv) Let $\{e,f\}\in A_4$ and $e=[(u_1,u_2),(u_1,v_2)], f=[(v_1,z_2),(v_1,t_2)]$. Thus $v_1\neq u_1$ and

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(v_1,z_2)|G_1[G_2]),d((u_1,u_2),(v_1,t_2)|G_1[G_2]),$$
$$d((u_1,v_2),(v_1,z_2)|G_1[G_2]),d((u_1,v_2),(v_1,t_2)|G_1[G_2])\}=$$

$1+\min\{d(u_1,v_1|G_1),d(u_1,v_1|G_1),d(u_1,v_1|G_1),d(u_1,v_1|G_1)\}=1+d(u_1,v_1|G_1)$,
so the proof is completed

In follow, we define five subsets B_1,B_2,B_3,B_4 and B_5 of the set B.

$B_1=\{\{e,f\}\in B: e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,u_2),(v_1,z_2)],\ u_1,v_1\in V(G_1),$
$u_2,v_2,z_2\in V(G_2)\}$

$B_2=\{\{e,f\}\in B: e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,z_2),(v_1,t_2)],\ u_1,v_1\in V(G_1),$
$u_2,v_2,z_2,t_2\in V(G_2),\ z_2\neq u_2,\ t_2\neq v_2\}$

$B_3=\{\{e,f\}\in B: e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,u_2),(z_1,z_2)],\ u_1,v_1,z_1\in V(G_1),$
$u_2,v_2,z_2\in V(G_2),\ z_1\neq v_1\}$

$B_4=\{\{e,f\}\in B: e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,t_2),(z_1,z_2)],\ u_1,v_1,z_1\in V(G_1),$
$u_2,v_2,t_2,z_2\in V(G_2),\ z_1\neq v_1,\ t_2\neq u_2\}$

$B_5=\{\{e,f\}\in B: e=[(u_1,u_2),(v_1,v_2)], f=[(z_1,z_2),(t_1,t_2)],\ u_1,v_1\in V(G_1),$
$z_1,t_1\in V(G_1)-\{u_1,v_1\},\ u_2,v_2,z_2,t_2\in V(G_2)\}$

It is clear that, each pair of the above sets is disjoint and $B=\bigcup_{i=1}^{5}B_i$.

The next Proposition, characterizes $d_0(e,f|G_1[G_2])$ for all $\{e,f\}\in B$.

Proposition 2. Let $\{e,f\}\in B$.

(i) If $\{e,f\}\in B_1$, then $d_0(e,f|G_1[G_2])=1$

(ii) If $\{e,f\}\in B_2$, then $d_0(e,f|G_1[G_2])=2$

(iii) If $\{e,f\}\in B_3$, then $d_0(e,f|G_1[G_2])=d_0([u_1,v_1],[u_1,z_1]|G_1)$,
where $e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,u_2),(z_1,z_2)]$

(iv) If $\{e,f\}\in B_4$, then $d_0(e,f|G_1[G_2])=d_0([u_1,v_1],[u_1,z_1]|G_1)+1$,
where $e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,t_2),(z_1,z_2)]$

(v) If $\{e,f\}\in B_5$, then $d_0(e,f|G_1[G_2])=d_0([u_1,v_1],[z_1,t_1]|G_1)$,
where $e=[(u_1,u_2),(v_1,v_2)], f=[(z_1,z_2),(t_1,t_2)]$

Proof. (i) Let $\{e,f\}\in B_1$ and $e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,u_2),(v_1,z_2)]$.
Using the definition of $d_0(e,f)$, we have:

$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,u_2)|G_1[G_2]),d((u_1,u_2),(v_1,z_2)|G_1[G_2]),$
$d((v_1,v_2),(u_1,u_2)|G_1[G_2]),d((v_1,v_2),(v_1,z_2)|G_1[G_2])\}=$
$1+\min\{0,1,1,d((v_1,v_2),(v_1,z_2)|G_1[G_2])\}=1+0=1$.

(ii) Let $\{e,f\}\in B_2$ and $e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,z_2),(v_1,t_2)]$. By definition of B_2, $z_2\neq u_2$, $t_2\neq v_2$. So due to distance between two vertices in $G_1[G_2]$, the distances $d((u_1,u_2),(u_1,z_2)|G_1[G_2])$ and $d((v_1,v_2),(v_1,t_2)|G_1[G_2])$ are either 1 or 2. Therefore,

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,z_2)|G_1[G_2]),d((u_1,u_2),(v_1,t_2)|G_1[G_2]),$$
$$d((v_1,v_2),(u_1,z_2)|G_1[G_2]),d((v_1,v_2),(v_1,t_2)|G_1[G_2])\}=$$
$$1+\min\{d((u_1,u_2),(u_1,z_2)|G_1[G_2]),1,1,d((v_1,v_2),(v_1,t_2)|G_1[G_2]\}=1+1=2$$

(iii) Let $\{e,f\}\in B_3$ and $e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,u_2),(z_1,z_2)]$. By the definition of B_3 we have $z_1\neq v_1$ and hence

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,u_2)|G_1[G_2]),d((u_1,u_2),(z_1,z_2)|G_1[G_2]),$$
$$d((v_1,v_2),(u_1,u_2)|G_1[G_2]),d((v_1,v_2),(z_1,z_2)|G_1[G_2])\}=$$
$$1+\min\{d(u_1,u_1|G_1),d(u_1,z_1|G_1),d(v_1,u_1|G_1),d(v_1,z_1|G_1)\}=d_0([u_1,v_1],[u_1,z_1]|G_1)$$

(iv) Let $\{e,f\}\in B_4$ and $e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,t_2),(z_1,z_2)]$. By definition of B_4, $z_1\neq v_1$, $t_2\neq u_2$. So $d(v_1,z_1|G_1)\geq 1$ and $d((u_1,u_2),(u_1,t_2)|G_1[G_2])\geq 1$. Therefore

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(u_1,t_2)|G_1[G_2]),d((u_1,u_2),(z_1,z_2)|G_1[G_2]),$$
$$d((v_1,v_2),(u_1,t_2)|G_1[G_2]),d((v_1,v_2),(z_1,z_2)|G_1[G_2])\}=$$
$$1+\min\{d((u_1,u_2),(u_1,t_2)|G_1[G_2]),1,1,d(v_1,z_1|G_1)\}=1+1=d_0([u_1,v_1],[u_1,z_1]|G_1)+1$$

(v) Let $\{e,f\}\in B_5$ and $e=[(u_1,u_2),(v_1,v_2)], f=[(z_1,z_2),(t_1,t_2)]$. By the definition of B_5, $z_1\neq u_1$, $z_1\neq v_1$, $t_1\neq u_1$ and $t_1\neq v_1$. So the edges $[u_1,v_1]$ and $[z_1,t_1]$ of G_1 are distinct. Therefore

$$d_0(e,f|G_1[G_2])=1+\min\{d((u_1,u_2),(z_1,z_2)|G_1[G_2]),d((u_1,u_2),(t_1,t_2)|G_1[G_2]),$$
$$d((v_1,v_2),(z_1,z_2)|G_1[G_2]),d((v_1,v_2),(t_1,t_2)|G_1[G_2])\}=$$
$$1+\min\{d(u_1,z_1|G_1),d(u_1,t_1|G_1),d(v_1,z_1|G_1),d(v_1,t_1|G_1)\}=d_0([u_1,v_1],[z_1,t_1]|G_1)$$

and the proof is completed

Now, we consider three subsets C_1, C_2 and C_3 of the set C as follows:

$C_1=\{\{e,f\}\in C: e=[(u_1,u_2),(u_1,v_2)], f=[(u_1,u_2),(z_1,z_2)],\ u_1,z_1\in V(G_1),$ where $u_2,v_2,z_2\in V(G_2)\}$

$C_2=\{\{e,f\}\in C: e=[(u_1,u_2),(u_1,v_2)], f=[(u_1,t_2),(z_1,z_2)],\ u_1,z_1\in V(G_1),$ where $u_2,v_2,t_2,z_2\in V(G_2),\ t_2\neq u_2,\ t_2\neq v_2\}$

$C_3 = \{\{e,f\} \in C : e = [(u_1,u_2),(u_1,t_2)], f = [(v_1,v_2),(z_1,z_2)],\ u_1,v_1,z_1 \in V(G_1),$
where $u_2,t_2,v_2,z_2 \in V(G_2),\ v_1 \neq u_1,\ z_1 \neq u_1\}$

Clearly, every pair of the above sets is disjoint and $C = \bigcup_{i=1}^{3} C_i$.

In the following Proposition, we find $d_0(e,f|G_1[G_2])$ for all $\{e,f\} \in C$.

Proposition 3. Let $\{e,f\} \in C$.

(i) If $\{e,f\} \in C_1$, then $d_0(e,f|G_1[G_2]) = 1$

(ii) If $\{e,f\} \in C_2$, then $d_0(e,f|G_1[G_2]) = 2$

(iii) If $\{e,f\} \in C_3$, then

$d_0(e,f|G_1[G_2]) = 1 + \min\{d(u_1,v_1|G_1), d(u_1,z_1|G_1)\}$,

where $e = [(u_1,u_2),(u_1,t_2)], f = [(v_1,v_2),(z_1,z_2)]$

Proof. (i) Let $\{e,f\} \in C_1$ and $e = [(u_1,u_2),(u_1,v_2)], f = [(u_1,u_2),(z_1,z_2)]$.
By definition of $d_0(e,f)$, we have:

$$d_0(e,f|G_1[G_2]) = 1 + \min\{d((u_1,u_2),(u_1,u_2)|G_1[G_2]), d((u_1,u_2),(z_1,z_2)|G_1[G_2]),$$
$$d((u_1,v_2),(u_1,u_2)|G_1[G_2]), d((u_1,v_2),(z_1,z_2)|G_1[G_2])\} = 1 + \min\{0,1,1,1\} = 1 + 0 = 1$$

(ii) Let $\{e,f\} \in C_2$ and $e = [(u_1,u_2),(u_1,v_2)], f = [(u_1,t_2),(z_1,z_2)]$. By definition of C_2 , $t_2 \neq u_2$, $t_2 \neq v_2$. Thus, due to the distance between two vertices in $G_1[G_2]$, the distances $d((u_1,u_2),(u_1,t_2)|G_1[G_2])$ and $d((u_1,v_2),(u_1,t_2)|G_1[G_2])$ are either 1 or 2. So

$$d_0(e,f|G_1[G_2]) = 1 + \min\{d((u_1,u_2),(u_1,t_2)|G_1[G_2]), d((u_1,u_2),(z_1,z_2)|G_1[G_2]),$$
$$d((u_1,v_2),(u_1,t_2)|G_1[G_2]), d((u_1,v_2),(z_1,z_2)|G_1[G_2])\} =$$
$$1 + \min\{d((u_1,u_2),(u_1,t_2)|G_1[G_2]),1, d((u_1,v_2),(u_1,t_2)|G_1[G_2],1\} = 1 + 1 = 2 .$$

(iii) Let $\{e,f\} \in C_3$ and $e = [(u_1,u_2),(u_1,t_2)], f = [(v_1,v_2),(z_1,z_2)]$. By definition of C_3 , $v_1 \neq u_1$, $z_1 \neq u_1$. Therefore

$$d_0(e,f|G_1[G_2]) = 1 + \min\{d((u_1,u_2),(v_1,v_2)|G_1[G_2]), d((u_1,u_2),(z_1,z_2)|G_1[G_2]),$$
$$d((u_1,t_2),(v_1,v_2)|G_1[G_2]), d((u_1,t_2),(z_1,z_2)|G_1[G_2])\} =$$
$$1 + \min\{d(u_1,v_1|G_1), d(u_1,z_1|G_1), d(u_1,v_1|G_1), d(u_1,z_1|G_1)\} =$$

$1 + \min\{d(u_1,v_1|G_1), d(u_1,z_1|G_1)\}$, and the proof is completed

Definition 4. Let $G=(V(G),E(G))$ be a graph.

(i) Let $u\in V(G)$. Set; $\Delta_u=\{z\in V(G):[z,u]\in E(G)\}$. In fact, Δ_u is the set of all vertices of G, which are adjacent to u. Suppose that, δ_u is the number of all vertices of G, which are adjacent to u. Clearly, $\delta_u=|\Delta_u|=\deg(u|G)$.

(ii) For each pair of distinct vertices $u,v\in V(G)$, let $\delta_{(u,v)}$ be the number of all vertices of G, which are adjacent both to u and v. Obviously, $\delta_{(u,v)}=|\Delta_u\cap\Delta_v|$.

(iii) Let u, v and z be three vertices of G, which every pair of them is distinct. Assume that, $\delta_{(u,v,z)}$ denotes the number of all vertices of G which are adjacent to vertices u, v and z. It is easy to see that, $\delta_{(u,v,z)}=|\Delta_u\cap\Delta_v\cap\Delta_z|$.

(iv) Suppose that, u, v and z be three vertices of graph G, which every pair of them is distinct. Denote by $N_{(z,\tilde{u},\tilde{v})}$, the number of all vertices of G, which are adjacent to z, but neither to u nor to v. By the definition of $N_{(z,\tilde{u},\tilde{v})}$, we have:

$$N_{(z,\tilde{u},\tilde{v})}=|\Delta_z-(\Delta_u\cup\Delta_v)|=|\Delta_z|-|\Delta_z\cap(\Delta_u\cup\Delta_v)|=|\Delta_z|-|(\Delta_z\cap\Delta_u)\cup(\Delta_z\cap\Delta_v)|=$$
$$|\Delta_z|-(|\Delta_z\cap\Delta_u|+|\Delta_z\cap\Delta_v|-|\Delta_z\cap\Delta_u\cap\Delta_v|)=\delta_z-\delta_{(z,u)}-\delta_{(z,v)}+\delta_{(z,u,v)}.$$

Proposition 5.

$$\sum_{\{e,f\}\in A}d_0(e,f|G_1[G_2])=|E(G_2)|^2\left(\binom{|V(G_1)|+1}{2}+W(G_1)\right)-\frac{1}{4}|V(G_1)|\ (2M_1(G_2)-N(G_2)),$$

where, $N(G_2)=\sum_{[u_2,v_2]\in E(G_2)}\sum_{z_2\in V(G_2)-(\Delta_{u_2}\cup\Delta_{v_2})}N_{(z_2,\tilde{u}_2,\tilde{v}_2)}$.

Proof. At first, we need to find $|A_2|$ and $|A_2\cup A_3|$. It is easy to see that

$$|A_2|=\frac{1}{4}|V(G_1)|\sum_{[u_2,v_2]\in E(G_2)}\sum_{z_2\in V(G_2)-(\Delta_{u_2}\cup\Delta_{v_2})}N_{(z_2,\tilde{u}_2,\tilde{v}_2)}=\frac{1}{4}|V(G_1)|N(G_2),$$

$$|A_2\cup A_3|=\frac{1}{2}|V(G_1)|\sum_{[u_2,v_2]\in E(G_2)}(|E(G_2)|-(\delta_{u_2}+\delta_{v_2}-1))=$$
$$\frac{1}{2}|V(G_1)|\left(\sum_{[u_2,v_2]\in E(G_2)}|E(G_2)|-\sum_{[u_2,v_2]\in E(G_2)}(\delta_{u_2}+\delta_{v_2})+\sum_{[u_2,v_2]\in E(G_2)}1\right)=$$
$$\frac{1}{2}|V(G_1)|\ (|E(G_2)|^2+|E(G_2)|-M_1(G_2)),$$

Recall that, each pair of the sets $A_i (1 \le i \le 4)$ is disjoint and $A = \bigcup_{i=1}^{4} A_i$, then by
Proposition 1, we have:

$$\sum_{\{e,f\}\in A} d_0(e,f|G_1[G_2]) = \sum_{i=1}^{4} \sum_{\{e,f\}\in A_i} d_0(e,f|G_1[G_2]) = |A_1| + 3|A_2| + 2|A_3| +$$

$$\sum \{1 + d(u_1,v_1|G_1) : \{e,f\} \in A_4, e = [(u_1,u_2),(u_1,v_2)], f = [(v_1,z_2),(v_1,t_2)]\} =$$
$$|A_1| + 3|A_2| + 2|A_3| + |A_4| +$$

$$\sum \{d(u_1,v_1|G_1) : \{e,f\} \in A_4, e = [(u_1,u_2),(u_1,v_2)], f = [(v_1,z_2),(v_1,t_2)]\} =$$
$$\sum_{i=1}^{4} |A_i| + (|A_2| + |A_3|) + |A_2| + |E(G_2)|^2 \sum_{\{u_1,v_1\}\subseteq V(G_1)} d(u_1,v_1|G_1) =$$
$$\left|\bigcup_{i=1}^{4} A_i\right| + |A_2 \cup A_3| + |A_2| + |E(G_2)|^2 W(G_1) = |A| + |A_2 \cup A_3| + |A_2| + |E(G_2)|^2 W(G_1) =$$
$$\binom{|V(G_1)|\ |E(G_2)|}{2} + \frac{1}{2}|V(G_1)|\ (|E(G_2)|^2 + |E(G_2)| - M_1(G_2)) + \frac{1}{4}|V(G_1)|N(G_2) +$$
$$|E(G_2)|^2 W(G_1) =$$
$$\frac{1}{2}\left(|V(G_1)|^2 |E(G_2)|^2 - |V(G_1)|\ |E(G_2)| + |V(G_1)|\ |E(G_2)|^2 + |V(G_1)|\ |E(G_2)| \right) -$$
$$\frac{1}{2}|V(G_1)|M_1(G_2) + \frac{1}{4}|V(G_1)|N(G_2) + |E(G_2)|^2 W(G_1) =$$
$$|E(G_2)|^2 \left(\binom{|V(G_1)|+1}{2} + W(G_1) \right) - \frac{1}{4}|V(G_1)|\ (2M_1(G_2) - N(G_2))|$$

Proposition 6.

$$\sum_{\{e,f\}\in B} d_0(e,f|G_1[G_2]) = |V(G_2)|^2 \binom{|V(G_2)|}{2} M_1(G_1) + |V(G_2)|^4 W_{e_0}(G_1)$$

Proof. For the proof of this proposition, we need to obtain $|B_1|, |B_2|$ and $|B_4|$. It is easy to see that:

$$\left|B_1\right| = 2\left|E(G_1)\right| \left|V(G_2)\right| \binom{|V(G_2)|}{2},\ \left|B_2\right| = 2\left|E(G_1)\right| \binom{|V(G_2)|}{2}^2,$$

$$\left|B_4\right| = \left|V(G_2)\right|^3(\left|V(G_2)\right|-1)\sum_{u_1\in V(G_1)}\binom{\delta_{u_1}}{2} = \left|V(G_2)\right|^2\binom{|V(G_2)|}{2}(M_1(G_1)-2\left|E(G_1)\right|)$$

Afterwards, we find $\sum_{\{e,f\}\in B_3\cup B_4\cup B_5} d_0(e,f\,|G_1[G_2])$. By Proposition 2.2, we have:

$$\sum_{\{e,f\}\in B_3} d_0(e,f\,|G_1[G_2]) =$$

$$\sum\{d_0([u_1,v_1],[u_1,z_1]|G_1):\{e,f\}\in B_3, e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,u_2),(z_1,z_2)]\} =$$

$$\left|V(G_2)\right|^3 \sum_{u_1\in V(G_1)}\ \sum_{\{[u_1,v_1],[u_1,z_1]\}\subseteq E(G_1)} d_0([u_1,v_1],[u_1,z_1]|G_1) =$$

$$\frac{1}{2}\left|V(G_2)\right|^3 \sum_{[u_1,v_1]\in E(G_1)}\ \sum_{\substack{z_1\in\{u_1,v_1\},\\ [z_1,t_1]\in E(G_1)}} d_0([u_1,v_1],[z_1,t_1]|G_1),$$

$$\sum_{\{e,f\}\in B_4} d_0(e,f\,|G_1[G_2]) =$$

$$\sum\{d_0([u_1,v_1],[u_1,z_1]|G_1)+1:\{e,f\}\in B_4, e=[(u_1,u_2),(v_1,v_2)], f=[(u_1,t_2),(z_1,z_2)]\} =$$

$$(\left|V(G_2)\right|^4-\left|V(G_2)\right|^3)\sum_{u_1\in V(G_1)}\ \sum_{\{[u_1,v_1],[u_1,z_1]\}\subseteq E(G_1)} d_0([u_1,v_1],[u_1,z_1]|G_1)+\left|B_4\right| =$$

$$\frac{1}{2}(\left|V(G_2)\right|^4-\left|V(G_2)\right|^3)\sum_{[u_1,v_1]\in E(G_1)}\ \sum_{\substack{z_1\in\{u_1,v_1\},\\ [z_1,t_1]\in E(G_1)}} d_0([u_1,v_1],[z_1,t_1]|G_1)+\left|B_4\right|,$$

$$\sum_{\{e,f\}\in B_5} d_0(e,f\,|G_1[G_2]) =$$

$$\sum\{d_0([u_1,v_1],[z_1,t_1]|G_1):\{e,f\}\in B_5, e=[(u_1,u_2),(v_1,v_2)], f=[(z_1,z_2),(t_1,t_2)]\} =$$

$$\frac{1}{2}\left|V(G_2)\right|^4 \sum_{[u_1,v_1]\in E(G_1)}\ \sum_{\substack{[z_1,t_1]\in E(G_1),\\ z_1,t_1\notin\{u_1,v_1\}}} d_0([u_1,v_1],[z_1,t_1]|G_1).$$

Based on the above computations and since each pair of $B_i(1\le i\le 5)$ is disjoint, we have:

$$\sum_{\{e,f\}\in B_3\cup B_4\cup B_5} d_0(e,f\,|G_1[G_2]) =$$

$$\sum_{i=3}^{5}\sum_{\{e,f\}\in B_i} d_0(e,f\,|G_1[G_2]) = \frac{1}{2}\left|V(G_2)\right|^3 \sum_{[u_1,v_1]\in E(G_1)}\ \sum_{\substack{z_1\in\{u_1,v_1\},\\ [z_1,t_1]\in E(G_1)}} d_0([u_1,v_1],[z_1,t_1]|G_1)+$$

$$\frac{1}{2}\left(|V(G_2)|^4-|V(G_2)|^3\right)\sum_{[u_1,v_1]\in E(G_1)}\sum_{\substack{z_1\in\{u_1,v_1\},\\ [z_1,t_1]\in E(G_1)}} d_0([u_1,v_1],[z_1,t_1]|G_1)+|B_4|+$$

$$\frac{1}{2}|V(G_2)|^4\sum_{[u_1,v_1]\in E(G_1)}\sum_{\substack{[z_1,t_1]\in E(G_1),\\ z_1,t_1\notin\{u_1,v_1\}}} d_0([u_1,v_1],[z_1,t_1]|G_1)=$$

$$\frac{1}{2}|V(G_2)|^4\sum_{[u_1,v_1]\in E(G_1)}\sum_{\substack{z_1\in\{u_1,v_1\},\\ [z_1,t_1]\in E(G_1)}} d_0([u_1,v_1],[z_1,t_1]|G_1)+|B_4|+$$

$$\frac{1}{2}|V(G_2)|^4\sum_{[u_1,v_1]\in E(G_1)}\sum_{\substack{[z_1,t_1]\in E(G_1),\\ z_1,t_1\notin\{u_1,v_1\}}} d_0([u_1,v_1],[z_1,t_1]|G_1)=$$

$$|B_4|+\frac{1}{2}|V(G_2)|^4(2W_{e_0}(G_1))=|B_4|+|V(G_2)|^4W_{e_0}(G_1).$$

Now, since $B=\bigcup_{i=1}^{5}B_i$, we have:

$$\sum_{\{e,f\}\in B} d_0(e,f|G_1[G_2])=\sum_{\{e,f\}\in B_1} d_0(e,f|G_1[G_2])+\sum_{\{e,f\}\in B_2} d_0(e,f|G_1[G_2])+\sum_{\{e,f\}\in B_3\cup B_4\cup B_5} d_0(e,f|G_1[G_2])=$$

$$|B_1|+2|B_2|+|B_4|+|V(G_2)|^4W_{e_0}(G_1)=2|E(G_1)|\binom{|V(G_2)|}{2}\left(|V(G_2)|+2\binom{|V(G_2)|}{2}-|V(G_2)|^2\right)+$$

$$|V(G_2)|^2\binom{|V(G_2)|}{2}M_1(G_1)+|V(G_2)|^4W_{e_0}(G_1)=|V(G_2)|^2\binom{|V(G_2)|}{2}M_1(G_1)+|V(G_2)|^4W_{e_0}(G_1)\cdot$$

Proposition 7.

$$\sum_{\{e,f\}\in C} d_0(e,f|G_1[G_2])=$$

$$|E(G_1)|\ |E(G_2)|\ |V(G_2)|\ (|V(G_1)|\ |V(G_2)|+2|V(G_2)|-4)+|E(G_2)|\ |V(G_2)|^2\ Min(G_1)$$

where, $Min(G_1)=\sum_{u_1\in V(G_1)}\sum_{[v_1,z_1]\in E(G_1)}\min\{d(u_1,v_1|G_1),d(u_1,z_1|G_1)\}$

Proof. First, we find $|C_2|$ and $\sum_{\{e,f\}\in C_3} d_0(e,f|G_1[G_2])$. It is easy to see that:

$$|C_2|=|V(G_2)|\ (|V(G_2)|-2)\ |E(G_2)|\sum_{u_1\in V(G_1)}\delta_{u_1}=2|E(G_1)|\ |E(G_2)|\ |V(G_2)|\ (|V(G_2)|-2)$$

and by Proposition 3, we have:

$$\sum_{\{e,f\}\in C_3} d_0(e,f|G_1[G_2]) =$$
$$\sum\{1+\min\{d(u_1,v_1|G_1), d(u_1,z_1|G_1) : \{e,f\}\in C_3, e=[(u_1,u_2),(u_1,t_2)], f=[(v_1,v_2),(z_1,z_2)]\} =$$
$$|C_3| + |E(G_2)|\ |V(G_2)|^2 \sum_{u_1\in V(G_1)} \sum_{\substack{[v_1,z_1]\in E(G_1)\\ v_1\neq z_1, z\neq_1 u_1}} \min\{d(u_1,v_1|G_1), d(u_1,z_1|G_1)\} =$$
$$|C_3| + |E(G_2)|\ |V(G_2)|^2 \sum_{u_1\in V(G_1)} \sum_{[v_1,z_1]\in E(G_1)} \min\{d(u_1,v_1|G_1), d(u_1,z_1|G_1)\} =$$
$$|C_3| + |E(G_2)|\ |V(G_2)|^2 Min(G_1).$$

Since each pair of the sets $C_i (1\le i\le 3)$ is disjoint and $C=\bigcup_{i=1}^{3} C_i$, we have:

$$\sum_{\{e,f\}\in C} d_0(e,f|G_1[G_2]) = \sum_{\{e,f\}\in C_1\cup C_2} d_0(e,f|G_1[G_2]) + \sum_{\{e,f\}\in C_3} d_0(e,f|G_1[G_2]) =$$
$$|C_1| + 2|C_2| + |C_3| + |E(G_2)|\ |V(G_2)|^2 Min(G_1) =$$
$$\sum_{i=1}^{3} |C_i| + |C_2| + |E(G_2)|\ |V(G_2)|^2 Min(G_1) =$$
$$\left|\bigcup_{i=1}^{3} C_i\right| + |C_2| + |E(G_2)|\ |V(G_2)|^2 Min(G_1) =$$
$$|C| + |C_2| + |E(G_2)|\ |V(G_2)|^2 Min(G_1) = |V(G_1)|\ |V(G_2)|^2 |E(G_1)|\ |E(G_2)| +$$
$$2|E(G_1)|\ |E(G_2)|\ |V(G_2)|\ (|V(G_2)|-2) + |E(G_2)|\ |V(G_2)|^2 Min(G_1) =$$
$$|E(G_1)|\ |E(G_2)|\ |V(G_2)|\ (|V(G_1)|\ |V(G_2)| + 2|V(G_2)| - 4) + |E(G_2)|\ |V(G_2)|^2 Min(G_1)$$

Now, as the main purpose of this paper, we express the following theorem, which characterizes the first edge Wiener index of the composition of two graphs.

Theorem. Let $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$ be two simple undirected connected finite graphs, then

$$W_{e_0}(G_1[G_2]) = |E(G_2)|^2 \binom{|V(G_1)|+1}{2} +$$
$$|E(G_1)|\ |E(G_2)|\ |V(G_2)|\ (|V(G_1)|\ |V(G_2)| + 2|V(G_2)| - 4) +$$
$$|E(G_2)|^2 W(G_1) + |V(G_2)|^4 W_{e_0}(G_1) + |V(G_2)|^2 \binom{|V(G_2)|}{2} M_1(G_1) +$$
$$|E(G_2)|\ |V(G_2)|^2 Min(G_1) -$$

$\frac{1}{4}|V(G_1)|\ (2M_1(G_2)-N(G_2))$,

where $Min(G_1)=\sum_{u_1\in V(G_1)}\sum_{[v_1,z_1]\in E(G_1)}\min\{d(u_1,v_1|G_1),d(u_1,z_1|G_1)\}$ and

$N(G_2)=\sum_{[u_2,v_2]\in E(G_2)}\sum_{z_2\in V(G_2)-(\Delta_{u_2}\cup\Delta_{v_2})}N_{(z_2,\widetilde{u}_2,\widetilde{v}_2)}$.

Proof. Recall that, each pair of the sets A, B and C is disjoint and union of them is the set of all two element subsets of $E(G_1[G_2])$. Now, using the definition of the first edge Wiener index, we obtain:

$W_{e_0}(G_1[G_2])=\sum_{\{e,f\}\subseteq E(G_1[G_2])}d_0(e,f|G_1[G_2])=\sum_{\{e,f\}\in A}d_0(e,f|G_1[G_2])+$

$\sum_{\{e,f\}\in B}d_0(e,f|G_1[G_2])+\sum_{\{e,f\}\in C}d_0(e,f|G_1[G_2])$. Now, by the above Lemmas, the proof is completed.

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